

# Extrapolation Methods for Approximating Arc Length and Surface Area

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## Abstract

A well-known method of estimating the length of a parametric curve in  $\mathbb{R}^d$  is to sample some points from it and compute the length of the polygon passing through them. In this paper we show that for uniform sampling of regular smooth curves, Richardson extrapolation can be applied repeatedly giving a sequence of derivative-free length estimates of arbitrarily high orders of accuracy. A similar result is derived for the approximation of the area of parametric surfaces.

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*Keywords:* Richardson extrapolation, arc length, surface area

## 1 Introduction

Computing the arc length of a parametric curve is a standard task in applied geometry, and many papers have been written about it or related issues, including [12], [17], [16], [11], [13], [19], [18], [2], [5], [3], [8], [9] and [10]. The purpose of this paper is to analyze the accumulated chord-length approximation to arc length, and show that it has an asymptotic  $h^2$ -expansion making it suitable for Richardson extrapolation.

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Let  $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}^d$ ,  $d \geq 2$ , be a regular parametric curve, by which we mean that  $\mathbf{f}$  is differentiable and  $\mathbf{f}'(t) \neq 0$  for all  $t \in [0, 1]$ . Its length is given by the integral formula

$$L(\mathbf{f}) := \int_0^1 |\mathbf{f}'(t)| dt,$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^d$ . It is typically difficult or even impossible to find a close-form solution to this integral, and one has to resort to numerical approximation. For example, polynomial curves of degree  $\geq 2$  are never arc-length parametrized according to [7] and their speed functions are only easily integrated if they are PH-curves, see [6]. One way of approximating  $L(\mathbf{f})$  is to apply one of the many known quadrature methods to the scalar function  $q(t) := |\mathbf{f}'(t)|$ . If  $\mathbf{f}$  is smooth enough, one can obtain arbitrarily high rates of approximation this way. For an analysis of this, see [10].

However, approximating the integral of  $q$  requires the computation of (first) derivatives of  $\mathbf{f}$  which can be awkward or time-consuming in practice. A popular alternative which avoids this problem is to use the ‘chordal’ method: we choose some uniform partition  $0 = t_0 < t_1 < \dots < t_n = 1$ , where  $t_i = ih$  and  $h = 1/n$ , and sum up the lengths of the chords (line segments) with end points  $\mathbf{f}(t_i), \mathbf{f}(t_{i+1})$ , giving the approximation

$$L_h(\mathbf{f}) := \sum_{i=0}^{n-1} |\mathbf{f}(t_{i+1}) - \mathbf{f}(t_i)| \approx L(\mathbf{f}), \quad (1)$$

which is simply the length of the polygon passing through the sample points  $\mathbf{f}(t_i)$ . It can be shown that

$$L(\mathbf{f}) - L_h(\mathbf{f}) = O(h^2)$$

as  $h \rightarrow 0$ , provided  $\mathbf{f} \in C^2[0, 1]$ . Thus the method has second order accuracy. A proof of this and generalizations can be found in [10].

In this paper we show that we can apply a standard extrapolation technique, often known as Richardson extrapolation, to the chord length method on equally spaced points in order to raise the order of accuracy first to  $O(h^4)$ , then to  $O(h^6)$  and so on, provided  $\mathbf{f}$  has enough bounded derivatives. For example, consider the first extrapolation. For small  $h$ , we would expect the finer approximation  $L_{h/2}(\mathbf{f})$  to be more accurate by a factor of  $\approx 4$ , but in order to increase the *order* of approximation, we take a linear combination of it and the coarser approximation. It turns out that if we weight the two approximations by the coefficients  $4/3$  and  $-1/3$ , giving

$$L(\mathbf{f}) \approx \frac{4L_{h/2}(\mathbf{f}) - L_h(\mathbf{f})}{3}, \quad (2)$$

the order of approximation is raised to  $O(h^4)$ . In fact it was shown in [10] that the error of the method (2) is  $O(h^4)$  provided  $\mathbf{f} \in C^4[0, 1]$ . This was done by

recognizing this method as the result of using quadratic polynomial interpolation and a certain open Newton-Cotes quadrature rule.

