

Fractional derivatives

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n-th order derivative

- Sequence of n-fold integrals and n-fold derivatives:

$$\dots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1) d\tau_1, \int_a^t f(\tau_1) d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2 f(t)}{dt^2}, \dots$$

- Fourier transform: $FT \left(\frac{d^n f(t)}{dt^n} \right) = (i\omega)^n F(\omega)$

$$\dots, (i\omega)^{-2} F(\omega), (i\omega)^{-1} F(\omega), F(\omega), i\omega F(\omega), (i\omega)^2 F(\omega), \dots$$

- I Podlubny, Fractional Differential Equations, Academic Press, 1999

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A physical fractional device: Capacitor soakage

- $i = Cd^a u/dt^a$, $a \lesssim 1$
- $Z = u/i = 1/C(j\omega)^a$
- Example in video:
 - 220 μF /63 Volt
 - 10 Volts for 60 sec
 - Shorted for 6 sec
 - http://www.youtube.com/watch?v=vhHog_yCQ4Q
- Dielectric absorption
 - Westerlund and Ekstam. "Capacitor theory," IEEE Trans. Dielectrics and Electrical Insulation, 1994



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Derivative of arbitrary order

- Derivative of order α :

$$\frac{d^\alpha f(t)}{dt^\alpha} =_a D_t^\alpha f(t)$$

- $\alpha < 0 \Leftrightarrow$ integration
- a and t: limits in defining integral

- Fourier transform (neglecting initial cond's):

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

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Fourier approach to fractional operator (1)

- Integer ($m > \alpha$) + fraction ($\alpha - m < 0$):

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega) = (i\omega)^m F(\omega) (i\omega)^{\alpha-m}$$

- First part: ordinary derivative
- Second part: fractional part
 - What is its inverse Fourier transform?

Fourier transform

$$h(t) = \frac{1}{\Gamma(\beta)} \frac{1}{t^{1-\beta}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{-\beta}$$

- $0 < \beta < 1$ Podlubny, 1999, pp. 110-
- $\Gamma(\cdot)$ is the gamma function
 - Generalization of the factorial: $\Gamma(n+1)=n!$
- Let $\beta=m-\alpha$ and rewrite:

$$h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{\alpha-m}$$

Fourier approach to fractional operator (2)

- Fractional Fourier transform as a convolution of derivative of order first integer m larger than α and a memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^m f(t)}{dt^m} * \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}$$

Fractional derivative: Two flavors

- Riemann-Liouville: order $\alpha \in \mathbb{R}$, $m-1 \leq \alpha < m$:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt} \right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

– First convolution, then integer order derivation

- Caputo: order $m-1 \leq \alpha < m$:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

– First integer order derivative, then convolution

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Riemann-Liouville vs Caputo

- Riemann-Liouville requires initialization of derivatives of non-integer orders:

$$\lim_{t \rightarrow a} {}_a D_t^{\alpha-1} f(t), \lim_{t \rightarrow a} {}_a D_t^{\alpha-2} f(t), \dots$$

- Caputo requires initialization of integer order derivatives: $f^{(k)}(0)$, $k=0, 1, \dots, m-1$
 - Usually have physical meaning
 - Simpler to use in numerical solutions

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Fractional derivative for numerics

- Caputo (lower limit $a = -\infty$):

$${}_{-\infty}^C D_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial^m f(t)}{\partial t^m} * g_{m-\alpha}(t)$$

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0$$

- Convolution with a memory function



Memory function

- Convolution kernel:

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}$$

- $m-\alpha = \varepsilon^+$: no memory,
 $\Gamma(\varepsilon^+) \rightarrow \infty$ for $\varepsilon^+ \rightarrow 0$
 \Rightarrow kernel \rightarrow impulse
- $m-\alpha = 1$: infinite
memory

