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Abel's 1823 Paper on Fractional Derivatives



UiO : Fysisk institutt

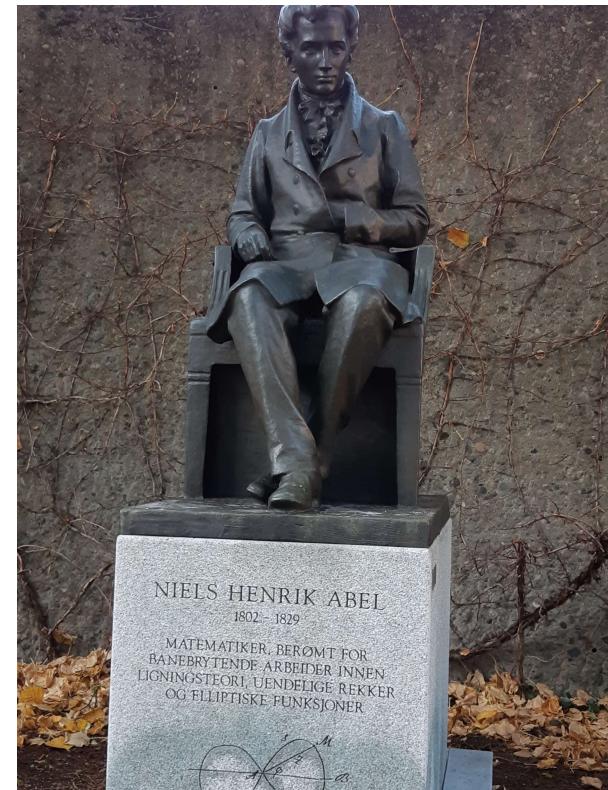
Det matematisk-naturvitenskapelige fakultet

An autumn day in Oslo ...

- Unfortunately, the inscription does not mention his contribution to the calculus of non-integer derivatives,
“Niels Henrik Abel and the birth of fractional calculus”

Twitter 30 Oct 2019

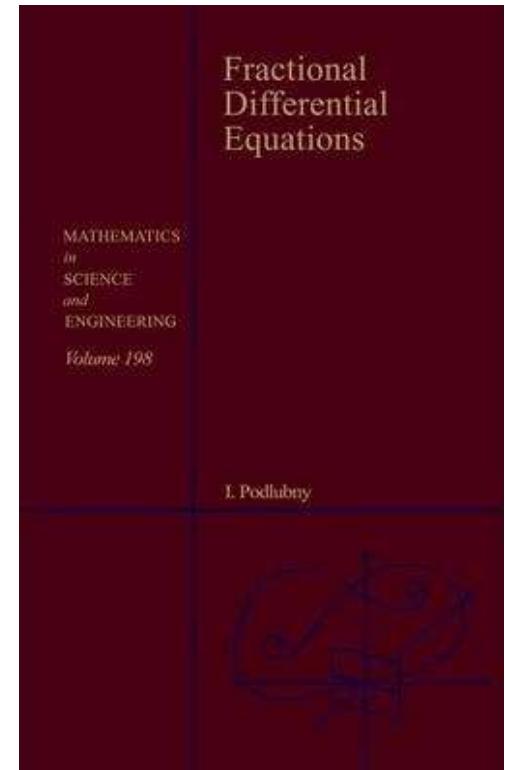
https://twitter.com/Sverre_Holm/status/1189524750055493634



Niels Henrik Abel and the birth of fractional calculus, 2017

“In his first paper on the generalization of the tautochrone problem, that was published in 1823, Niels Henrik Abel presented **a complete framework for fractional-order calculus**, and used the clear and appropriate notation for fractional-order integration and differentiation.”

Podlubny, Magin, Trymorus, (2017). Niels Henrik Abel and the birth of fractional calculus, *Fract. Calc. Appl. Anal.*, 1068-1075: <https://doi.org/10.1515/fca-2017-0057>



Oplösning af et Par Opgaver ved
Hjelp af bestemte Integraler.

Af
N. H. Abel.

Magazin for natur-videnskaberne, side 55-68,
Aargang I, Bind 2, Christiania, 1823



N.H. Abel, Solution de quelques problèmes
à l'aide d'intégrales définies:

- “Œuvres complètes de Niels Henrik Abel. Nouvelle édition”, L. Sylow and S. Lie, Grøndahl & Søn, Christiania, 1881, Ch. II, pp. 11–27.
- https://www.abelprize.no/nedlastning/verker/oeuvres_1839/oeuvres_completes_de_abel_1_kap04_opt.pdf

Caputo (1967)

Liouville (1832)

Abel (1823)

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Michele Caputo

These perfectly elastic fields are insufficient models for the physical phenomena because they do not allow us to explain the dis A more complete description of the actual elastic fields is obtain stress-strain relations which include also linear combinations of the strain and the stress. The numerical coefficients appearing in combinations of higher order derivatives are called viscoelastic order.

Elastic fields described by elastic constants of higher order ha by many authors (e.g. see Knopoff (1954), Caputo (1966a). Fo the various models of losses see Knopoff (1964).) Knopoff studied the stress-strain relations are of the type

$$\tau_{rs} = \lambda g^{hi} g_{rs} e_{hi} + 2\mu e_{rs} + \frac{d^m}{dt^m} [\lambda_m g^{hi} g_{rs} e_{hi} + 2\mu_m e_{rs}],$$

where λ_m and μ_m are constant; he obtained a condition for these vis of higher order analogous to those existing for the perfectly elas proved that in order to have a dissipative elastic field the stress-strai contain a time derivative of odd order.

A generalization of the relation (1) is

$$\tau_{rs} = \sum_{m=0}^p \frac{d^m}{dt^m} [\lambda_m g^{hi} g_{rs} e_{hi} + 2\mu_m e_{rs}],$$

where one can also consider λ_m and μ_m as functions of position.

