Wachspress and mean value coordinates

Michael S. Floater

Abstract This paper gives a brief survey of two kinds of generalized barycentric coordinates, Wachspress and mean value coordinates, and their applications. Applications include surface parameterization in geometric modelling, curve and surface deformation in computer graphics, and their use as nodal shape functions for polygonal and polyhedral finite element methods.

1 Introduction

There is no unique way to generalize barycentric coordinates to polygons and polyhedra. However, two specific choices have turned out to be useful in several applications: Wachspress and mean value coordinates, and the purpose of this paper is to survey their main properties, applications, and generalizations.

For convex polygons the coordinates of Wachspress and their generalizations due to Warren and others [30, 22, 32, 33, 15], are arguably the simplest since they are rational functions (quotients of bivariate polynomials) and it is relatively simple to evaluate them and their derivatives. Some simple bounds on their gradients have been found recently in [6], justifying their use as shape functions for polygonal finite elements.

For star-shaped polygons, and arbitrary polygons, Wachspress coordinates are not well-defined, and mean value coordinates are perhaps the most popular choice, due to their generality and surprising robustness over complex geometric shapes [4, 8, 16, 13, 2, 1], even though they are no longer positive if the polygon is not star-shaped. They have been employed in various tasks in geometric modelling, such as surface parameterization, and plane and space deformation, as well as to shading and animation in computer graphics.

Michael S. Floater
Department of Mathematics, University of Oslo, PO Box 1053, Blindern, 0316 Oslo, Norway, e-mail: michaelf@ifi.uio.no

While most of this paper surveys previous results, we add two new ones. The first is a new formula for the gradients of mean value coordinates, which could be used in finite element methods. The second is an alternative formula for the mean value coordinates themselves, which is valid on the boundary of the polygon. Though it may not be of practical value, it offers an alternative way of of showing that these coordinates extend continuously to the polygon boundary.

2 Barycentric coordinates on polygons

Let $P \subset \mathbb{R}^2$ be a convex polygon, viewed as an open set, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, $n \geq 3$, in some anticlockwise ordering. Figure 1 shows an example with $n = 5$. We

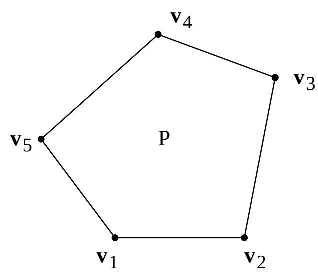

Fig. 1 Vertex ordering for a polygon.

call any functions $\phi_i : P \to \mathbb{R}$, $i = 1, \ldots, n$, (generalized) barycentric coordinates if, for $\mathbf{x} \in P$, $\phi_i(\mathbf{x}) \geq 0$, $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1, \qquad \sum_{i=1}^{n} \phi_i(\mathbf{x}) \mathbf{v}_i = \mathbf{x}. \tag{1}$$

For $n = 3$, the functions ϕ_1, ϕ_2, ϕ_3 are uniquely determined and are the usual triangular barycentric coordinates w.r.t. the triangle with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. For $n \geq 4$, the choice of $\phi_1, \ldots, \phi_n$ is no longer unique. However, they share some basic properties, derived in [7]:

- The functions ϕ_i have a unique continuous extension to ∂P, the boundary of P.
- Lagrange property: $\phi_i(\mathbf{v}_j) = \delta_{ij}$.
- Piecewise linearity on ∂P:

$$\phi_i((1-\mu)\mathbf{v}_j + \mu \mathbf{v}_{j+1}) = (1-\mu)\phi_i(\mathbf{v}_j) + \mu \phi_i(\mathbf{v}_{j+1}), \qquad \mu \in [0,1]. \tag{2}$$

  (Here and throughout, vertices are indexed cyclically, i.e., $\mathbf{v}_{n+1} := \mathbf{v}_1$ etc.)
- Interpolation: if

$$g(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) f(\mathbf{v}_i), \qquad \mathbf{x} \in P, \tag{3}$$

then $g(\mathbf{v}_i) = f(\mathbf{v}_i)$. We call g a barycentric interpolant to f.

- Linear precision: if f is linear then $g = f$.
- $\ell_i \le \phi_i \le L_i$ where $L_i, \ell_i : P \to \mathbb{R}$ are the continuous, piecewise linear functions over the partitions of P shown in Figure 2 satisfying $L_i(\mathbf{v}_j) = \ell_i(\mathbf{v}_j) = \delta_{ij}$.

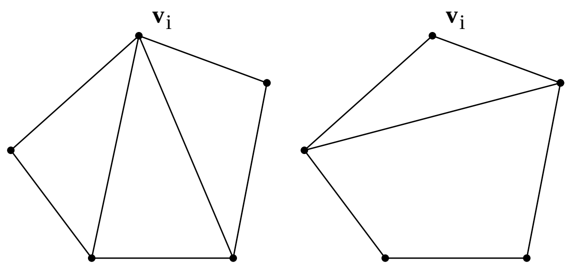

Fig. 2 Partitions for L_i and ℓ_i.