In the next section, we derive more information about the error in the fourth order rule (2) and show that for smooth  $\mathbf{f}$  Richardson extrapolation can be applied indefinitely, giving methods with accuracy  $O(h^6)$ ,  $O(h^8)$  and so on. This method is an alternative to the general approach suggested in [10]. In Section 3, we formulate an algorithm and assess its performance. Then, in Section 4, we apply the idea of extrapolation to approximate surface area, where the basic approximation scheme uses cross products of diagonals of quadrilateral patches obtained by uniform sampling. We compare this quad-based method with the triangle-based method of Lyness [14].

## 2 Approximation of Arc Length

Validating Richardson extrapolation according to (2) for the chord length rule (1) is simply a matter of showing that its error  $L(\mathbf{f}) - L_h(\mathbf{f})$  can be expanded in powers of  $h$ , where the coefficients are *independent* of  $h$ . We will show in fact that the expansion contains only even powers of  $h$ . Moreover the coefficient of the  $h^2$  term can be expressed in terms of the curvature vector

$$\boldsymbol{\kappa} := \frac{1}{|\mathbf{f}'|} \left( \frac{\mathbf{f}'}{|\mathbf{f}'|} \right)', \quad (3)$$

which is the derivative of the normalized tangent vector with respect to arc length. A key part in proving this is the following expansion for the length of the vector

$$\mathbf{d}_h(t) := \mathbf{f}(t + h/2) - \mathbf{f}(t - h/2)$$

which, for given  $h \in (0, 1/2)$ , is regarded as a function of  $t \in [h/2, 1 - h/2]$ .

**Lemma 1** *Let  $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}^d$ ,  $d \geq 2$ , be a regular parametric curve with  $\mathbf{f} \in C^{2k+1}[0, 1]$ . Then there exist functions  $c_r \in C^{2k-2r}[0, 1]$  such that*

$$|\mathbf{d}_h(t)|/h = \sum_{r=0}^k c_r(t)h^{2r} + o(h^{2k}), \quad t \in [h/2, 1 - h/2]. \quad (4)$$

*The first two coefficients are*

$$c_0 = |\mathbf{f}'|, \quad c_1 = \frac{\mathbf{f}' \cdot \mathbf{f}'''}{24 |\mathbf{f}'|^3}. \quad (5)$$

Throughout, for an expression  $g(t, h)$  depending on  $t$  and  $h$ , the Landau symbol  $o(h^\ell)$  refers to *uniform convergence* with respect to  $t$ . That is, we write  $g(t, h) = o(h^\ell)$  if for all  $\varepsilon > 0$ , there is some  $h_0 > 0$  such that for all  $h \in (0, h_0]$  and all  $t \in [h/2, 1 - h/2]$ ,

$$|g(t, h)| \leq \varepsilon h^\ell.$$

*Proof.* By Taylor's theorem,

$$\mathbf{f}(t \pm h/2) = \sum_{j=0}^{2k+1} \left(\frac{\pm h}{2}\right)^j \frac{\mathbf{f}^{(j)}(t)}{j!} + o(h^{2k+1}),$$

and so

$$\mathbf{d}_h(t)/h = \sum_{j=0}^k \mathbf{a}_j(t) h^{2j} + o(h^{2k}), \quad \mathbf{a}_j(t) := \frac{\mathbf{f}^{(2j+1)}(t)}{2^{2j} (2j+1)!}. \quad (6)$$

The first two  $\mathbf{a}_j$  are

$$\mathbf{a}_0 = \mathbf{f}', \quad \mathbf{a}_1 = \frac{1}{24} \mathbf{f}'''.$$

Next, taking the inner product of  $\mathbf{d}_h$  with itself gives

$$|\mathbf{d}_h(t)|^2/h^2 = \sum_{j=0}^k b_j(t) h^{2j} + o(h^{2k}), \quad b_j(t) := \sum_{\ell=0}^j \mathbf{a}_\ell(t) \cdot \mathbf{a}_{j-\ell}(t). \quad (7)$$

The first two  $b_j$  are

$$b_0 = \mathbf{a}_0 \cdot \mathbf{a}_0 = |\mathbf{f}'|^2, \quad b_1 = 2 \mathbf{a}_0 \cdot \mathbf{a}_1 = \frac{1}{12} \mathbf{f}' \cdot \mathbf{f}'''.$$

