Exercises

4.1

a) \( f(0) = e^0 - 1 = 0 \)

b) Newton’s method; \( f'(x) = e^x \):

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 - \frac{e^1 - 1}{e} = 0.367879 \\
x_2 &= 0.367879 - \frac{e^{0.367879} - 1}{e^{0.367879}} = 0.060080 \\
x_3 &= 0.001769 \\
x_4 &= 0.000002
\end{align*}
\]

4.2

a) \( f(0) = 0 - \sin 0 = 0 \)

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 - \frac{1 - \sin 1}{1 - \cos 1} = 0.655145 \\
x_2 &= 0.433590 \\
x_3 &= 0.288148 \\
x_4 &= 0.191832
\end{align*}
\]

b) \( f(0) = 1 - \cos 0 = 0 \)

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 - \frac{1 - \cos 1}{\sin 1} = 0.453678 \\
x_2 &= 0.222866 \\
x_3 &= 0.110969 \\
x_4 &= 0.055427
\end{align*}
\]
4.3

a)

\[
\begin{align*}
x_0 &= 3 \\
x_1 &= 3 - \frac{3^2 - 4}{2 \cdot 3} = \frac{13}{6} \\
x_2 &= \frac{313}{156} = 2.006410256 \\
x_3 &= 2.00001024 \\
x_4 &= 2.0000000 \\
\end{align*}
\]

b)

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 - \frac{1^2}{2 \cdot 1} = \frac{1}{2} \\
x_2 &= \frac{1}{4} \\
x_3 &= \frac{1}{8} \\
x_4 &= \frac{1}{16} \\
\end{align*}
\]

d)

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 - \frac{1^6}{5 \cdot 1^5} = \frac{5}{6} \\
x_2 &= \left(\frac{5}{6}\right)^2 \\
x_3 &= \left(\frac{5}{6}\right)^3 \\
x_4 &= \left(\frac{5}{6}\right)^4 \\
\end{align*}
\]

4.4

a) Inserting \( x = y = 0 \):

\[
\begin{align*}
ed^x - x &= e^0 - 0 = 1 \\
x^2 - y &= 0^2 - 0 = 0 \\
\end{align*}
\]

b) \((x_0, y_0) = (0.5, 0.5), (x_1, y_1) = (0.9061, 0.6561), (x_2, y_2) = (0.7701, 0.5746)\)

(Note: The system converges towards a different solution than the one in exercise a.)
4.5

a) Explicit Euler:

\[ u_{n+1} = \Delta te^{-u_n} + u_n \]

b) Implicit Euler:

\[ u_{n+1} - \Delta te^{-u_{n+1}} - u_n = 0 \]

c) See the program `ex45.m`.

d) Using \( u(t) = \ln(1 + t) \):

\[ u'(t) = \frac{1}{1 + t} = e^{-\ln(1 + t)} = e^{-u} \]

e)+f) See the program `ex45.m`.

4.6

a) Newton’s method:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{c - \frac{1}{x_k}}{\frac{1}{x_k^2}} = (2 - cx_k)x_k \]

b)

\[
\begin{align*}
x_0 &= 0.2 \\
x_1 &= (2 - 4 \cdot 0.2)0.2 = 0.24 \\
x_2 &= 0.2496 \\
x_3 &= 0.24999936 \\
x_4 &= 0.25000000
\end{align*}
\]

Projects

4.8; Convergence of Newton’s Method

a) Newton’s method for \( f(x) = 0 \) when \( f(x) = x^2 - 4 \):

\[
\begin{align*}
x_0 &= 3.000000000000000 \\
x_1 &= 2.16666666666667 \\
x_2 &= 2.00641025641026 \\
x_3 &= 2.00001024002621 \\
x_4 &= 2.00000000002621
\end{align*}
\]
Figure 1: The function $g(x) = e^x - \cos(x)$ is zero at the point where the two lines intersect, at $x = 0$.

b) The ratio $c_k = \frac{k}{k+1}$:

\[
\begin{align*}
  c_0 &= 0.16666666666667 \\
  c_1 &= 0.23076923076923 \\
  c_2 &= 0.24920127795669 \\
  c_3 &= 0.2499625922877
\end{align*}
\]

It appears that $c_k$ converges toward $c = 0.25$.

c) See figure 1.

\[
g(0) = e^0 - \cos(0) = 1 - 1 = 0
\]

d) Newton’s method for $g(x) = 0$:

\[
\begin{align*}
  x_0 &= 0.250000000000000 \\
  x_1 &= 0.04423602555409 \\
  x_2 &= 0.00182264870657 \\
  x_3 &= 0.00000331198793 \\
  x_4 &= 0.00000000001097
\end{align*}
\]

\[
c_0 = 0.70777640886547
\]
\[ c_1 = 0.93143117823681 \]
\[ c_2 = 0.99697163563298 \]
\[ c_3 = 0.99999172671747 \]