3 Wachspress coordinates

Wachspress coordinates were developed by Wachspress [30], and Warren [32]. They can be defined by the formula

$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})}, \tag{4}$$

where

$$w_i(\mathbf{x}) = \frac{A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1})}{A(\mathbf{x}, \mathbf{v}_{i-1}, \mathbf{v}_i) A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1})},$$

and $A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ denotes the signed area of the triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$,

$$A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\mathbf{x}_k = (x_k, y_k)$; see Figure 3. The original proof that these coordinates are barycentric was based on the so-called adjoint of P; see Wachspress [30], and Warren [32]. The following proof is due to Meyer et al. [22]. Due to (4), it is sufficient to show that

$$\sum_{i=1}^n w_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0. \tag{5}$$

Fix $\mathbf{x} \in P$ and let

$$A_i = A_i(\mathbf{x}) = A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}) \qquad \text{and} \qquad B_i = A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

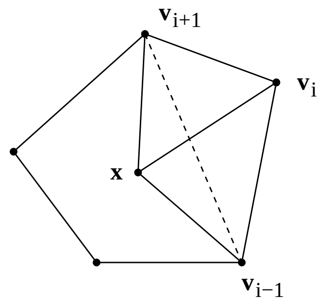

Fig. 3 Triangles defining Wachspress coordinates.

Then we can express $\mathbf{x}$ as a barycentric combination of $\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}$:

$$\mathbf{x} = \frac{A_i}{B_i}\mathbf{v}_{i-1} + \frac{(B_i - A_{i-1} - A_i)}{B_i}\mathbf{v}_i + \frac{A_{i-1}}{B_i}\mathbf{v}_{i+1},$$

regardless of whether $\mathbf{x}$ lies inside or outside the triangle formed by $\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}$. This equation can be rearranged in the form

$$\frac{B_i}{A_{i-1}A_i}(\mathbf{v}_i - \mathbf{x}) = \frac{1}{A_{i-1}}(\mathbf{v}_i - \mathbf{v}_{i-1}) - \frac{1}{A_i}(\mathbf{v}_{i+1} - \mathbf{v}_i).$$

Summing both sides of this over i, and observing that the right hand side then cancels to zero, gives

$$\sum_{i=1}^{n} \frac{B_i}{A_{i-1}A_i}(\mathbf{v}_i - \mathbf{x}) = 0,$$

which proves (5).

3.1 Rational functions

Another way of expressing these coordinates is clearly in the form

$$\phi_i(\mathbf{x}) = \frac{\hat{w}_i(\mathbf{x})}{\sum_{j=1}^{n} \hat{w}_j(\mathbf{x})}, \qquad \hat{w}_i(\mathbf{x}) = B_i \prod_{j \neq i-1,i} A_j(\mathbf{x}), \tag{6}$$

and since each area $A_j(\mathbf{x})$ is linear in $\mathbf{x}$, we see from this that ϕ_i is a rational (bivariate) function, with total degree $\leq n-2$ in the numerator and denominator. In fact, the denominator, $W = \sum_{j=1}^{n} \hat{w}_j$, has total degree $\leq n-3$ due to linear precision: since (5) holds with w_i replaced by $\hat{w}_i$, it implies that

$$\sum_{i=1}^{n} \hat{w}_i(\mathbf{x})\mathbf{v}_i = W(\mathbf{x})\mathbf{x}.$$

The left hand side is a (vector-valued) polynomial of degree $\leq n-2$ in $\mathbf{x}$ and since $\mathbf{x}$ has degree 1, the degree of W must be at most $n-3$.

The degrees, $n-2$ and $n-3$, of the numerator and denominator of ϕ_i agree with the triangular case where $n=3$ and the coordinates are linear functions.

We note that the 'global' form of $\phi_i(\mathbf{x})$ in (6) is also valid for $\mathbf{x} \in \partial P$, unlike the 'local' form (4), though it requires more computation for large n.

3.2 Perpendicular distances to edges

An alternative way of expressing Wachspress coordinates is in terms of the perpendicular distances of $\mathbf{x}$ to the edges of P. This is the form used by Warren et al. [33] and it generalizes in a natural way to higher dimension.

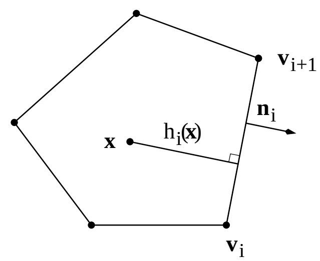

Fig. 4 Perpendicular distances.