We can generalize (2) to the case when the operation d^m/dt^m is a real number z and also further by replacing the summation Σ follows:

$$\tau_{rs} = \int_{a_1}^{b_1} f_1(r, z) \frac{d^z}{dt^z} (g^{hi} g_{rs} e_{hi}) dz + 2 \int_{a_2}^{b_2} f_2(r, z) \frac{d^z}{dt^z} (e,$$

Relations (1) and (2) are a special case of (3). They are obtained b

$$f_1(r, z) = \sum_{m=q}^p \delta(z-m) \lambda_m,$$

$$f_2(r, z) = \sum_{m=q}^p \delta(z-m) \mu_m,$$

where $\delta(z-m)$ are unitary delta functions.

If $a_i=p=q=0$ then we have the case of a perfectly elastic field; i have a perfectly viscous field; if $a_i=q=0$ and $p=1$ then we have a

We have now to establish a few relations which we shall have to

Let $f(t)$ and its i th order derivatives ($i=1, 2, \dots, m+1$) be continu $(0, +\infty)$ and also let z be a real number ($0 < z < 1$).

We shall assume the definition of the derivative of $f(t)$ of order

$$\frac{d^{m+z}}{dt^{m+z}} f(t) = \frac{1}{\Gamma(1-z)} \int_0^t (t-\xi)^{-z} f^{(m+1)}(\xi) d\xi.$$

We want to prove that if $|f^{(i+1)}(t)| e^{-pt}$, ($p>0$) ($i=0, 1, \dots, m$) $(0, +\infty)$, then

$$\int_0^\infty \left[\frac{d^{m+z}}{dt^{m+z}} f(t) \right] e^{-pt} dt = p^z \left(p^m \int_0^\infty f(\xi) e^{-p\xi} d\xi - p^{m-1} f(0) - p^{m-2} \dots - f^{(m-1)}(0) \right)$$

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QUESTIONS DE GÉOMÉTRIE

On aura, en effet, d'après cela:

$$\frac{d^\mu \Phi(x)}{dx^\mu} = \frac{d^{n-p} \Phi(x)}{dx^{n-p}} = \frac{d^n \int^p \Phi(x) dx^p}{dx^n}.$$

L'indice p étant positif, on peut faire $\mu=p$ dans la formule qui donne :

$$\int^p \Phi(x) dx^p = \frac{1}{(-1)^p \Gamma(p)} \int_0^\infty \Phi(x+\alpha) \alpha^{p-1} d\alpha.$$

Mettant au lieu de $\int^p \Phi(x) dx^p$, cette valeur, on obtient donc :

$$[B] \quad \frac{d^\mu \Phi(x)}{dx^\mu} = \frac{1}{(-1)^p \Gamma(p)} \cdot \int_0^\infty \frac{d^n \Phi(x+\alpha)}{dx^n} \alpha^{p-1} d\alpha;$$

formule par laquelle la dérivée à indice fractionnaire $\frac{d^\mu \Phi(x)}{dx^\mu}$ se exprimée en intégrale définie.

A l'aide des deux formules [A], [B], il est donc toujours facile culer les dérivées à indices quelconques, positifs ou négatifs fonction $\Phi(x)$, pourvu que cette fonction soit zéro pour $x=\infty$ fonctions que nous considérons dans ce mémoire jouissent, en de cette propriété.

[7] Si dans la formule [A] on change x en x^z et α en α^z , elle d

$$\int^\mu \Phi(x^z) d(x^z)^\mu = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_0^\infty \Phi(x^z + \alpha^z) \alpha^{2\mu-1} d\alpha,$$

et donne par conséquent la formule nouvelle :

$$[C] \quad \int_0^\infty \Phi(x^z + \alpha^z) \alpha^{2\mu-1} d\alpha = \frac{(-1)^\mu \Gamma(\mu)}{z} \int^\mu \Phi(x^z) d(x^z)^\mu$$

dans laquelle on remarquera que l'intégrale

$$\int^\mu \Phi(x^z) d(x^z)^\mu$$

doit être prise en y regardant x^z comme la variable indépendante à-dire que cette intégrale n'est autre chose que ce que devient ion de z :

$$\int^\mu \Phi(z) dz^\mu,$$

quand on y pose $z=x^z$, après avoir effectué l'intégration.

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Abel, Opgavers Oplösning

nu er

$$\frac{\Gamma(m+1)}{\Gamma(m-k+1)} = \frac{1}{\Gamma(-k)} \int \frac{t^m dt}{(1-t)^{1+k}} (t=0, t=1)$$

altsaa:

$$\frac{d^k \psi x}{dx^k} = \frac{1}{x^k \Gamma(-k)} \int \frac{\sum a^{(m)} (xt)^m dt}{(1-t)^{1+k}} (t=0, t=1)$$

men $\sum a^{(m)} (xt)^m = \psi(xt)$, altsaa

$$\frac{d^k \psi x}{dx^k} = \frac{1}{x^k \Gamma(-k)} \cdot \int \frac{\psi(xt) dt}{(1-t)^{1+k}} (t=0, t=1)$$

Sættes $k=-n$ saa faaer man

$$\frac{x^n}{\Gamma(n)} \cdot \int \frac{\psi(xt) dt}{(1-t)^{1-n}} (t=0, t=1) = \frac{d^{-n} \psi x}{dx^{-n}}$$

Men vi have seet at

$$s = \frac{x^n}{\Gamma(n) \cdot \Gamma(1-n)} \int \frac{\psi(xt) dt}{(1-t)^{1-n}} (t=0, t=1)$$

altsaa

$$s = \frac{1}{\Gamma(1-n)} \cdot \frac{d^{-n} \psi x}{dx^{-n}}, \text{naar } \psi = \int \frac{ds}{(a-x)^n} (x=0, x=a),$$

q.e.d.