\( c \) converges towards 1, so the convergence is quadratic:

\[ |e_{k+1}| \approx c_k^2 \]

e) Newtons method for \( f(x) = x^2 - 4 \):

\[ x_{k+1} = x_k - \frac{f(x)}{f'(x)} = x_k - \frac{x_k^2 - 4}{2x_k} = \frac{x_k^2 + 4}{2x_k} \]

f) \( h'(x) = \frac{(x^2 + 4)'(2x) - (2x)'(x^2 + 4)}{(2x)^2} = \frac{2x(2x) - 2(x^2 - 4)}{4x^2} = \frac{x^2 - 4}{2x^2} \)

\( h'(x) \) is nonnegative for all \( x \geq 2 \). This means that the function \( h \) is increasing for \( x \geq 2 \), and as \( h(2) = 2 \), \( h(x) \) will be greater than or equal to 2 for all \( x \geq 2 \).

h) Using the result from e: If, for any \( k \), \( x_k \geq 2 \) then \( x_{k+1} \geq 2 \), since \( x_{k+1} = h(x_k) \).

i) If \( x_k \) is always 2 or more, then \( (x_k - 2) \) is always positive. This means we can replace \( e_k \) with \( |e_k| \).

j) \[ |e_{k+1}| = \frac{e_{k+1}^2}{2x_k} \]

\[ = \frac{e_k^2}{2x_k} \]

k) We know from exercise h that \( x_k \geq 2 \). Combining this with the result from exercise j gives the result

\[ |e_{k+1}| = \frac{e_k^2}{2x_k} \leq \frac{1}{4} \frac{e_k^2}{2x_k} \]

This is consistent with the ratio \( c_k \) converging towards 0.25 in exercise b.

l) Combining (4.207) and (4.208):

\[ e_{k+1} = x_{k+1} - x^* \]
\[ = x_k - \frac{f(x_k)}{f'(x_k)} - 2 \]
\[ = e_k - \frac{f(x_k)}{f'(x_k)} \]
\[ = \frac{e_k f'(x_k) - f(x_k)}{f'(x_k)} \]
m) The Taylor series for $f$ on the interval $(x^*, x_k)$ is:

$$f(x^*) = f(x_k) + (x^* - x_k)f'(x_k) + \frac{(x^* - x_k)^2}{2}f''(\xi)$$

$$= f(x_k) - e_kf'(x_k) + \frac{e_k^2}{2}f''(\xi)$$

n) We know that $x^*$ is a root of $f$ so $f(x^*) = 0$. Inserting this into the result from exercise m gives:

$$0 = f(x_k) - e_kf'(x_k) + \frac{e_k^2}{2}f''(\xi)$$

$$e_kf'(x_k) - f(x_k) = \frac{e_k^2}{2}f''(\xi)$$

o) Combining equations (4.212) and (4.214):

$$e_{k+1} = \frac{e_kf'(x_k) - f(x_k)}{f'(x_k)} = \frac{f''(\xi)}{2f'(x_k)} e_k^2$$

p) Using (4.210) and (4.211) (And $e_k = |e_k|$):

$$|e_{k+1}| = \frac{f''(\xi)}{f'(x_k)} e_k^2 \leq \frac{\beta}{2\alpha} e_k^2$$

q) See figure 2.

r) From figure 2 we can easily see that the tangent of the function at $x = 1$ will come closer to the desired point for the function $\sinh(x)$ than for the function $(\cosh(x) - 1)$.

s)\[
\begin{align*}
    f'(x) &= \sinh(x), \quad \sinh(0) = 0 \\
    f''(x) &= \cosh(x), \quad \cosh(0) = 1 \\
    g'(x) &= \cosh(x), \quad \cosh(0) = 1 \\
    g''(x) &= \sinh(x), \quad \sinh(0) = 0
\end{align*}
\]

We know from exercise o that

$$e_{k+1} = \frac{f''(\xi)}{2f'(x_k)} e_k^2$$

where $\xi \in [x^*, x_k]$. For $g(x)$ this means the error will decrease rapidly, since $g''(\xi)$ goes towards zero as we approach the exact solution. For $f(x)$ however, we instead have $f'(x_k)$ becoming very small, and the error hardly decreases at all.
t) Newton’s method for \( f(x) = \cosh(x) - 1 \):
\[
\begin{array}{l|l}
  x_0 & 1.0000 \\
  x_1 & 0.5379 \\
  x_2 & 0.2752 \\
  x_3 & 0.1385 \\
  x_4 & 0.0694 \\
\end{array}
\]
\[ \frac{\Delta x}{x} = 0.5397 \]
\[ \frac{\Delta x}{x} = 0.5117 \]
\[ \frac{\Delta x}{x} = 0.5031 \]
\[ \frac{\Delta x}{x} = 0.5008 \]