For each i, let $\mathbf{n}_i \in \mathbb{R}^2$ be the outward unit normal to the edge $e_i = [\mathbf{v}_i, \mathbf{v}_{i+1}]$, and for any $\mathbf{x} \in P$ let $h_i(\mathbf{x})$ be the perpendicular distance of $\mathbf{x}$ to the edge e_i, so that

$$h_i(\mathbf{x}) = (\mathbf{v}_i - \mathbf{x}) \cdot \mathbf{n}_i = (\mathbf{v}_{i+1} - \mathbf{x}) \cdot \mathbf{n}_i;$$

see Figure 4. Then the coordinates in (4) can be expressed as

$$\phi_i(\mathbf{x}) = \frac{\tilde{w}_i(\mathbf{x})}{\sum_{j=1}^n \tilde{w}_j(\mathbf{x})}, \tag{7}$$

where

$$\tilde{w}_i(\mathbf{x}) := \frac{\mathbf{n}_{i-1} \times \mathbf{n}_i}{h_{i-1}(\mathbf{x}) h_i(\mathbf{x})}, \tag{8}$$

and

$$\mathbf{x}_1 \times \mathbf{x}_2 := \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.$$

for $\mathbf{x}_k = (x_k, y_k)$. To see this, observe that with $L_j = |\mathbf{v}_{j+1} - \mathbf{v}_j|$ (and $|\cdot|$ the Euclidean norm) and β_i the interior angle of the polygon at $\mathbf{v}_i$,

$$A(\mathbf{v}_{i-1},\mathbf{v}_i,\mathbf{v}_{i+1}) = \frac{1}{2}\sin\beta_i L_{i-1}L_i,$$

and

$$A(\mathbf{x},\mathbf{v}_{i-1},\mathbf{v}_i) = \frac{1}{2}h_{i-1}(\mathbf{x})L_{i-1}, \qquad A(\mathbf{x},\mathbf{v}_i,\mathbf{v}_{i+1}) = \frac{1}{2}h_i(\mathbf{x})L_i,$$

so that

$$w_i(x) = 2\tilde{w}_i(\mathbf{x}).$$

3.3 Gradients

The gradient of a Wachspress coordinate can be found quite easily from the perpendicular form (7–8). Since $\nabla h_i(\mathbf{x}) = -\mathbf{n}_i$, the gradient of $\tilde{w}_i$ is [6]

$$\nabla\tilde{w}_i(\mathbf{x}) = \tilde{w}_i(\mathbf{x})\left(\frac{\mathbf{n}_{i-1}}{h_{i-1}(\mathbf{x})} + \frac{\mathbf{n}_i}{h_i(\mathbf{x})}\right). \tag{9}$$

Thus the (vector-valued) ratio $\mathbf{R}_i := \nabla\tilde{w}_i/\tilde{w}_i$ is simply

$$\mathbf{R}_i(x) = \frac{\mathbf{n}_{i-1}}{h_{i-1}(\mathbf{x})} + \frac{\mathbf{n}_i}{h_i(\mathbf{x})}.$$

Using the formula [6]

$$\nabla\phi_i = \phi_i(\mathbf{R}_i - \sum_{j=1}^{n}\phi_j\mathbf{R}_j) \tag{10}$$

for any function ϕ_i of the form (7), we thus obtain $\nabla\phi_i(\mathbf{x})$ for $\mathbf{x}\in P$.

3.4 Curve deformation

While Wachspress's motivation for these coordinates was finite element methods over polygonal partitions, Warren suggested their use in deforming curves. The coordinates can be used to define a barycentric mapping of one polygon to another, and such a mapping will then map, or deform, a curve embedded in the first polygon into a new one, with the vertices of the polygon acting as control points, with an effect similar to those of Bézier and spline curves and surfaces.

Assuming the second polygon is P' with vertices $\mathbf{v}'_1,\ldots,\mathbf{v}'_n$, the barycentric mapping $\mathbf{g}: P \to P'$ is defined as follows. Given $\mathbf{x}\in P$,

1. express $\mathbf{x}$ in Wachspress coordinates, $\mathbf{x} = \sum_{i=1}^n \phi_i(\mathbf{x})\mathbf{v}_i$,
2. set $\mathbf{g}(\mathbf{x}) = \sum_{i=1}^n \phi_i(\mathbf{x})\mathbf{v}'_i$.

Figure 5 shows such a mapping. Figure 6 shows the effect of using the mapping to deform a curve (a circle in this case).

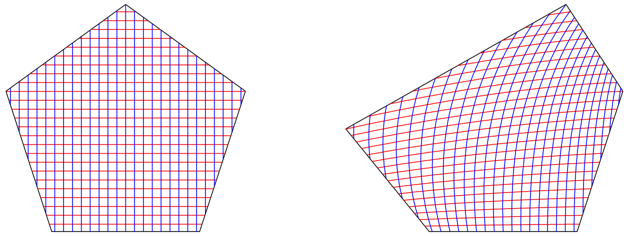

Fig. 5 Barycentric mapping.