Differentieres Værdien for s nGange, saa faaer man

$$\frac{d^n s}{dx^n} = \frac{1}{\Gamma(1-n)} \cdot \psi x,$$

altsaa naar s sættes = ψx

$$\frac{d^n \psi x}{dx^n} = \frac{1}{\Gamma(1-n)} \cdot \int \frac{\phi^1 x dx}{(a-x)^n} (x=0, x=a)$$

Man maa lægge Mærke til, at i det Foregaaende n altid maa være mindre end 1.

Sættes $n=\frac{1}{2}$ saa har man

$$\psi x = \int \frac{ds}{\sqrt{a-x}} (x=0, x=a)$$

$$\text{og } s = \frac{1}{\sqrt{\pi}} \cdot \frac{d^{-\frac{1}{2}} \psi x}{dx^{-\frac{1}{2}}} = \frac{1}{\sqrt{\pi}} \int \frac{\psi x dx}{x^{\frac{1}{2}}}.$$

Fundamental theorem of calculus

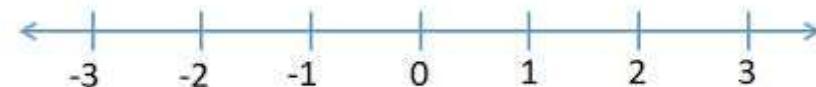
- Differentiation and integration cancel:

$$\int^n(\cdot)d\tau^n = \frac{d^{-n}}{dt^{-n}}(\cdot), \quad I^n(\cdot) = D^{-n}(\cdot)$$

[Abel's and Liouville's notation for repeated integration]

- Sequence of n-fold integrals and derivatives:

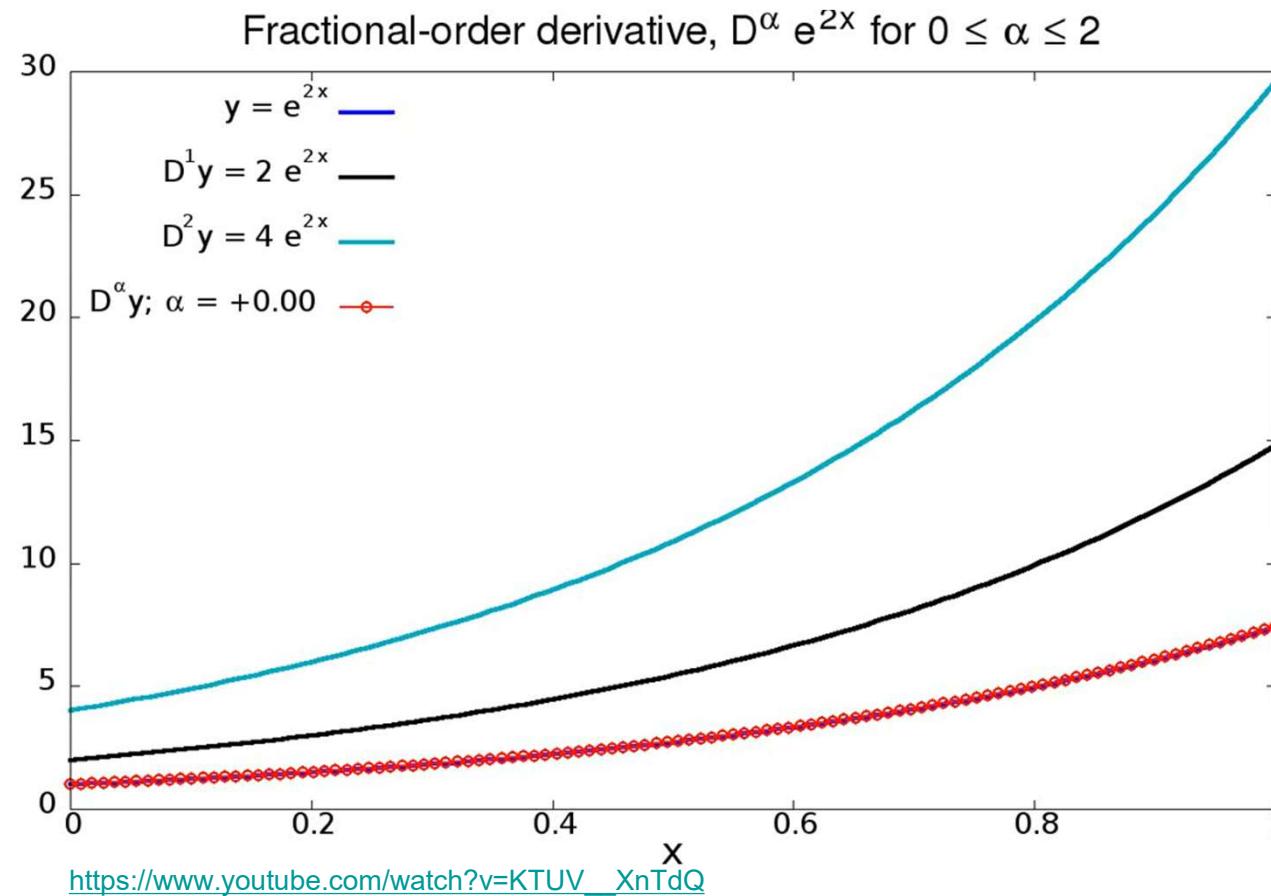
$$\dots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1) d\tau_1, \int_a^t f(\tau_1) d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2f(t)}{dt^2}, \dots$$



1. Fractional derivative
2. Power laws are everywhere
3. Fractional differential equations
4. Abel's Danish-Norwegian paper from 1823