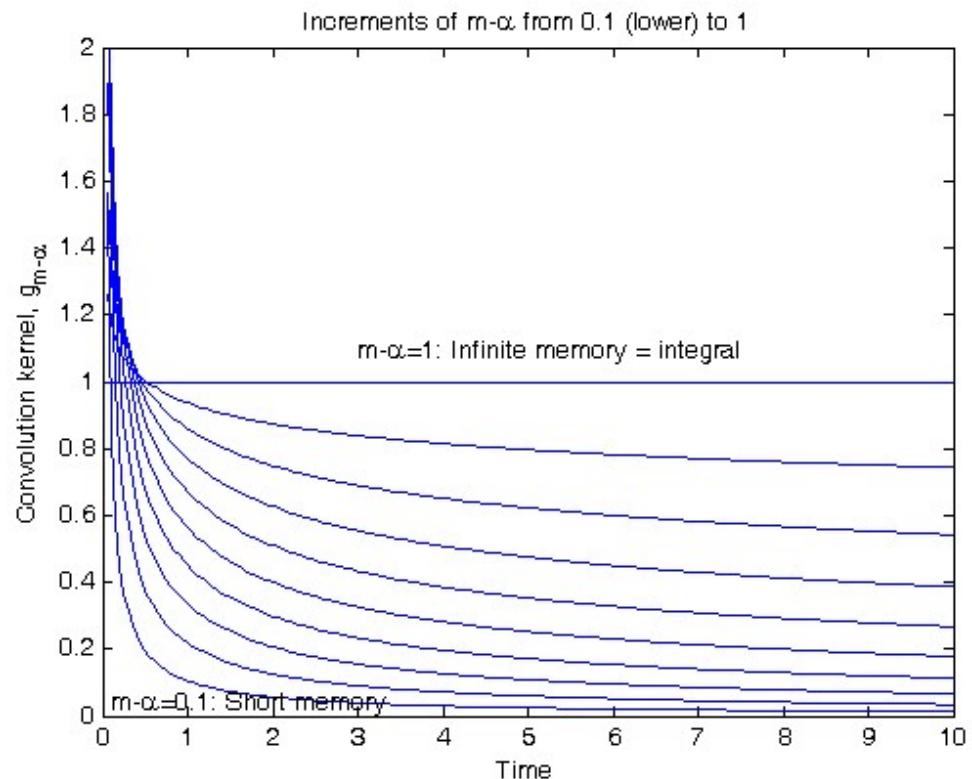


Figure based on Treeby and Cox, "Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian", J. Acoust. Soc. Amer, 2010

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Fractional derivative of order 0..1

- Example: $0 \leq \alpha < 1$ (Caputo with $m=1$):

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial f(t)}{\partial t} * g_{1-\alpha}(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{-\infty}^t \frac{f^{(1)}(\tau)}{(t - \tau)^\alpha} d\tau$$

- Limits:

- $\alpha \rightarrow 0 \Rightarrow \frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^t \frac{f^{(1)}(\tau)}{(t - \tau)^0} d\tau = f(t)$

- $\alpha \rightarrow 1 \Rightarrow \frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^t f^{(1)}(\tau) \delta(t - \tau) d\tau = f^{(1)}(t)$

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Fractional Integral

- Definition via Fourier relationship:

$$F f I^{\circledR}[f(t)]g = (j!)^{1-\circledR} F f f g$$

- Cancel derivative:

$$\frac{d}{dt^{\circ}}[I^{\circledR}] = \begin{cases} \frac{d}{dt^{\circ}} I^{\circledR} & ; 0 < \circledR < \infty \\ I^{\circledR} & ; 0 < \infty < \circledR \end{cases}$$

- Integral representation:

$$I^{\circledR}[f(t)] = \frac{1}{\Gamma(\circledR)} \int_0^t (t - \zeta)^{(\circledR-1)} f(\zeta) d\zeta; \quad 0 < \circledR$$

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha+1-m}} d\tau$$

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Conclusion

Two interpretations of fractional derivative:

1. Fourier:

$$FT \left(\frac{d^\alpha f(t)}{dt^\alpha} \right) = (i\omega)^\alpha F(\omega)$$

2. Convolution of ordinary derivative of order $m > \alpha$ and causal memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$