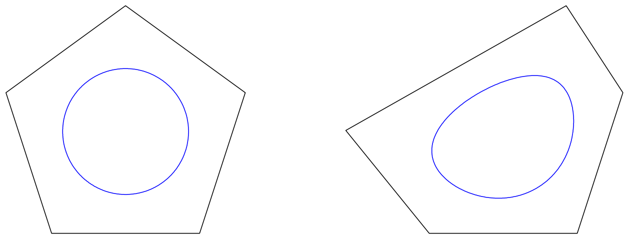

Fig. 6 Curve deformation.

It is now known that Wachspress mappings between convex polygons are always injective; as shown in [9]. The basic idea of the proof is to show that $\mathbf{g}$ has a positive Jacobian determinant $J(\mathbf{g})$. To do this one first shows that $J(\mathbf{g})$ can be expressed as

$$J(\mathbf{g}) = 2 \sum_{1\le i<j<k\le n} \begin{vmatrix} \phi_i & \phi_j & \phi_k \\ \partial_1\phi_i & \partial_1\phi_j & \partial_1\phi_k \\ \partial_2\phi_i & \partial_2\phi_j & \partial_2\phi_k \end{vmatrix} A(\mathbf{v}'_i, \mathbf{v}'_j, \mathbf{v}'_k).$$

By the convexity of P', the signed areas $A(\mathbf{v}'_i, \mathbf{v}'_j, \mathbf{v}'_k)$ in the sum are all positive, and so $J(\mathbf{g}) > 0$ if all the 3×3 determinants in the sum are positive, and this turns out to be the case for Wachspress coordinates ϕ_i.

4 Mean value coordinates

As we have seen, Wachspress coordinates are relatively simple functions, and lead to well-behaved barycentric mappings. They are, however, limited to convex polygons. For a non-convex polygon they are not well-defined, since the denominator in the rational expression becomes zero at certain points in the polygon. An alternative set of coordinates for convex polygons is the mean value coordinates [4], which have a simple generalization to non-convex polygons, though positivity is in general lost. Suppose initially that P is convex as before. Then the mean value (MV) coordinates are defined by (4) and

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{|\mathbf{v}_i - \mathbf{x}|}, \tag{11}$$

with the angles $\alpha_j = \alpha_j(\mathbf{x})$, with $0 < \alpha_j < \pi$, as shown in Figure 7. To show that

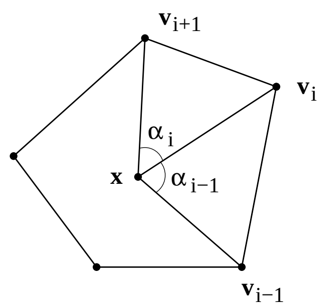

Fig. 7 Notation for mean value coordinates.

these coordinates are barycentric, it is sufficient, as in the Wachspress case, to show that the w_i in (11) satisfy (5). This can be done in four steps:

1. Express the unit vectors $\mathbf{e}_i := (\mathbf{v}_i - \mathbf{x})/|\mathbf{v}_i - \mathbf{x}|$ in polar coordinates:

$$\mathbf{e}_i = (\cos\theta_i, \sin\theta_i),$$

   and note that $\alpha_i = \theta_{i+1} - \theta_i$.
2. Use the fact that the integral of the unit normals $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$ on a circle is zero:

$$\int_0^{2\pi} \mathbf{n}(\theta)\, d\theta = 0.$$

3. Split this integral according to the θ_i:

$$\int_0^{2\pi} \mathbf{n}(\theta)\, d\theta = \sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \mathbf{n}(\theta)\, d\theta. \tag{12}$$

4. Show by trigonometry that

$$\int_{\theta_i}^{\theta_{i+1}} \mathbf{n}(\theta)\, d\theta = \frac{1 - \cos\alpha_i}{\sin\alpha_i}(\mathbf{e}_i + \mathbf{e}_{i+1}) = \tan(\alpha_i/2)(\mathbf{e}_i + \mathbf{e}_{i+1}).$$

   Substituting this into the sum in (12) and rearranging gives (5).

   We can compute $\tan(\alpha_i/2)$ from the formulas

$$\cos\alpha_i = \mathbf{e}_i \cdot \mathbf{e}_{i+1}, \qquad \sin\alpha_i = \mathbf{e}_i \times \mathbf{e}_{i+1}. \tag{13}$$

Figure 8 compares the contour lines of a Wachspress coordinate, on the left, with the corresponding MV coordinate, on the right.