Taking square roots, (7) gives

$$|\mathbf{d}_h(t)|/h = (b_0(t) + \Delta)^{1/2}, \quad \Delta := \sum_{j=1}^k b_j(t) h^{2j} + o(h^{2k})$$

where  $\Delta \rightarrow 0$  as  $h \rightarrow 0$ . The Taylor expansion of the right hand side with respect to  $\Delta$  reads

$$(b_0(t) + \Delta)^{1/2} = \sum_{\ell=0}^k \gamma_\ell b_0(t)^{1/2-\ell} \Delta^\ell + o(\Delta^k), \quad \gamma_\ell := \frac{1}{\ell!} \prod_{p=0}^{\ell-1} \left(\frac{1}{2} - p\right).$$

Again, the convergence of the remainder term  $o(\Delta^k)$  is uniform with respect to  $t$  because, by regularity of  $\mathbf{f}$ ,  $b_0$  is bounded from below by a positive constant.

Substituting in the definition of  $\Delta$  and collecting together powers of  $h^2$ , we obtain the expansion

$$|\mathbf{d}_h(t)|/h = \sum_{r=0}^k c_r(t)h^{2r} + o(h^{2k})$$

with coefficients

$$c_0 := b_0^{1/2} = |\mathbf{f}'|, \quad c_1 := \frac{b_1}{2|\mathbf{f}'|} = \frac{\mathbf{f}' \cdot \mathbf{f}'''}{24|\mathbf{f}'|},$$

and in general, for  $r \geq 1$ ,

$$c_r := \sum_{\ell=1}^r \gamma_\ell |\mathbf{f}'|^{1-2\ell} \sum_{\substack{j_1+\dots+j_\ell=r \\ j_p \geq 1}} b_{j_1} \cdots b_{j_\ell}. \quad (8)$$

By (6) and (7), we have  $b_j \in C^{2k-2j}[0, 1]$ . Hence, the least smooth term appearing in the definition of  $c_r$  is  $b_r$ , and thus,  $c_r \in C^{2k-2r}[0, 1]$ .  $\square$

Now, we are prepared to derive an expansion of the difference between the chord length approximation  $L_h(\mathbf{f})$  according to (1) and the true length  $L(\mathbf{f})$  of a curve  $\mathbf{f}$ .

**Theorem 1** *Let  $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}^d$ ,  $d \geq 2$ , be a regular parametric curve with  $\mathbf{f} \in C^{2k+1}[0, 1]$ . Then there exist coefficients  $d_\ell$  independent of  $h$  such that*

$$L(\mathbf{f}) - L_h(\mathbf{f}) = \sum_{\ell=1}^k d_\ell h^{2\ell} + o(h^{2k}). \quad (9)$$

*In particular,*

$$d_1 = \frac{1}{24} \int_0^1 |\kappa(t)|^2 |\mathbf{f}'(t)|^3 dt. \quad (10)$$

In [10], results closely resembling the theorem above were shown for  $k = 1$  and  $k = 2$ , requiring only  $\mathbf{f} \in C^2$  and  $\mathbf{f} \in C^4$ , respectively. We conjecture that in Theorem 1, the smoothness requirement may be relaxed for all  $k$  to  $\mathbf{f} \in C^{2k}$ . This conjecture is supported by the following observation: One can show that the coefficients  $d_\ell$  depend only on derivatives of  $\mathbf{f}$  up to order  $2\ell$ . This is remarkable since the corresponding coefficients  $c_\ell$  appearing in the expansion for a single segment involve also derivatives of order  $2\ell + 1$ . Hence, the sum  $\sum_{\ell=1}^k d_\ell h^{2\ell}$  in (9) is well defined for  $\mathbf{f} \in C^{2k}$ .

*Proof.* We apply (4) with  $t = u_i := t_i + h/2$  and sum over  $i$  to obtain

$$L_h(\mathbf{f}) = \sum_{i=0}^{n-1} |\mathbf{d}(u_i)| = \sum_{j=0}^k h^{2j+1} \sum_{i=0}^{n-1} c_j(u_i) + o(h^{2k}). \quad (11)$$

where, as above,  $n = 1/h$ . In order to convert the terms in this expansion into integrals, let us briefly review the second Euler-Maclaurin expansion concerning quadrature by the midpoint rule. We start from the more familiar first Euler-Maclaurin expansion concerning quadrature by the trapezoidal rule. From Theorem 2.2 of [1], p. 64, it can be shown that if  $\phi \in C^{2m}[0, 1]$ , then