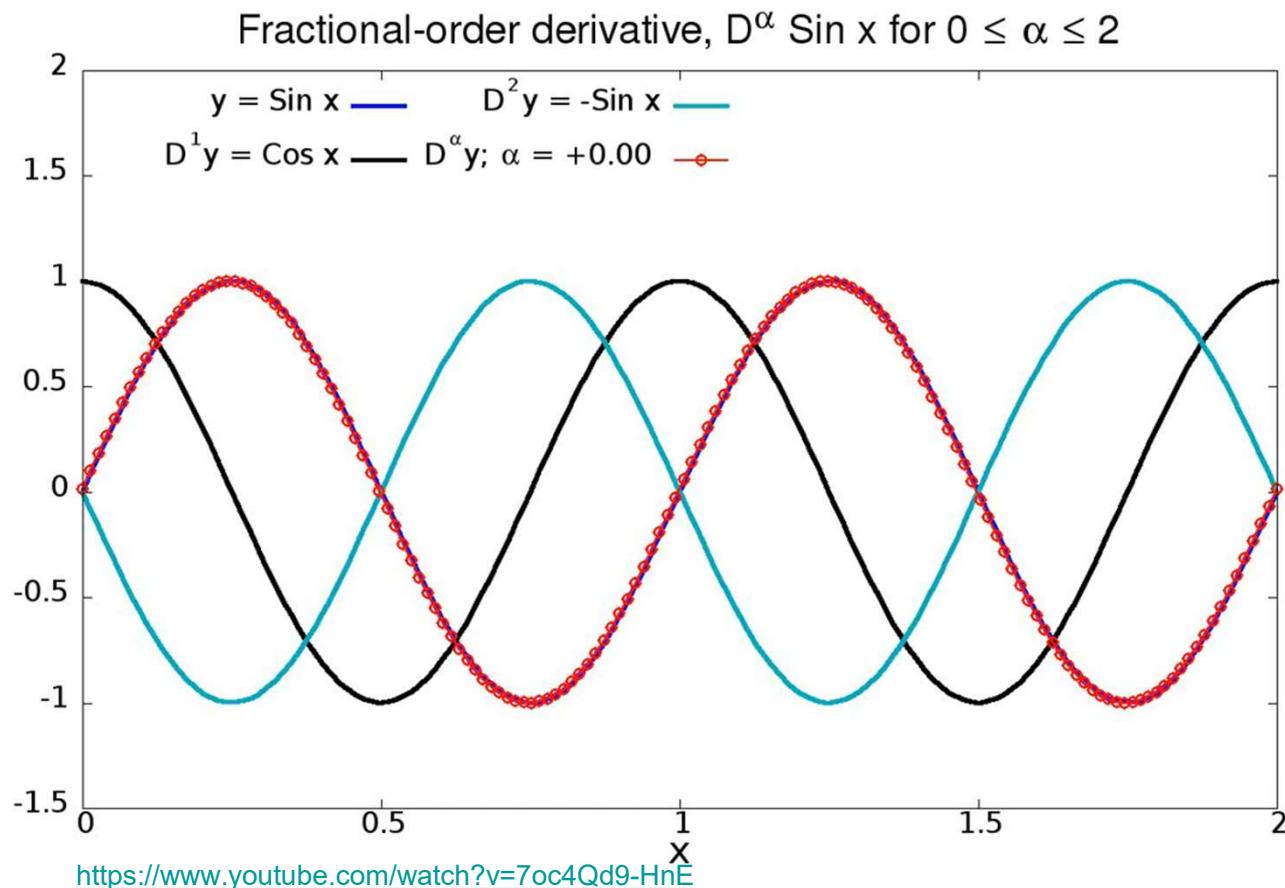
Exponential function

$$\frac{d^\alpha}{dx^\alpha} e^{kx} = k^\alpha e^{kx}, \quad k \geq 0$$



Sinusoid

$$\frac{d^\alpha}{dx^\alpha} \sin kx = k^\alpha \sin \left(kx + \frac{\pi}{2}\alpha \right), \quad k \geq 0$$



Power law

$$\frac{dx^k}{dx} = kx^{k-1} \Rightarrow \frac{d^n x^k}{dx^n} = \frac{k!}{(k-n)!} x^{k-n} \Rightarrow \frac{d^\alpha x^k}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}$$