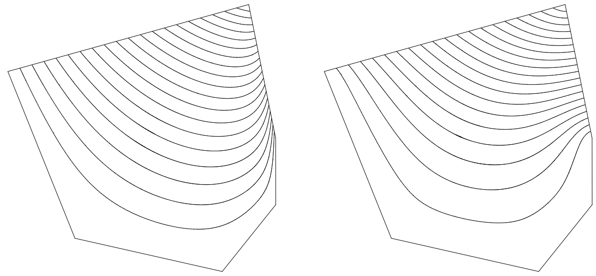

Fig. 8 Wachspress (left). Mean value (right).

4.1 Gradients

Similar to the Wachspress case, the gradient $\nabla\phi_i$ of the MV coordinate ϕ_i can be computed from the formula (10) if we can find the ratio $\mathbf{R}_i := \nabla w_i / w_i$, with w_i in (11). Let $r_i = |\mathbf{v}_i - \mathbf{x}|$ and $t_i = \tan(\alpha_i/2)$ so that

$$w_i = \frac{t_{i-1} + t_i}{r_i}.$$

Further, define

$$\mathbf{c}_i = \frac{\mathbf{e}_i}{r_i} - \frac{\mathbf{e}_{i+1}}{r_{i+1}},$$

and for a vector $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, let $\mathbf{a}^\perp := (-a_2, a_1)$.

Theorem 1. *For the MV coordinates,*

$$\mathbf{R}_i = \left(\frac{t_{i-1}}{t_{i-1} + t_i}\right) \frac{\mathbf{c}_{i-1}^\perp}{\sin\alpha_{i-1}} + \left(\frac{t_i}{t_{i-1} + t_i}\right) \frac{\mathbf{c}_i^\perp}{\sin\alpha_i} + \frac{\mathbf{e}_i}{r_i}.$$

We will show this using two lemmas.

Lemma 1. *For* $\mathbf{u} \in \mathbb{R}^2$, *let* $\mathbf{e} = (e_1, e_2) = (\mathbf{u} - \mathbf{x})/|\mathbf{u} - \mathbf{x}|$ *and* $r = |\mathbf{u} - \mathbf{x}|$. *Then*

$$\nabla e_1 = \frac{e_2 \mathbf{e}^\perp}{r}, \qquad \nabla e_2 = -\frac{e_1 \mathbf{e}^\perp}{r}.$$

Proof. If $\mathbf{d} = (d_1, d_2) = \mathbf{u} - \mathbf{x}$, then using the fact that

$$\nabla d_1 = (-1, 0), \quad \nabla d_2 = (0, -1), \quad \text{and} \quad \nabla r = -\mathbf{d}/r,$$

the result follows from the quotient rule:

$$\nabla e_k = \nabla\left(\frac{d_k}{r}\right) = \frac{r \nabla d_k - d_k \nabla r}{r^2}, \qquad k = 1, 2.$$

□

Lemma 2. *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, *and let*

$$\mathbf{e} = (\mathbf{u} - \mathbf{x})/|\mathbf{u} - \mathbf{x}|, \qquad r = |\mathbf{u} - \mathbf{x}|,$$
$$\mathbf{f} = (\mathbf{v} - \mathbf{x})/|\mathbf{v} - \mathbf{x}|, \qquad s = |\mathbf{v} - \mathbf{x}|.$$

Then

$$\nabla(\mathbf{e} \cdot \mathbf{f}) = -(\mathbf{e} \times \mathbf{f})\mathbf{c}^\perp \qquad \textit{and} \qquad \nabla(\mathbf{e} \times \mathbf{f}) = (\mathbf{e} \cdot \mathbf{f})\mathbf{c}^\perp,$$

where

$$\mathbf{c} = \frac{\mathbf{e}}{r} - \frac{\mathbf{f}}{s}.$$

Proof. With $\mathbf{e} = (e_1, e_2)$ and $\mathbf{f} = (f_1, f_2)$,

$$\nabla(\mathbf{e} \cdot \mathbf{f}) = f_1 \nabla e_1 + e_1 \nabla f_1 + f_2 \nabla e_2 + e_2 \nabla f_2,$$
$$\nabla(\mathbf{e} \times \mathbf{f}) = f_2 \nabla e_1 + e_1 \nabla f_2 - f_1 \nabla e_2 - e_2 \nabla f_1,$$

and applying Lemma 1 to ∇e_k and ∇f_k, $k = 1, 2$, gives the result. □

We now prove Theorem 1. Recalling (13), Lemma 2 shows that

$$\nabla(\cos \alpha_i) = -(\sin \alpha_i)\mathbf{c}_i^\perp, \qquad \nabla(\sin \alpha_i) = (\cos \alpha_i)\mathbf{c}_i^\perp. \tag{14}$$

