On the monotonicity of generalized barycentric coordinates on convex polygons

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Abstract

We show that four well-known kinds of generalized barycentric coordinates in convex polygons share a simple monotonicity property: the coordinate function associated with a vertex is increasing along any line from the polygon boundary to that vertex. This shows that the coordinate functions have no local extrema and that their contours are single curves connecting pairs of points on the two edges adjacent to the vertex.

Keywords: monotonicity, Wachspress coordinates, harmonic coordinates, Gordon-Wixom coordinates, mean value coordinates, GBCs.

1 Introduction

Let \( P \subset \mathbb{R}^2 \) be a convex polygon, with vertices \( v_1, v_2, \ldots, v_n, \ n \geq 3 \), in some anticlockwise ordering. Figure 1 shows an example with \( n = 5 \). Any functions

![Convex polygon](image)

Figure 1: Convex polygon

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\( \phi_i : P \rightarrow \mathbb{R}, \ i = 1, \ldots, n, \) are \textit{generalized barycentric coordinates} (GBCs) if, for all \( x \in P, \ \phi_i(x) \geq 0, \ i = 1, \ldots, n, \) and

\[
\sum_{i=1}^{n} \phi_i(x) = 1, \quad \sum_{i=1}^{n} \phi_i(x)v_i = x. \tag{1}
\]

From this definition one can show that all GBCs have the same values on the boundary of the polygon, \( \partial P. \) Specifically, \( \phi_i|_{\partial P} = f_i, \) where the boundary function \( f_i : \partial P \rightarrow \mathbb{R} \) has the values \( f_i|_{e_j} = 0, \ j \neq i-1, 1, \)

\[
f_i((1 - \mu)v_{i \pm 1} + \mu v_i) = \mu, \quad \mu \in [0, 1], \tag{2}
\]

and \( e_j \) is the \( j \)-th edge, \( e_j := [v_j, v_{j+1}]. \) Here and throughout, vertices, edges, and so on are indexed cyclically, i.e., \( v_{n+1} := v_1 \) etc.

We can see that \( \phi_i \) is increasing along the edges \( e_{i-1} \) and \( e_i \) in the direction towards \( v_i. \) In this note we show that several well known GBCs share a more general monotonicity property. We will say that \( \phi_1, \ldots, \phi_n \) are \textit{monotonic} if, for all \( i = 1, \ldots, n, \) and for all \( y \in \partial P, \ y \neq v_i, \) the coordinate \( \phi_i \) is increasing along the line segment from \( y \) to \( v_i. \) Figure 2 shows an example. Thus, by monotonic we mean that if

\[
x = (1 - \lambda)y + \lambda v_i \quad \text{and} \quad \tilde{x} = (1 - \tilde{\lambda})y + \tilde{\lambda} v_i, \tag{3}
\]

with \( 0 < \lambda < \tilde{\lambda} < 1, \) then \( \phi_i(x) < \phi_i(\tilde{x}), \) and we need only check this property for boundary points \( y \notin e_{i-1}, e_i. \) We will show that four distinct kinds of GBCs are monotonic: Wachspress coordinates, harmonic coordinates, Gordon-Wixom coordinates, and mean value coordinates.

Monotonicity does not follow from the definition (1) alone, since it is local to each point \( x. \) We could construct an example of ‘strange’ GBCs that are neither continuous nor monotonic. If both coordinates of \( x \) are rational numbers, let \( \phi_1(x), \ldots, \phi_n(x) \) be the Wachspress coordinates of \( x. \) Otherwise, if either coordinate is non-rational, let \( \phi_1(x), \ldots, \phi_n(x) \) be the mean value coordinates of \( x. \) This defines a set of GBCs
φ₁, . . . , φₙ for all x in P that are clearly not continuous (except on the boundary of P), and clearly not monotonic either.

The monotonicity property shows that each contour of φᵢ is a single curve connecting a point on the edge eᵢ₋₁ to the corresponding point on the edge eᵢ. Despite the fact that numerous contour plots of GBCs have appeared in the literature over several years, and they tend to exhibit this behaviour, a mathematical proof seems to be missing. Another consequence of monotonicity is that φᵢ cannot have any extrema in P.

It seems reasonable to use the simple term ‘monotonic’ because (a) this is the property that φᵢ has along each line passing through vᵢ and (b) it seems unlikely that φᵢ will be monotonic, in general, along any other line. Certainly, if we take any two points y₁ and y₂ on the boundary of P that lie on distinct edges other than eᵢ and eᵢ₋₁, then φᵢ(y₁) = φᵢ(y₂) = 0 and φᵢ is positive on the line between y₁ and y₂, from which we conclude that φᵢ is not monotonic on that line.