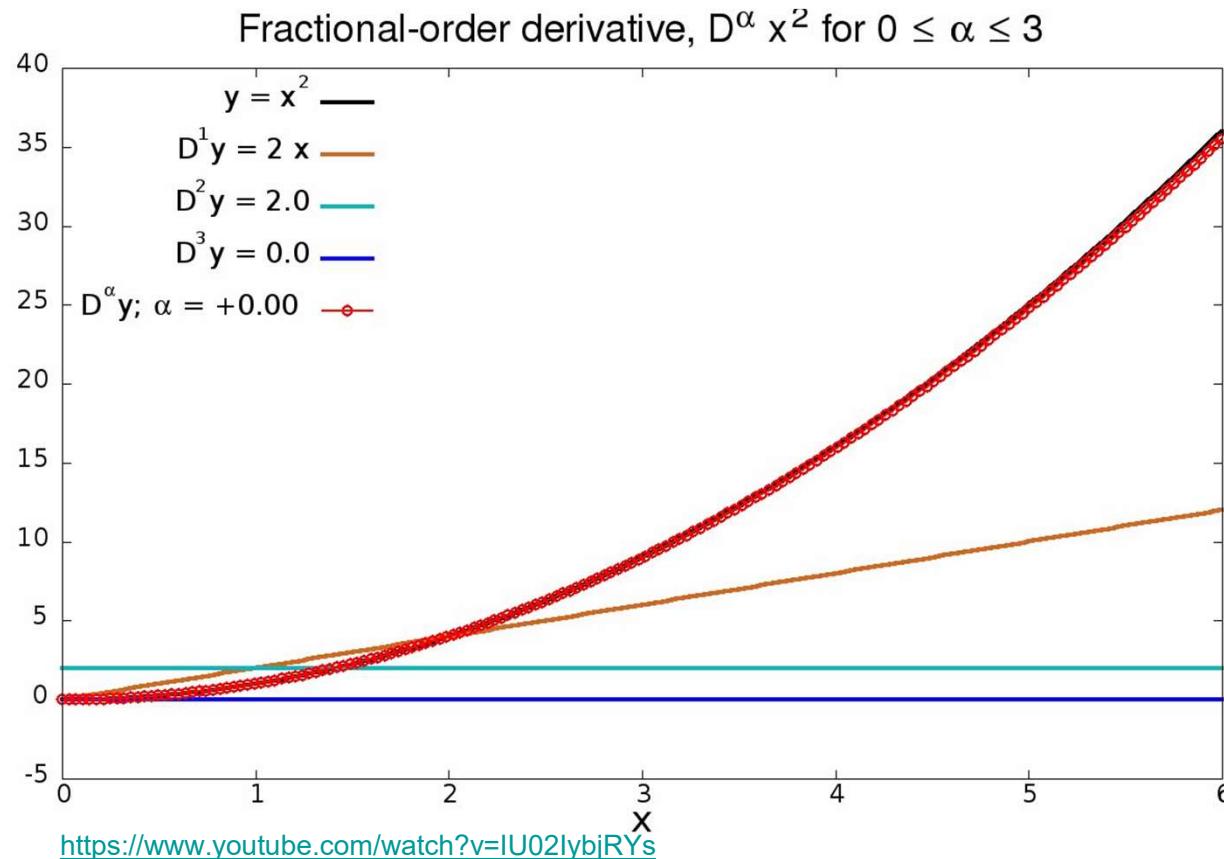


Fig:
Vikash Pandey,
YouTube

Oops!

- Derivative of a constant, 1:

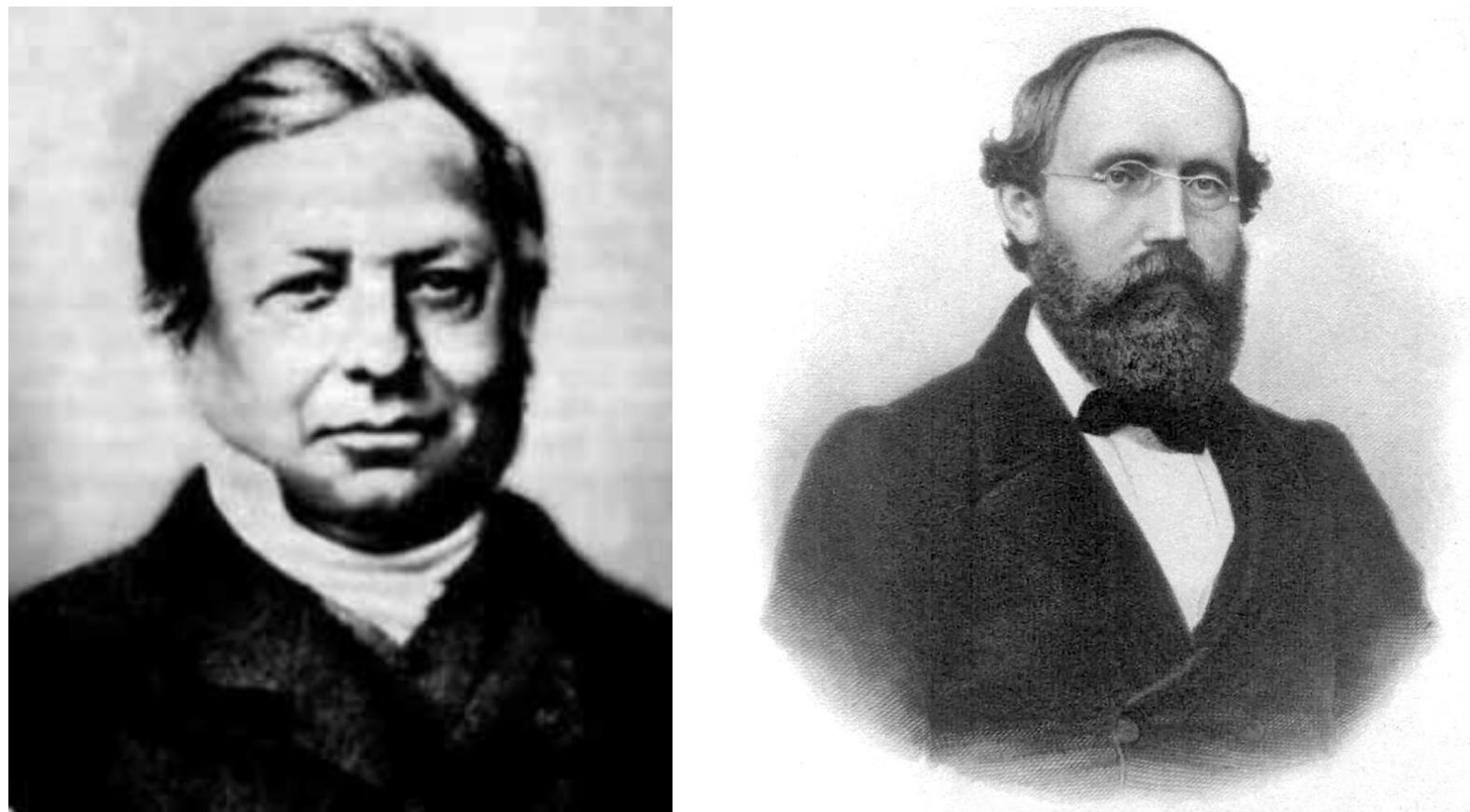
$$\frac{d^\alpha x^k}{dx^\alpha} \Big|_{k=0} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \Big|_{k=0} \Rightarrow \frac{d^\alpha}{dx^\alpha} 1 = \frac{1}{1-\alpha} x^{-\alpha} \neq 0$$

Cauchy formula for repeated integration

$$I^n[f(t)] = \int_a^n f(t) dt^n = \frac{1}{(n-1)!} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau, \quad n = 1, 2, \dots$$



Augustin Louis Cauchy
(1789 – 1857)



Joseph Liouville
(1809 – 1882)

Georg Friedrich Bernhard
Riemann (1826 – 1866)