$$h \left( \frac{1}{2}\phi(0) + \sum_{i=1}^{n-1} \phi(t_i) + \frac{1}{2}\phi(1) \right) = \sum_{\ell=0}^m \frac{B_{2\ell}}{(2\ell)!} h^{2\ell} \int_0^1 \phi^{(2\ell)}(t) dt + o(h^{2m}),$$

where the coefficients  $B_{2k}$  are the Bernoulli numbers

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \dots$$

Now, following Davis and Rabinowitz [4], pp. 108–109 and 327–331, we subtract this formula from twice the formula obtained by replacing  $h$  by  $h/2$  to get an expansion for the midpoint rule,

$$h \sum_{i=0}^{n-1} \phi(u_i) = - \sum_{\ell=0}^m C_\ell h^{2\ell} \int_0^1 \phi^{(2\ell)}(t) dt + o(h^{2m}), \quad (12)$$

where  $C_\ell := (1 - 2^{1-2\ell})B_{2\ell}/(2\ell)!$ . Applying the expansion (12) to the function  $c_j \in C^{2k-2j}[0, 1]$  and setting  $m = k - j$  gives

$$h \sum_{i=0}^{n-1} c_j(u_i) = - \sum_{\ell=0}^{k-j} C_\ell h^{2\ell} \int_0^1 c_j^{(2\ell)}(t) dt + o(h^{2k-2j}),$$

and substituting this into (11) yields

$$L_h(\mathbf{f}) = - \sum_{\ell=0}^k d_\ell h^{2\ell} - o(h^{2k}), \quad d_\ell := \int_0^1 \sum_{j=0}^{\ell} C_j c_{\ell-j}^{(2j)}(t) dt. \quad (13)$$

Since  $c_0(t) = |\mathbf{f}'(t)|$ , we have

$$d_0 = - \int_0^1 |\mathbf{f}'(t)| dt = -L(\mathbf{f}),$$

and therefore

$$L(\mathbf{f}) - L_h(\mathbf{f}) = \sum_{\ell=1}^k d_\ell h^{2\ell} + o(h^{2k}),$$

which is the required expansion. Finally, we compute  $d_1$ . Since

$$d_1 = \int_0^1 \left( -c_1(t) + \frac{1}{24} c_0''(t) \right) dt,$$

and

$$c_1 = \frac{1}{24} \frac{\mathbf{f}' \cdot \mathbf{f}'''}{|\mathbf{f}'|}, \quad c_0'' = \frac{\mathbf{f}' \cdot \mathbf{f}'' + |\mathbf{f}''|^2}{|\mathbf{f}'|} - \frac{(\mathbf{f}' \cdot \mathbf{f}'')^2}{|\mathbf{f}'|^3},$$

we find that

$$d_1 = \frac{1}{24} \int_0^1 \left( \frac{|\mathbf{f}''(t)|^2}{|\mathbf{f}'(t)|} - \frac{(\mathbf{f}'(t) \cdot \mathbf{f}''(t))^2}{|\mathbf{f}'(t)|^3} \right) dt.$$

As noticed above, the terms involving third derivatives of  $\mathbf{f}$  are canceled, and the formula (10) follows easily using the definition (3) of the curvature vector.  $\square$

### 3 Numerical Examples

Due to the  $h^2$  expansion of Theorem 1, we can use repeated Richardson extrapolations to obtain numerical approximations to curve length of any order, assuming sufficient smoothness of the curve, raising the approximation order by 2 at each step. In numerical integration, applying Richardson extrapolation indefinitely to the composite trapezoidal rule is known as Romberg integration, and we can apply the same process of ‘Romberg extrapolation’ to curve length. We compute a table of the form

$$\begin{array}{ccccccc} R(0,0) & & & & & & \\ R(1,0) & R(1,1) & & & & & \\ R(2,0) & R(2,1) & R(2,2) & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ R(N,0) & R(N,1) & R(N,2) & \cdots & R(N,N) & & \end{array} \quad (14)$$

where  $R(i,0)$  is the composite chordal approximation over  $2^i$  intervals and the other table entries are given by