2 Wachspress’ rational coordinates

To define Wachspress’ coordinates (Wachspress 1975), (Warren 1996), (Meyer, Barr, Lee & Desbrun 2002), (Warren, Schaefer, Hirani & Desbrun 2007), let nᵢ denote the outward unit normal to the edge eᵢ, and for x ∈ P, let hᵢ(x) be the perpendicular distance of x to eᵢ, i.e., by the scalar product,

\[ hᵢ(x) = (vᵢ − x) \cdot nᵢ = (vᵢ₊₁ − x) \cdot nᵢ. \] (4)

Then the i-th Wachspress coordinate is

\[ φᵢ(x) = \frac{wᵢ(x)}{W}, \]

where

\[ wᵢ(x) = cᵢhᵢ(x)hᵢ₋₁(x), \quad cᵢ = nᵢ₋₁ \times nᵢ, \quad W = \sum_{j=1}^{n} wⱼ, \]

and × is the scalar-valued cross product, so nᵢ₋₁ × nᵢ = det[nᵢ₋₁, nᵢ].

To derive the monotonicity property consider the gradient ∇φᵢ of φᵢ for some fixed i. By the quotient rule,

\[ \nabla φᵢ = \frac{1}{W^2} (W \nabla wᵢ − wᵢ \nabla W) = \frac{wᵢ}{W^2} \sum_{j=1}^{n} wⱼ(Rᵢ − Rⱼ), \]

where Rⱼ := ∇wⱼ/wⱼ. Therefore, the directional derivative of φᵢ in the (not necessarily unit) direction vᵢ − x is

\[ D_{vᵢ − x} φᵢ(x) = (vᵢ − x) \cdot \nabla φᵢ(x) = \frac{wᵢ(x)}{W^2(x)} \sum_{j \neq i} wⱼ(x)(aᵢᵢ(x) − aᵢⱼ(x)), \]

where

\[ aᵢⱼ(x) := (vᵢ − x) \cdot Rⱼ(x). \]

Thus to show that D_{vᵢ − x} φᵢ(x) > 0, and since wⱼ(x) > 0 for all k, it is sufficient to show that

\[ aᵢᵢ(x) > aᵢⱼ(x), \quad j \neq i. \] (5)
Since $\nabla h_j = -n_j$, we find
\[ \nabla w_j(x) = \left( \frac{n_{j-1}}{h_{j-1}(x)} + \frac{n_j}{h_j(x)} \right) w_j(x), \]
and so
\[ R_j(x) = \frac{n_{j-1}}{h_{j-1}(x)} + \frac{n_j}{h_j(x)}, \]
as shown in (Floater, Gillette & Sukumar 2014). Now observe that for any $k = 1, \ldots, n$,
\[ (v_i - x) \cdot n_k = (v_k - x) \cdot n_k - (v_k - v_i) \cdot n_k = h_k(x) - h_k(v_i), \]
and therefore, for any $j = 1, \ldots, n$,
\[ a_{ij}(x) = 2 - \frac{h_{j-1}(v_i)}{h_{j-1}(x)} - \frac{h_j(v_i)}{h_j(x)}. \]
It follows that $a_{ii}(x) = 2$ and $a_{ij}(x) < 2, j \neq i$, which implies (5).

3 Harmonic coordinates

The harmonic coordinates (Floater, Hormann & Kós 2006), (Joshi, Meyer, DeRose, Green & Sanocki 2007) are defined by the Laplace equation with Dirichlet boundary condition:
\[ \Delta \phi_i = 0, \text{ in } P, \]
\[ \phi_i = f_i, \text{ on } \partial P. \]
Here, $\Delta$ is the Laplace operator, $\partial^2 / \partial x^2 + \partial^2 / \partial y^2$, where $x = (x, y)$.

To show monotonicity, it is sufficient to show that $\psi_i > 0$ in the interior of $P$, where
\[ \psi_i(x) := D_{v_i-x} \phi_i(x) = (v_i - x) \cdot \nabla \phi_i(x), \quad x \in P. \]
To show this we can use the maximum principle (Protter & Weinberger 1967) for the Laplace operator. By differentiating $\psi_i$ we find
\[ \Delta \psi_i = -2\Delta \phi_i + (v_i - x) \cdot \nabla(\Delta \phi_i), \]
and therefore, since $\phi_i$ is harmonic, so is $\psi_i$. Considering the boundary values of $\psi_i$, if $x$ belongs to $e_{i+1}$ then
\[ \psi_i(x) = \frac{|v_i - x|}{|v_{i+1} - v_i|} \geq 0. \]
If, on the other hand, $x$ belongs to an edge $e_j, j \neq i - 1, i$, then, since $\phi_i(x) = 0$ and $\phi_i \geq 0$ in $P$, we see that, by the convexity of $P$, $\psi_i(x) \geq 0$. Thus $\psi_i \geq 0$ on $\partial P$. Since $\Delta \psi_i = 0$ in $P$, and $\psi_i$ cannot be constant, it cannot attain its minimum in the interior of $P$, and so $\psi_i > 0$ in $P$. 
To define Gordon-Wixom coordinates (Gordon & Wixom 1974), (Belyaev 2006), (Manson, Li & Schaefer 2011) let \( x \in P \) and for any angle \( \theta \in \mathbb{R} \), let \( y_\theta \) be the unique point of intersection between the ray \( L_\theta := \{ x + s(\cos \theta, \sin \theta) : s \geq 0 \} \) (6) and the polygon boundary \( \partial P \), and let \( s_\theta = |y_\theta - x| \), the Euclidean distance from \( x \) to \( y_\theta \). The ray \( L_{\theta+\pi} \), in the direction opposite to \( L_\theta \), also meets \( \partial P \) uniquely, at the point \( z_\theta := y_{\theta+\pi} \), whose distance from \( x \) is \( t_\theta := s_{\theta+\pi} \). Then the \( i \)-th coordinate at \( x \) is defined as the integral mean of linear interpolants,