Cauchy → Riemann (1847) - Liouville (1832) fractional derivative

$$I^n[f(t)] = D^{-n}[f(t)] = \frac{1}{(n-1)!} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau, \quad n = 1, 2, \dots$$

- Generalize to non-integer order, $n \rightarrow \alpha$:
 - Fractional integral

$$I^\alpha[f(t)] = D^{-\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0$$

- Riemann-Liouville derivative, $m = \lceil \alpha \rceil$
 - differentiate a «little too much», m ; then integrate by $m-\alpha$

$$D^\alpha[f(t)] = \frac{d^m}{dt^m} D^{\alpha-m} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

Fractional derivative: Two flavors

- Riemann-Liouville: order $m-1 \leq \alpha < m$:

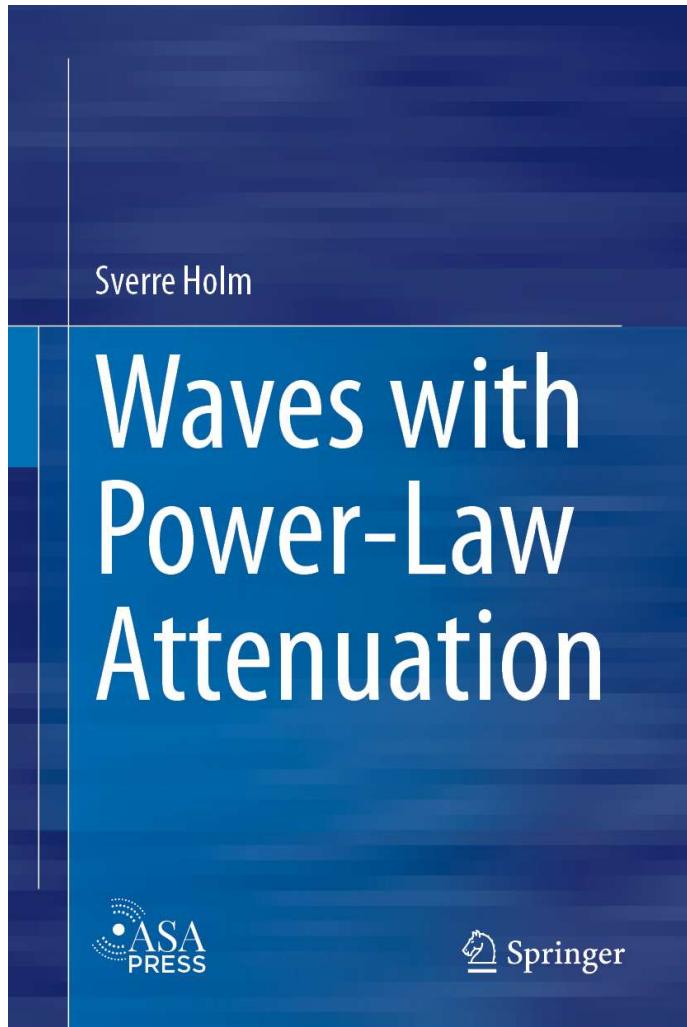
$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

- First convolution, then integer order derivation

- Caputo: order $m-1 \leq \alpha < m$ (1967):

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

- First integer order derivative, then convolution
 - Derivative of a constant is zero for all orders
 - PDE's will require initialization at integer order derivatives
 - Useful for numerical solvers



1 Introduction

Part I Acoustics and linear viscoelasticity

- 2 Classical wave equations
- 3 Models of linear viscoelasticity
- 4 Absorption mechanisms and physical constraints

Part II Modeling of power-law media

- 5 Power-law wave equations from constitutive equations
- 6 Phenomenological power-law wave equations
- 7 Justification for power laws and fractional models
- 8 Power laws and porous media
- 9 Power laws and fractal scattering media

Appendices

- A Mathematical background
- B Wave and heat equations

<http://folk.uio.no/sverre/Waves/Corrigendum-WavesPowerLaws.pdf>

1. Fractional derivative
2. Power laws are everywhere
3. Fractional differential equations
4. Abel's paper from 1823

Oplösning af et Par Opgaver ved Hjelp af bestemte Integraler.

Af
N. H. Abel.

I.

Det er som bekjendt ofte Tilfældet, at man ved Hjelp af bestemte Integraler (*intégrales definies*) kan oplöse mange Opgaver, som man paa anden Maade enten aldeles ikke eller dog meget vanskelig kan op löse, og især har man anvendt dem med Held paa Oplösningen af flere vanskelige Opgaver i Mechaniken, f. Ex. om Bevægelsen af en elastisk Flade, i Bølgetheorien &c. En anden Anwendung af disse Integraler vil jeg vise i Oplösningen af følgende Opgave:



Abel, N. H. "Oplösning af et par opgaver ved hjælp af bestemte integraler." Magazin for naturvidenskaberne (1823), pp. 55-68

Podlubny, Magin, Trymorush, (2017). Niels Henrik Abel and the birth of fractional calculus. *Fractional Calculus and Applied Analysis*

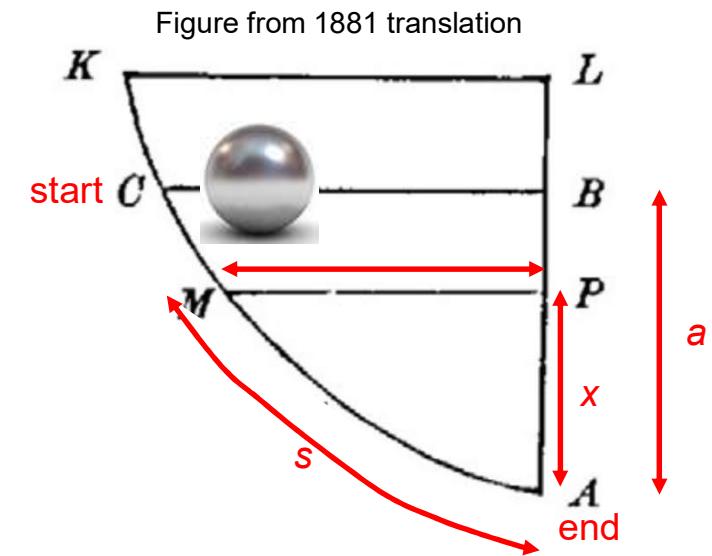
Supplement to arXiv version has an English translation of Abel's paper