$$R(i,j) = \frac{4^j R(i,j-1) - R(i-1,j-1)}{4^j - 1}. \quad (15)$$

In the following pseudocode, arrays are indexed from 0. Algorithm 1 takes an array  $P$  of points and returns the accumulated chord-length. Algorithm 2 takes an array  $P$  of points and returns a new array of size  $2 \cdot \text{ARRAYLENGTH}(P) - 1$ . The new array contains at its even indices, all the points of  $P$ , and at its odd indices points evaluated half-way between the original points. Algorithm 3 computes an approximation to the arc-length of the parametric curve  $f$  on the interval  $[0,1]$ . The maximum number of Romberg table rows computed is  $N + 1$ , like in (14). The optional tolerance parameter  $\tau$  together with the code block marked as optional give a way to terminate the iteration once the error estimate  $e := |R(i,i) - R(i-1,i-1)|$  is less than  $\tau$ . But like in Romberg integration, a tolerance on  $e$  must be treated with care because  $e$  is only an estimate, not an error bound.

**Algorithm 1:** CHORDAL( $P$ )

```

 $n = \text{ARRAYLENGTH}(P) - 1$ 
 $l = 0$ 
for  $i = 1$  to  $n$ 
  do  $l = l + |P(i) - P(i - 1)|$ 
return ( $l$ )

```

**Algorithm 2:** NEWPOINTS( $P, f$ )

```

 $n = \text{ARRAYLENGTH}(P) - 1$ 
for  $i = 0$  to  $n - 1$ 
  do  $\begin{cases} Q(2i) = P(i) \\ Q(2i + 1) = f((2i + 1)/(2n)) \end{cases}$ 
 $Q(2n) = P(n)$ 
return ( $Q$ )

```

**Algorithm 3:** ROMBERG CHORD-LENGTH( $f, N, [\tau]$ )

```

 $P = [f(0), f(1)]$ 
 $R(0, 0) = \text{CHORDAL}(P)$ 
for  $i = 1$  to  $N$ 
  do  $\begin{cases} P = \text{NEWPOINTS}(P, f) \\ R(i, 0) = \text{CHORDAL}(P) \\ \text{for } j = 1 \text{ to } i \\ \text{do } \{ R(i, j) = (4^j R(i, j - 1) - R(i - 1, j - 1)) / (4^j - 1) \} \\ \text{optional } \begin{cases} e = |R(i, i) - R(i - 1, i - 1)| \\ \text{if } e < \tau \\ \text{then return } (R(i, i), e) \end{cases} \end{cases}$ 
return ( $R(N, N), [e]$ )

```

Next consider some examples. We begin with Pythagorean hodograph curves (PH-curves) [6]. These are polynomial curves  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^d$  with the property that their speed functions  $|\mathbf{f}'|$  are polynomials, so their arc length is easy to compute exactly. For  $\alpha \in \mathbb{R}$ , let

$$\mathbf{f}_\alpha(t) = \left( \frac{1}{3}(t + \alpha)^3 - \frac{1}{5}(t + \alpha)^5, \frac{1}{2}(t + \alpha)^4 \right), \quad t \in [0, 1], \quad (16)$$

be the PH-curve with speed function  $|\mathbf{f}'_\alpha(t)| = (t+\alpha)^2 + (t+\alpha)^4$ . Then  $L(\mathbf{f}_0) = 8/15$  and  $L(\mathbf{f}_1) = 128/15$ .

Consider the case  $\alpha = 1$ . Figure 1, left, shows the actual error obtained for the Romberg table generated by Algorithm 3 with  $N = 5$ . Remarkably, the true length  $L(\mathbf{f}_1) = 128/15$  is approximated with an error of less than  $3\text{e-}13$  using only 33 function evaluations. In Table 1 we show the experimental rates of convergence (ERC)

$$\varrho_{i,j} := \log_2 \frac{L(\mathbf{f}) - R(i,j)}{L(\mathbf{f}) - R(i+1,j)}$$

for the schemes found in the columns of the Romberg table. At least in the first four columns, the theoretical prediction  $\varrho(i,j) \approx 2j + 2$  is matched quite well. The ratios of the actual errors of  $R(i,i)$  on the diagonal and its estimates  $e = |R(i+1, i+1) - R(i,i)|$  all lie in the interval  $[1, 1.02]$  indicating that, in this case,  $e$  is a reliable estimate. Of course, the actual accuracy  $L(\mathbf{f}_1) - R(4,4) \approx 3\text{e-}13$  of the returned value is much higher than the final estimate  $e \approx 1\text{e-}09$ .