\[
\phi_i(x) = \frac{1}{\pi} \int_0^\pi \left( \frac{s_\theta}{t_\theta + s_\theta} f_i(z_\theta) + \frac{t_\theta}{t_\theta + s_\theta} f_i(y_\theta) \right) d\theta,
\]
or equivalently,

\[
\phi_i(x) = \frac{1}{\pi} \int_0^{2\pi} \frac{t_\theta}{t_\theta + s_\theta} f_i(y_\theta) d\theta.
\]

To show monotonicity, suppose that \( y, x, \bar{x}, \lambda, \text{ and } \bar{\lambda} \) are as in (3), and define \( \bar{y}_\theta, \bar{s}_\theta, \text{ and } \bar{t}_\theta \) with respect to the point \( \bar{x} \), in the same way as \( y_\theta, s_\theta, \text{ and } t_\theta \) are defined with respect to \( x \). Then, from (7), it is sufficient to show that

\[
\frac{t_\theta}{t_\theta + s_\theta} f_i(y_\theta) \leq \frac{\bar{t}_\theta}{\bar{t}_\theta + s_\theta} f_i(\bar{y}_\theta)
\]

for all \( \theta \), with strict inequality for at least one \( \theta \). Suppose first that \( y_\theta \not\in e_{i-1}, e_i \). Then \( f_i(y_\theta) = 0 \) and (8) trivially holds. Suppose otherwise that \( y_\theta \in e_j \) for some \( j \in \{ i-1, i \} \). Referring to Figure 3, since \( \bar{\lambda} \in [\lambda, 1] \), we have \( \bar{x} \in [x, v_i] \) and \( \bar{y}_\theta \in [y_\theta, v_i] \) and \( f_i(y_\theta) \leq f_i(\bar{y}_\theta) \leq 1 \).

![Figure 3: Points involved in Gordon-Wixom coordinates.](image)

So \( f_i(y_\theta) \leq f_i(\bar{y}_\theta) \) and (8) holds if

\[
\frac{t_\theta}{t_\theta + s_\theta} \leq \frac{\bar{t}_\theta}{\bar{t}_\theta + s_\theta}.
\]
or equivalently (by considering the reciprocal),

\[
\frac{t_\theta}{s_\theta} \leq \frac{\tilde{t}_\theta}{\tilde{s}_\theta}.
\]  

(9)

The distance \(\tilde{s}_\theta\) decreases linearly with \(\tilde{\lambda}\) for \(\tilde{\lambda} \in [\lambda, 1]\), from \(s_\theta\) to 0, and so

\[
\tilde{s}_\theta = \frac{1 - \tilde{\lambda}}{1 - \lambda} s_\theta.
\]  

(10)

On the other hand, by the convexity of \(P\), the line segment \([z_\theta, v_i]\) intersects the line segment \([\tilde{x}, \tilde{z_\theta}]\). Let \(q\) be the point of intersection, shown in Figure 3. Then

\[
\tilde{t}_\theta = |\tilde{z}_\theta - \tilde{x}| \geq |q - \tilde{x}| = \frac{1 - \tilde{\lambda}}{1 - \lambda} t_\theta,
\]  

(11)

which, combined with (10), implies (9). This proves (8). We can also see that if \(y_\theta \in c_j\) for some \(j \in \{i - 1, i\}\) and \(y_\theta \neq v_i\) then \(f_i(y_\theta) < f_i(\tilde{y}_\theta)\) and the inequality in (8) is strict.

5 Mean value coordinates

The \(i\)-th mean value coordinate at \(x \in P\) (Floater 2003) can be defined as the ratio of integrals

\[
\phi_i(x) = \int_0^{2\pi} \frac{f_i(y_\theta)}{s_\theta} d\theta \bigg/ \int_0^{2\pi} \frac{1}{s_\alpha} d\alpha,
\]  

(12)

with \(y_\theta\) and \(s_\theta\) as defined in Section 4.