<https://doi.org/10.1515/fca-2017-0057>
<https://arxiv.org/pdf/1802.05441.pdf>

Energy balance

$$\frac{1}{2}mv^2 = mg(a - x) \Rightarrow v = \sqrt{2g(a - x)}$$

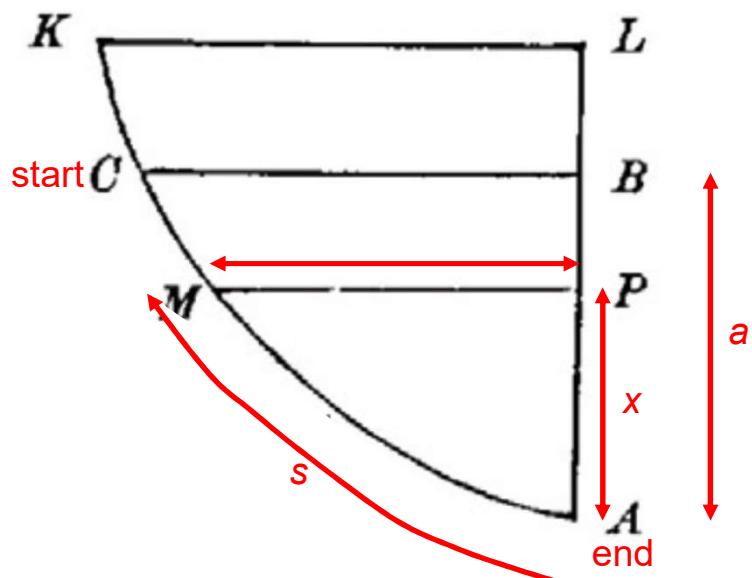
$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{(2g(a - x))^{1/2}}$$



Integrate for time for descent as a f. of arc s:

$$T = \int dt = \frac{1}{\sqrt{2g}} \int_0^a \frac{ds}{(a - x)^{1/2}} = \frac{1}{\sqrt{2g}} \int_0^a \frac{s'}{(a - x)^{1/2}} dx$$

"Lad CB Tayle 1, Fig. 4, være en horizontal Linie, A et givet Punkt; AB lodret paa BC , AM en krum Linie, hvis retvinklede Koordinater ere $AP=x$, $PM=y$. Endvidere være $AB=a$ og $KM=s$. Tænker man sig at et Legeme gjennemlöber Buen CA med en



$$T = \psi a = \psi(a) = \int_0^a \frac{ds}{\sqrt{a-x}}$$

Abel, 1823, side 56

"Initial-Hastighed $= o$, saa vil Tiden T , som det bruger til at gjennemlöbe hele Buen CA , være afhængig af Kurvens Natur og af a . Man forlanger at bestemme den krumme Linie KCA saaledes, at Tiden T bliver lig en given Funktion af a , f. Ex. ψa ."

Som bekjendt er, naar Hastigheden af Legemet i M kaldes h og Tiden, som det bruger til at gjennemlöbe Buen CM , t

$$h = \sqrt{BP} = \sqrt{a-x} \text{ og } dt = -\frac{ds}{h}$$

altsaa,

$$dt = -\frac{ds}{\sqrt{a-x}}$$

og naar man integrerer:

$$t = - \int \frac{ds}{\sqrt{a-x}}$$

For at have T maa man tage Integralet fra $x=a$ til $x=o$ og man har altsaa:

$$T = \int \frac{ds}{\sqrt{a-x}} \text{ (fra } x=o \text{, til } x=a)$$

Da nu T er lig ψa , saa bliver altsaa Ligningen

$$\psi a = \int \frac{ds}{\sqrt{a-x}} \text{ (fra } x=o \text{ til } x=a)$$

Generalizes

Da nu T er lig Ψa , saa bliver altsaa Ligningen

$$\Psi a = \int \frac{ds}{\sqrt{a-x}} \text{ (fra } x=0 \text{ til } x=a)$$

Istedsfor at oplöse denne Ligning, vil jeg i Almindelighed vise, hvorledes man kan finde s af Ligningen:

$$\Psi a = \int \frac{ds}{(a-x)^n} \text{ (fra } x=0 \text{ til } x=a)$$

hvor n er mindre end 1, for at ikke Integralet skal blive uendeligt mellem de givne Grændser; Ψa er en hvilkensomhelst Funktion, der ikke bliver uendelig naar $x=0$.

Abel, 1823, side 56

Auflösung einer mechanischen Aufgabe.

(Von Herrn N. H. Abel.)

Abel, Auflösung einer mechanischen
Aufgabe, J. Reine u. Angew. Math,
pp. 153-157. 1826

$$f^{\alpha} = \int_0^a \frac{ds}{\sqrt{(a-x)}},$$

aus welcher Gleichung s , durch x ausgedrückt, gefunden werden muß.

Statt dieser Gleichung wollen wir die allgemeinere

$$f^{\alpha} = \int_0^a \frac{ds}{(a-x)^n}$$

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Abel, Auflösung einer mechanischen Aufgabe.

Man setze nun $n = \frac{1}{2}$, so ist

$$\varphi^{\alpha} = \int_0^a \frac{ds}{\sqrt{(a-x)}}$$

und

$$s = \frac{1}{\pi} \int_0^x \frac{\varphi^{\alpha} da}{\sqrt{(x-a)}}.$$

Diese Gleichung giebt, wie bekannt, den Bogen s durch die Abscisse x , und folglich ist die Curve nunmehr völlig bestimmt.