From this it follows that

$$\nabla t_i = \frac{t_i}{\sin \alpha_i} \mathbf{c}_i^\perp.$$

Since, $\nabla r_i = -\mathbf{e}_i$, this means that

$$\nabla \left(\frac{t_j}{r_i} \right) = \frac{t_j}{r_i} \left(\frac{\mathbf{c}_j^\perp}{\sin \alpha_i} + \frac{\mathbf{e}_i}{r_i} \right), \qquad j = i-1, i.$$

Therefore,

$$\nabla w_i = \frac{t_{i-1}}{r_i} \left(\frac{\mathbf{c}_{i-1}^\perp}{\sin \alpha_{i-1}} \right) + \frac{t_i}{r_i} \left(\frac{\mathbf{c}_i^\perp}{\sin \alpha_i} \right) + w_i \frac{\mathbf{e}_i}{r_i},$$

which, after dividing by w_i, proves Theorem 1.

Incidentally, though we did not use it, we note that both equations in (14) imply that

$$\nabla \alpha_i = \mathbf{c}_i^\perp.$$

Another derivative formula for MV coordinates can be found in [28].

4.2 Alternative formula

We saw that Wachspress coordinates can be expressed in the 'global form' (6) in which $\phi_i(\mathbf{x})$ is well-defined for $\mathbf{x} \in \partial P$ as well as for $\mathbf{x} \in P$. It turns out that MV coordinates also have a global form with the same property, though for large n, the resulting expression requires more computation, and involves more square roots, than the local form based on (11). Let $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$, $i = 1, \ldots, n$.

Theorem 2. *The MV coordinates in (4) can be expressed as*

$$\phi_i(\mathbf{x}) = \frac{\hat{w}_i(\mathbf{x})}{\sum_{j=1}^n \hat{w}_j(\mathbf{x})}, \tag{15}$$

where

$$\hat{w}_i = (r_{i-1}r_{i+1} - \mathbf{d}_{i-1} \cdot \mathbf{d}_{i+1})^{1/2} \prod_{j \neq i-1,i} (r_j r_{j+1} + \mathbf{d}_j \cdot \mathbf{d}_{j+1})^{1/2}. \tag{16}$$

Proof. From the addition formula for sines, we have

$$w_i = \frac{1}{r_i}\left(\frac{\sin(\alpha_{i-1}/2)}{\cos(\alpha_{i-1}/2)} + \frac{\sin(\alpha_i/2)}{\cos(\alpha_i/2)}\right) = \frac{\sin((\alpha_{i-1}+\alpha_i)/2)}{r_i \cos(\alpha_{i-1}/2)\cos(\alpha_i/2)}.$$

Then, to get rid of the half-angles we use the identities

$$\sin(A/2) = \sqrt{(1-\cos A)/2},$$
$$\cos(A/2) = \sqrt{(1+\cos A)/2},$$

to obtain

$$w_i = \frac{1}{r_i}\left(\frac{2(1-\cos(\alpha_{i-1}+\alpha_i))}{(1+\cos\alpha_{i-1})(1+\cos\alpha_i)}\right)^{1/2}.$$

Now we substitute in the scalar product formula,

$$\cos(\alpha_{i-1}+\alpha_i) = \frac{\mathbf{d}_{i-1} \cdot \mathbf{d}_{i+1}}{r_{i-1}r_{i+1}},$$

and similarly for $\cos\alpha_{i-1}$ and $\cos\alpha_i$, and the $1/r_i$ term cancels out:

$$w_i = \left(\frac{2(r_{i-1}r_{i+1} - \mathbf{d}_{i-1} \cdot \mathbf{d}_{i+1})}{(r_{i-1}r_i + \mathbf{d}_{i-1} \cdot \mathbf{d}_i)(r_i r_{i+1} + \mathbf{d}_i \cdot \mathbf{d}_{i+1})}\right)^{1/2},$$

which gives (15–16). □

One can easily check that this formula gives the correct values (2) for $\mathbf{x} \in \partial P$.

4.3 Star-shaped polygons

The original motivation for these coordinates was for parameterizing triangular meshes [29, 3, 5]. In this application, the point $\mathbf{x}$ is a vertex in a planar triangulation, with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ its neighbouring vertices. Thus, in this case, the polygon P

(with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$) is not necessarily convex, but always star-shaped, with $\mathbf{x}$ a point in its kernel, i.e., every vertex $\mathbf{v}_i$ is 'visible' from $\mathbf{x}$; see Figure 9. In this case

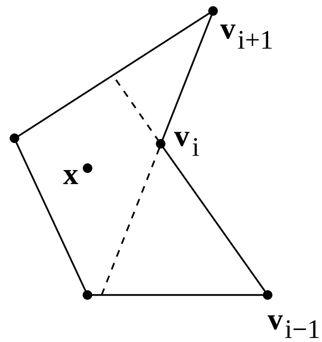

Fig. 9 A star-shaped polygon and its kernel.

the angles α_i in (11) are again positive, and the weight $w_i(\mathbf{x})$ is again positive. Thus the MV coordinates of $\mathbf{x}$ remain positive in this star-shaped case. The advantage of this is that when these coordinates are applied to the parameterization of triangular meshes, the piecewise linear mapping is guaranteed to be injective, i.e., none of the triangles 'fold over', when the boundary of the mesh is mapped to a convex polygon.