Regarding monotonicity, let \(y, x,\) and \(\lambda\) be as in (3). To show that \(\phi_i(x)\) is increasing in \(\lambda\) at \(x\) it is sufficient to show that

\[
\frac{f_i(y_\theta)}{s_\theta} \bigg/ \int_0^{2\pi} \frac{1}{s_\alpha} d\alpha,
\]

is non-decreasing in \(\lambda\) for all \(\theta\), and increasing for some \(\theta\). Then, since the reciprocal of an increasing function is decreasing, and vice versa, a similar consideration of the angle \(\alpha\) shows that it is sufficient that

\[
\frac{s_\alpha}{s_\theta} f_i(y_\theta)
\]

is non-decreasing for all \(\theta\) and \(\alpha\), and increasing for at least one choice of \(\theta\). Thus it is sufficient to show that with \(\tilde{x}\) also as in (3),

\[
\frac{s_\alpha}{s_\theta} f_i(y_\theta) \leq \frac{s_\alpha}{s_\theta} f_i(\tilde{y}_\theta)
\]  

(13)

for all \(\theta\) and \(\alpha\), with strict inequality for some choice of \(\theta\).
The proof of (13) is similar to that of (8). If $y_\theta \notin e_{i-1}, e_i$ then (13) trivially holds because $f_i(y_\theta) = 0$. Otherwise, with $y_\theta \in e_j$ for some $j \in \{i-1, i\}$, it is sufficient to show that

$$\frac{s_\alpha}{s_\theta} \leq \frac{\tilde{s}_\alpha}{\tilde{s}_\theta}.$$ 

This follows from equation (10) and the fact that, in analogy to (11),

$$\tilde{s}_\alpha \geq \frac{1 - \tilde{\lambda}}{1 - \lambda} s_\alpha,$$

with $y_\alpha$ and $\tilde{y}_\alpha$ playing the role of $z_\theta$ and $\tilde{z}_\theta$, and this establishes (13). Similar to Section 4, if $y_\theta \in e_j$ for some $j \in \{i-1, i\}$ but with $y_\theta \neq v_i$ then $f_i(y_\theta) < f_i(\tilde{y}_\theta)$ and the inequality in (13) is strict.

6 Higher dimensions

As one might expect, the monotonicity property studied here extends in the obvious way to convex polyhedra and, more generally, convex polytopes in higher dimensions for the four coordinates considered here, if we make the right definitions.

Considering first Wachspress coordinates, the proof of monotonicity of Section 2 generalizes at least to simple convex polytopes in $\mathbb{R}^d$, i.e., convex polytopes in which every vertex has $d$ adjacent $(d-1)$-dimensional faces. Using the definitions derived in (Warren et al. 2007), and applying the notation and gradient formulas of Sec. 2 of (Floater et al. 2014), the argument is similar, and inequality (5) now follows from the fact that $a_{ii}(x) = d$ and $a_{ij}(x) < d$, $j \neq i$.

For the remaining three kinds of GBCs: harmonic, GW, and MV, we do not need to restrict to a simple polytope, and, moreover, we have a choice of how to generalize the boundary functions $f_i$ of (2). For each $(d-1)$-dimensional face of the convex polytope we could choose any set of GBCs for that face that are themselves monotonic. These face GBCs then define the boundary functions $f_i$ for the vertices $v_i$ of the polytope. The proof of monotonicity of the harmonic coordinates is now as before, with $\psi_i$ harmonic in $P$ and $\psi_i \geq 0$ on $\partial P$ due to $f_i$ being monotonic on the faces adjacent to $v_i$. The proof of monotonicity for GW and MV coordinates is also similar to the 2-D case, with integration carried out over unit vectors $\mu$ on the unit sphere in $\mathbb{R}^d$ instead of angles around the unit circle. We would replace the ray $L_\theta$ of (6) by

$$L_\mu := \{x + s\mu : s \geq 0\},$$

and could denote by $y_\mu$ its point of intersection with $\partial P$. Then for any fixed $\mu$, the five points $v_i$, $x$, and $\tilde{x}$ of (3) and $y_\mu$ and $\tilde{y}_\mu$ would all lie in the same plane, and the steps used to prove (8) and (13) are the same.

7 Future work

Monotonicity could also be investigated for other GBCs, such as maximum entropy coordinates (Sukumar 2004), the families of coordinates of (Floater et al. 2006), and the inverse rational bilinear coordinates for quadrilaterals of (Floater 2015).
A shape-property related to monotonicity is convexity. Are there GBCs which are convex as well as monotonic along the lines considered here?

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References