On the right hand side, Figure 1 and Table 1 show the according results for the parameter value  $\alpha = 0$ . We see that the ERCs in the third and fourth column are much smaller than before. The values suggest that in this case convergence is limited to the order  $h^5$ . This behavior is due to the fact that  $\mathbf{f}_0$  is *not* a regular curve since  $\mathbf{f}'_0(0) = 0$ .

As a second example, consider the helix

$$\mathbf{g}(t) = (\cos(50t), \sin(50t), t), \quad t \in [0, 1].$$

It has approximately 8 windings and length  $L(\mathbf{g}) = \sqrt{2501} \approx 50.01$ . Calling Algorithm 3 with the optional tolerance parameter  $\tau = 1\text{e-}10$ , the iteration is stopped after only three steps, and the return value is  $R(3,3) \approx 1.03$ . This is completely off the target since the error estimate fails spectacularly here. The point is that, incidentally, the 9 points used to compute the approximations up to the third row are all very close to a straight line, see Figure 2. This shortcut is resolved only if more points are included, disregarding the over-optimistic error estimate. With  $N = 8$ , we obtain the true length with error  $< 2\text{e-}09$ . As always, the phenomenon of false error estimation cannot be excluded for a discrete method unless further information on the curve in question is available.

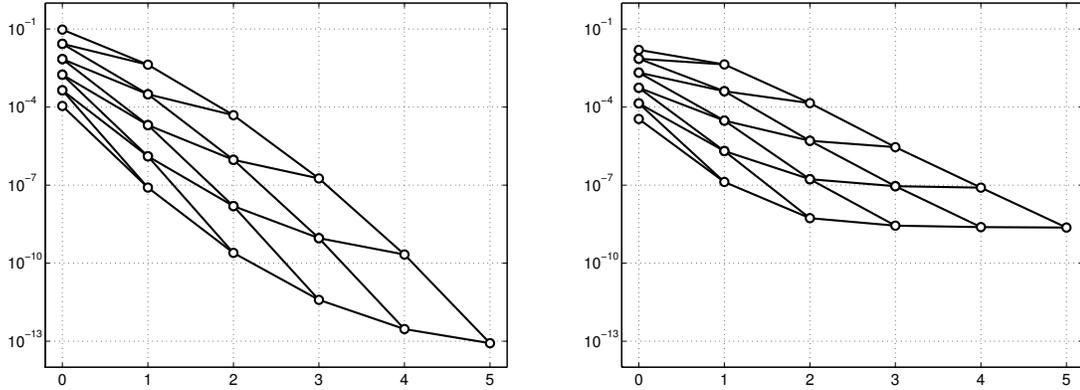


Figure 1: Error of Romberg table for PH-curves  $\mathbf{f}_1$  (left), and  $\mathbf{f}_0$  (right).

0	1	2	3	4	0	1	2	3	4
1.82					1.13				
1.95	3.77				1.78	3.43			
1.99	3.94	5.70			1.94	3.75	4.81		
2.00	3.98	5.91	7.62		1.98	3.88	4.91	4.99	
2.00	4.00	5.98	7.89	9.52	2.00	3.94	4.98	5.04	5.04

Table 1: ERC of Romberg table for PH-curves  $\mathbf{f}_1$  (left), and  $\mathbf{f}_0$  (right).

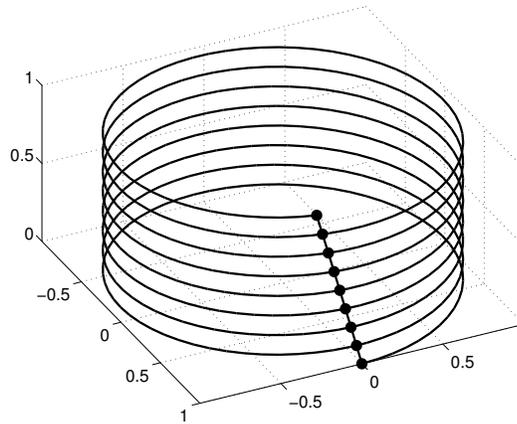


Figure 2: Helix with shortcut.