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The full general solution is in
the paper from 1826 –
no mention of link to non-
integer calculus.

Af det Foregaaende flyder følgende mærkværdige Theorem:

Vi have set at når

$$\Psi^a = \int \frac{ds}{(a-x)^n} \quad (x=0, x=a)$$

saa er

$$s = \frac{\sin n\pi}{\pi} \cdot x^n \cdot \int \frac{\psi(xt) \cdot dt}{(1-t)^{1-n}} \quad (t=0, t=1)$$

Man kan ogsaa udtrykke s paa en anden Maade, som jeg for dens Besynderligheds Skyld vil anføre, nemlig

$$s = \frac{1}{\Gamma(1-n)} \int_0^a \psi x \cdot dx^n = \frac{1}{\Gamma(1-n)} \cdot \frac{d^{-n} \psi x}{dx^{-n}}$$

hvor n er mindre end 1

s = arc shape

$T = \Psi \alpha$ = time

~ Caputo fractional derivative

Cauchy formula for repeated integration for non-integer order =>

n'th order integral of Liouville 1832

= derivative of negative order

Even non-integer order derivatives and integrals are inverse operations
 ⇔ Assumes the fundamental theorem of calculus for non-integer orders

Abel, 1823, side 59, 61

Pascal (1623-62): Principles are intuited, propositions are inferred

Pierre Duhem (1861-1916):

- Intuitive mind, ~non-scientific
 - guesswork, perfected by the practice of history
 - “esprit de finesse” which is either
 - Common sense, “bon sens”: physics - hypotheses
 - Common knowledge: mathematics - axioms
- Mathematical mind: deductive method
 - “esprit géométrique”, use of pure logic



<https://plato.stanford.edu/entries/duhem/#LatDev>

Extension to non-integer derivatives appears to be almost trivial to Abel

Differentieres Værdien for s nGange, saa faaer man

$$\frac{d^n s}{dx^n} = \frac{1}{\Gamma(1-n)} \cdot \Psi x,$$

altsaa naar s sættes $\equiv \phi x$

$$\frac{d^n \phi x}{da^n} = \frac{1}{\Gamma(1-n)} \cdot \int \frac{\phi^1 x \cdot dx}{(a-x)^n} (x=0, x=a)$$

“Differentiating the value of s , n times, one obtains ...”,

- Caputo derivative for order 0...1

Abel, 1823, side 62

Back to the original problem, $n=1/2$

Sættes $n = \frac{1}{2}$ saa har man

$$\psi a = \int \frac{ds}{\sqrt{a-x}} \quad (x=0, x=a)$$

$$\text{og } s = \sqrt{\pi} \cdot \frac{d^{-\frac{1}{2}} \psi x}{dx^{-\frac{1}{2}}} = \frac{1}{\sqrt{\pi}} \int^{\frac{1}{2}} \psi x \cdot dx^{\frac{1}{2}}$$

Dette er altsaa Ligningen for den sögte Kurve naar Tiden er $= \psi a$.

Af denne Ligning faaes:

$$\psi x = \sqrt{\pi} \cdot \frac{d^{\frac{1}{2}} s}{dx^{\frac{1}{2}}}$$

altsaa :

Naar Ligningen for en krum Linie er $s = \phi x$, saa er Tiden, som et Legeme bruger for at gjennemlöbe en Bue, hvis Höide $= a$, lig $\sqrt{\pi} \cdot \frac{d^{\frac{1}{2}} \phi a}{da^{\frac{1}{2}}}$; det næderste Punkt af a er fast.

Abel, 1823, side 62-63

s = arc shape

$T = \psi a$ = time

Fractional integral

Inverts the integral and gets fractional derivative

Assumes that the fundamental theorem of calculus is equally valid for order = 1/2

Podlubny, Magin, Trymorus, (2017)

Conclusion

“It is not clear why Niels Henrik Abel abandoned the direction of research so nicely formed in his 1823 paper, and one can only guess the reasons.

Abel had all the elements of the fractional-order calculus there:

- the idea of fractional-order integration and differentiation,
- the mutually inverse relationship between them,
- the understanding that fractional-order differentiation and integration can be considered as the same generalized operation,
- and even the unified notation for differentiation and integration of arbitrary real order.”

Wessel: Geometry, complex numbers

Den glemte norske helten av matematikk: Caspar Wessel

$$i = \sqrt{-1}$$

Juristen og landmåleren Caspar Wessel var den første nordmannen som leverte et sentralt bidrag til matematikken. Men i dag er det nesten ingen som husker ham.

Bjarne Røsjø
INFORMASJONSRÅDGIVER

Universitetet i Oslo

PUBLISERT Fredag 30. november 2018 - 05:00

«Han tegner Landkort og leser Loven. Han er saa flittig som jeg er doven», skrev forfatteren Johan Herman Wessel om den yngre broren Caspar.

Men i ettertid har den dovne Wessel satt større spor etter seg enn den flittige, for storebror Wessel huskes fortsatt som en viktig litterær skikkelse i Danmark-Norge på slutten av 1700-tallet.

Lillebroren er derimot stort sett glemt av de fleste – til tross for at han var den første i verden som ga det som kalles komplekse tall en geometrisk tolkning.

Annonsen er lukket av Google

- Wessel, Caspar (1799). ["Om Directionens analytiske Betegning, et Forsøg, anvendt fornemmelig til plane og sphæriske Polygoners Opløsning"](#) [On the analytic representation of direction, an effort applied in particular to the determination of plane and spherical polygons]. Copenhagen: Royal Danish Acad. Scienc. Letters. **5**: 469–518.
- [Argand](#) in 1806 and [Gauss](#) in 1831
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Classification of Abel's work

1. Solution of algebraic equations by radicals;
2. New transcendental functions, in particular elliptic integrals, elliptic functions, abelian integrals;
3. Functional equations;
4. Integral transforms;
5. Theory of series treated in a rigorous way.

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