4.4 Arbitrary polygons

It was later observed, in [13], that the coordinates are still well-defined, though not necessarily positive, when P is an arbitrary polygon, provided that the angles α_i are treated as signed angles: i.e., we take α_i in (11) to have the same sign as $\mathbf{e}_i \times \mathbf{e}_{i+1}$, which will be the case if we use the formulas (13). The reason for this is that even though $w_i(\mathbf{x})$ in (11) may be negative for some i, when P is arbitrary, the sum $\sum_{i=1}^n w_i(\mathbf{x})$ is nevertheless positive for any $\mathbf{x}$ in P. This was shown in [13], where it was also shown that these more general MV coordinates have the Lagrange and piecewise linearity properties on ∂P.

This generalization of MV coordinates allows the curve deformation method to be extended to arbitrary polygons. It was further observed in [13] that MV coordinates even have a natural generalization to any set of polygons, as long as the polygons do not intersect one another. The polygons may or may not be nested. These generalized MV coordinates were applied to image warping in [13].

5 Polygonal finite elements

There has been steadily growing interest in using generalized barycentric coordinates for finite element methods on polygonal (and polyhedral) meshes [11, 23, 26, 34, 6]. In order to establish the convergence of the finite element method one would need to derive a bound on the gradients of the coordinates in terms of the geometry of the polygon P. Various bounds on

$$\sup_{\mathbf{x}\in P} |\nabla \phi_i(\mathbf{x})|$$

were derived in [11] for Wachspress (and other) coordinates, and in [23] for MV coordinates. For the Wachspress coordinates, a simpler bound was derived in [6]. If we define, for $\mathbf{x} \in P$,

$$\lambda(\mathbf{x}) := \sum_{i=1}^{n} |\nabla \phi_i(\mathbf{x})|, \tag{17}$$

then λ plays a role similar to the Lebesgue function in the theory of polynomial interpolation because for g in (3),

$$|\nabla g(\mathbf{x})| \le \sum_{i=1}^{n} |\nabla \phi_i(\mathbf{x})||f(\mathbf{v}_i)| \le \lambda(\mathbf{x}) \max_{i=1,\ldots,n} |f(\mathbf{v}_i)|.$$

It was shown in [6] that with

$$\Lambda := \sup_{\mathbf{x}\in P} \lambda(\mathbf{x}) \tag{18}$$

the corresponding 'Lebesgue constant', and with ϕ_i the Wachspress coordinates,

$$\Lambda \le \frac{4}{h_*},$$

where

$$h_* = \min_{i=1,\ldots,n} \min_{j\neq i,i+1} h_i(\mathbf{v}_j).$$

6 Curved domains

Consider again the barycentric interpolant g in (3). Since g is piecewise linear on the boundary ∂P, it interpolates f on ∂P if f itself is piecewise linear on ∂P. Warren et al. [33] proposed a method of interpolating any continuous function f defined on the boundary of any convex domain, by, roughly speaking, taking a continuous 'limit' of the polygonal interpolants g in (3). Specifically, suppose that the boundary of some convex domain $P \subset \mathbb{R}^2$ is represented as a closed, parametric curve $\mathbf{c} : [a,b] \to \mathbb{R}^2$, with $\mathbf{c}(b) = \mathbf{c}(a)$. Then any sequence of parameter values, $t_1,\ldots,t_n$, with $a \le t_1 < t_2 < \cdots < t_n < b$, with mesh size $h = \max_i(t_{i+1} - t_i)$, defines a convex polygon P_h

with vertices $\mathbf{v}_i = \mathbf{c}(t_i)$; see Figure 10. The barycentric interpolant g in (3) with

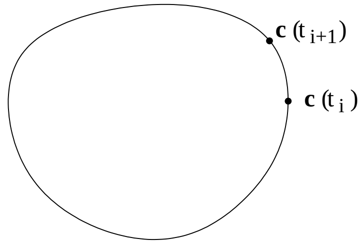

Fig. 10 From polygons to curved domains.

respect to this polygon is then

$$g_h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) f(\mathbf{c}(t_i)). \qquad (19)$$

Taking the limit $g = \lim_{h\to 0} g_h$ over a sequence of such polygons, and letting the ϕ_i be the Wachspress coordinates, gives

$$g(\mathbf{x}) = \int_a^b w(\mathbf{x},t) f(\mathbf{c}(t))\,dt \Big/ \int_a^b w(\mathbf{x},t)\,dt, \qquad \mathbf{x} \in P, \qquad (20)$$

where

$$w(\mathbf{x},t) = \frac{(\mathbf{c}'(t) \times \mathbf{c}''(t))}{((\mathbf{c}(t)-\mathbf{x}) \times \mathbf{c}'(t))^2}.$$

It was shown in [33] that the barycentric property also holds for this g: if $f : \mathbb{R}^2 \to \mathbb{R}$ is linear, i.e., $f(\mathbf{x}) = ax + by + c$, then $g = f$. However, it also follows from the fact that if f is linear, $g_h = f$ for all h.