## 4 Approximation of Surface Area

In this section, we consider the problem of computing the area

$$A(\mathbf{s}) := \int_D |\mathbf{s}_u(u, v) \times \mathbf{s}_v(u, v)| \, du \, dv, \quad (17)$$

of a parametric surface  $\mathbf{s} : D \rightarrow \mathbb{R}^3$  defined on the unit square  $D = [0, 1]^2$ . We assume that  $\mathbf{s}$  is regular, by which we mean that the integrand in (17) is never zero. For any chosen  $n$ , let  $u_i = (i + 1/2)h$  and  $v_j = (j + 1/2)h$ , where  $h = 1/n$ . We then make the approximation

$$A_h(\mathbf{s}) := \frac{1}{2} \sum_{i,j=0}^{n-1} |\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j)| \approx A(\mathbf{s}),$$

where

$$\delta_{\alpha,\beta}\mathbf{s}(u, v) := \mathbf{s}(u + \alpha/2, v + \beta/2) - \mathbf{s}(u - \alpha/2, v - \beta/2).$$

Thus the surface  $\mathbf{s}$  is sampled on a square grid, and the area of each quadrilateral patch is approximated by half the norm of the cross product of its two diagonals.

**Theorem 2** *Let  $\mathbf{s} : D \rightarrow \mathbb{R}^3$  be a regular parametric surface with  $\mathbf{s} \in C^{2k+1}(D)$ . Then there exist coefficients  $d_p$  independent of  $h$  such that*

$$A(\mathbf{s}) - A_h(\mathbf{s}) = \sum_{p=1}^k d_p h^{2p} + o(h^{2k}). \quad (18)$$

Unlike in the curve case, the first coefficient  $d_1$  involves third derivatives of  $\mathbf{s}$  as well as first and second ones and we do not write down an explicit expression.

We note that in [15] it was shown that the above method is second order, if the surface  $\mathbf{s} \in C^{2,2}$ . So as in the curve case, we conjecture that the smoothness requirement above may be relaxed, but make no attempt to prove this. Also, a similar expansion was studied by Lyness in [14] for the purpose of quadrature on surfaces. Applying the basic approximation in that paper to integrating the constant function 1 over a surface reduces to, for every quad, adding the areas of the two triangles obtained by splitting the quad. So the basic approximation is different. In particular, it requires the computation of two cross products and Euclidean norms per quad, instead of only one cross product and norm in our approach.

*Proof.* Using four Taylor expansions around the point  $\mathbf{s}(u_i, v_j)$  with remainder of order  $2k + 1$ , one easily finds that

$$\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j) = \sum_{p=0}^k \mathbf{a}_p(u_i, v_j) h^{2p+2} + o(h^{2k+2})$$

for certain coefficients  $\mathbf{a}_p(u, v)$ , where  $\mathbf{a}_0 := 2(\mathbf{s}_u \times \mathbf{s}_v)$ . The dot product of this with itself is

$$|\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j)|^2 = \sum_{p=0}^k b_p(u_i, v_j) h^{2p+4} + o(h^{2k+4}),$$

for coefficients  $b_p(u, v)$  with  $b_0 = 4|\mathbf{s}_u \times \mathbf{s}_v|^2$ . Then, analogous to the curve case, since the leading term  $h^4 b_0(u_i, v_j)$  is positive by the regularity of  $\mathbf{s}$ , there is an expansion

$$|\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j)| = \sum_{p=0}^k c_p(u_i, v_j) h^{2p+2} + o(h^{2k+2})$$

for some coefficients  $c_p(u, v)$  with  $c_0 = 2|\mathbf{s}_u \times \mathbf{s}_v|$ . Summing over  $i$  and  $j$  gives

$$\frac{1}{2} \sum_{i,j=0}^{n-1} |\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j)| = \frac{1}{2} \sum_{p=0}^k h^{2p+2} \sum_{i,j=0}^{n-1} c_p(u_i, v_j) + o(h^{2k}). \quad (19)$$

Finally, we apply the expansion for the midpoint rule in (12) in both parameters to get the bivariate midpoint rule expansion

$$h^2 \sum_{i,j=0}^{n-1} \phi(u_i, v_j) = \sum_{p=0}^m h^{2p} \sum_{q=0}^p C_q C_{p-q} \int_D \partial_u^{2q} \partial_v^{2p-2q} \phi(u, v) du dv + o(h^{2m})$$