Newton’s method for \( g(x) = \sinh(x) \):
\[
\begin{array}{l|l}
  x_0 & 1.0000 \\
  x_1 & 0.2384 \\
  x_2 & 4.416 \cdot 10^{-3} \\
  x_3 & 2.871 \cdot 10^{-8} \\
  x_4 & 9.926 \cdot 10^{-24} \\
\end{array}
\]
\[ \frac{\Delta x}{x} = 0.2384 \]
\[ \frac{\Delta x}{x} = 0.3259 \]
\[ \frac{\Delta x}{x} = 0.333 \]
\[ \frac{\Delta x}{x} = 0.4193 \]

u) Newton’s method does not converge if we use a value for \( x \) that results in \( f'(x) = 0 \).
In the graphical analysis of \( f(x) = 1 - x^2 \) this means we are looking for the point where the tangent to the maxima of \( f(x) \) intersects the x-axis. Obviously this point does not exist. If \( x_0 \) is close to zero, then the tangent will intersect the x-axis far away from the solution at \( x = \pm 1 \), and convergence will be slow (see fig 3).

v) When we choose \( x_0 = 0 \) we find that \( x_1 = 2 \), which is a solution. As seen in figure 4, Newton’s method ‘overshoots’ the solutions close to \( x_0 \) and we end up with an answer that is correct, but may not be the one we are looking for.

w) For \( f(x) = x - x^3 \) Newton’s method can be written as
Figure 3: $f(x) = 1 - x^2$

Figure 4: $f(x) = (x + 1)(x - 1)(x - 2)$
Putting $x_0 = 1/\sqrt{5}$ we get:

$$x_1 = \frac{-2 - \frac{1}{x_0}}{1 - \frac{3}{x_0}} = -\frac{1}{\sqrt{5}}.$$  

and

$$x_2 = \frac{2 + \frac{1}{x_0}}{1 - \frac{3}{x_0}} = \frac{1}{\sqrt{5}}.$$  

This means that instead converging towards a solution, the answers alternate between two values; $x = \pm \frac{1}{\sqrt{5}}$, as seen in figure 5.

**Programs (MATLAB)**

**ex45.m**

```matlab
function ex45(Dt);
% Solves the expression u’=exp(-u), using four different numerical tecniques.
% Compares with the exact solution.
% example: ex45(0.05)

eE(1)=0; iE(1)=0;
t(1)=0; S(1)=0;
Se(1)=0; Sf(1)=0;
```

Figure 5: $f(x) = x - x^3$
for i= 2:(1/Dt+1);
    t(i)=(i-1)*Dt;
% exact solution
    S(i)=log(1+t(i));
% explicit Euler
    eE(i)=eE(i-1)+Dt*exp(-eE(i-1));
% implicit Euler (using Newton)
    n_1=iE(i-1);
    while(abs(f(n_1,iE(i-1),Dt))>10^(-6))
        n_1=n_1-f(n_1,iE(i-1),Dt)/df(n_1,Dt);
    end
    iE(i)=n_1;
% scheme e (using Newton)
    e_1=Se(i-1);
    while(abs(f_e(e_1,Se(i-1),Dt))>10^(-6))
        e_1=e_1-f_e(e_1,Se(i-1),Dt)/df_e(e_1,Dt);
    end
    Se(i)=e_1;
% scheme f (using Newton)
    f_1=Sf(i-1);
    while(abs(f_f(f_1,Sf(i-1),Dt))>10^(-6))
        f_1=f_1-f_f(f_1,Sf(i-1),Dt)/df_f(f_1,Sf(i-1),Dt);
    end
    Sf(i)=f_1;
end

disp(sprintf('Error: explicit Euler: %g',abs(S(i)-eE(i))));
disp(sprintf('Error: implicit Euler: %g',abs(S(i)-iE(i))));
disp(sprintf('Error: scheme e: %g',abs(S(i)-Se(i))));
disp(sprintf('Error: scheme f: %g',abs(S(i)-Sf(i))));

plot(t,S,t,eE,'r--',t,iE,'g:',t,Se,'kx',t,Sf,'c+');
xlabel('t');
legend('Exact solution','Explicit Euler','Implicit Euler','Scheme e','Scheme f',2);

function val = f(u_n1, u_n, Dt);
    val = u_n1-Dt*exp(-u_n1)-u_n;
end

function der = df(u_n1,Dt);
    der = 1+Dt*exp(-u_n1);
end

function val_e = f_e(u_n1, u_n, Dt);
    val_e=u_n1-Dt*exp(-u_n1)*0.5-u_n-Dt*exp(-u_n)*0.5;
end

function der_e = df_e(u_n1,Dt)
    der_e = 1+Dt*0.5*exp(-u_n1);
end

function val_f = f_f(u_n1, u_n, Dt);
    val_f = Dt*exp(-0.5*u_n1-0.5*u_n)-u_n1+u_n;
function der_f = df_f(u_n1, u_n, Dt)
    der_f = -1-Dt*0.5*exp(-0.5*u_n1-0.5*u_n);