Substitution of this expansion with  $\phi = c_p$  and  $m = k - p$  into (19) gives

$$\frac{1}{2} \sum_{i,j=0}^{n-1} |\delta_{h,-h}\mathbf{s}(u_i, v_j) \times \delta_{h,h}\mathbf{s}(u_i, v_j)| = - \sum_{q=0}^k d_q h^{2q} + o(h^{2k}),$$

where

$$d_0 = - \int_D |\mathbf{s}_u \times \mathbf{s}_v| du dv.$$

□

## 5 Numerical example for surface area

We have estimated the area of the Bézier patch of Figure 3 by both the method given in the previous section and Lyness' method and then applied Romberg extrapolation. Since the exact area is unknown, we estimate the errors of entries in the Romberg table by columnwise differences. That is, the error of entry  $R(i, j)$  is estimated by  $|R(i, j) - R(i + 1, j)|$ . The resulting error estimates are in Tables 2

and 4. From this we have computed ERCs by first taking the 2-logarithm of the errors, then again taking columnwise differences. The result is printed in Tables 3 and 5. It is clear that for the first few columns, we approach the expected rates of convergence, whereas for the remaining columns, the pattern is not as clear. Still, the estimates seem to improve, to the point roundoff errors become dominant.

0	1	2	3	4	5	6	7
2.14e-01							
1.34e-01	1.07e-01						
3.94e-02	7.95e-03	1.35e-03					
1.04e-02	6.77e-04	1.92e-04	1.74e-04				
2.63e-03	5.06e-05	8.89e-06	5.98e-06	5.32e-06			
6.59e-04	3.42e-06	2.76e-07	1.39e-07	1.16e-07	1.11e-07		
1.65e-04	2.20e-07	5.90e-09	1.62e-09	1.08e-09	9.70e-10	9.43e-10	
4.12e-05	1.38e-08	1.03e-10	1.07e-11	4.35e-12	3.30e-12	3.07e-12	3.01e-12

Table 2: Error estimates, Romberg table for Bézier patch.

0	1	2	3	4	5	6
0.678						
1.763	3.750					
1.927	3.553	2.813				
1.979	3.741	4.435	4.863			
1.994	3.887	5.010	5.425	5.517		
1.999	3.963	5.545	6.422	6.746	6.839	
2.000	3.990	5.844	7.248	7.958	8.199	8.265

Table 3: ERCs, Romberg table for Bézier patch.

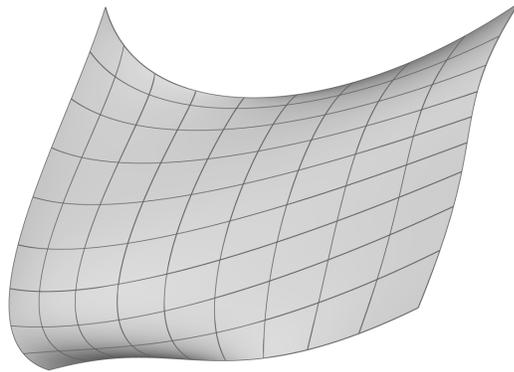


Figure 3: Bézier patch.

0	1	2	3	4	5	6	7
1.53e-01							
1.46e-02	3.15e-02						
1.26e-03	3.18e-03	1.29e-03					
2.10e-04	1.40e-04	6.31e-05	8.46e-05				
4.99e-05	3.52e-06	5.56e-06	4.64e-06	4.33e-06			
1.25e-05	3.66e-08	1.96e-07	1.10e-07	9.27e-08	8.85e-08		
3.11e-06	1.80e-09	4.36e-09	1.33e-09	9.01e-10	8.11e-10	7.90e-10	
7.79e-07	1.85e-10	7.68e-11	8.71e-12	3.53e-12	2.66e-12	2.46e-12	2.41e-12

Table 4: Error estimates, Romberg table for Bézier patch, Lyness method.

0	1	2	3	4	5	6
3.39						
3.53	3.31					
2.58	4.51	4.35				
2.07	5.31	3.51	4.19			
2.00	6.59	4.83	5.39	5.55		
2.00	4.34	5.49	6.38	6.68	6.77	
2.00	3.29	5.83	7.25	7.99	8.25	8.33

Table 5: ERCs, Romberg table for Bézier patch, Lyness method.

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