There is an analogous continuous MV interpolant, with g also given by (20), but with the weight function $w(\mathbf{x},t)$ replaced by

$$w(\mathbf{x},t) = \frac{(\mathbf{c}(t)-\mathbf{x}) \times \mathbf{c}'(t)}{|\mathbf{c}(t)-\mathbf{x}|^3}. \qquad (21)$$

One can also derive the barycentric property of this continuous interpolant by applying the unit circle construction of Section 4 directly to the curved domain P. Figure 11 shows the MV interpolant to the function $\cos(2\theta)$, $0 \le \theta < 2\pi$, on the boundary of the unit circle.

Similar to the generalization of MV coordinates to non-convex polygons, the continuous MV interpolant also extends to arbitrarily shaped curve domains: one simply applies the same formula (21). Even though the cross product,

$$(\mathbf{c}(t)-\mathbf{x}) \times \mathbf{c}'(t)$$

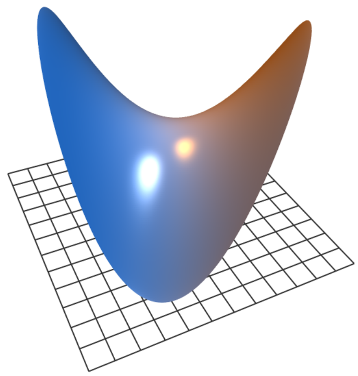

Fig. 11 An MV interpolant on a circle.

may be negative for some values of t, the integral $\int_a^b w(\mathbf{x},t)\,dt$ of w in (21) remains positive [2].

6.1 Hermite interpolation

If the normal derivative of f is also known on the boundary of the domain, we could consider matching both the values and normal derivatives of f. In [2] and [10] two distinct approaches were used to construct such a Hermite interpolant, both based on the construction of MV interpolants. To motivate this, let π_n denote the linear space of polynomials of degree $\leq n$ in one real variable. Suppose that $f : [0,1] \to \mathbb{R}$ has a first derivative at $x = 0$ and $x = 1$. Then there is a unique cubic polynomial, $p \in \pi_3$, such that

$$p^{(k)}(i) = f^{(k)}(i), \qquad i = 0,1, \quad k = 0,1.$$

There are various ways of expressing p. One is as

$$p = l_0(x) + \omega(x) l_1(x),$$

where

$$l_0(x) = (1-x) f(0) + x f(1), \quad \omega(x) = x(1-x), \quad l_1(x) = (1-x)m_0 + x m_1,$$

and

$$m_0 = f'(0) - (f(1) - f(0)), \qquad m_1 = (f(1) - f(0)) - f'(1).$$

The basic idea of the Hermite interpolant in [2] is to generalize this construction to a general planar domain, replacing the linear interpolants l_0 and l_1 by MV interpolants, and replacing the weight function ω by an MV 'weight' function. This gives a Hermite interpolant in 2-D, but it does not in general have cubic precision. Another way of expressing p above is as the minimizer of a functional. For a fixed $x \in (0,1)$, $p(x)$ is the value $s(x)$ of the spline s that minimizes the functional

$$E(s) = \int_0^1 (s''(y))^2 \, dy,$$

in the spline space

$$S = \{s \in C^1[0,1] : s|_{[0,x]}, s|_{[x,1]} \in \pi_3\},$$

subject to the boundary conditions

$$s^{(k)}(i) = f^{(k)}(i), \qquad i = 0,1, \quad k = 0,1.$$

A generalization of this minimization was used in [10] to generate a function on a curved domain that appears, numerically, to interpolate the boundary data, but a mathematical proof of this is still missing. The cubic construction in [10] was recently derived independently through certain mean value properties of biharmonic functions by Xianying Li et al. [19]. They also give a closed-form expression for the coordinates on a polygonal domain when a suitable definition of the boundary data is used along the edges.

7 Coordinates in higher dimensions

So far we have only considered coordinates for points in $\mathbb{R}^2$, but there are applications of barycentric coordinates for points in a polyhedron in $\mathbb{R}^3$, such as in Figure 12, or more generally for points in a polytope in $\mathbb{R}^d$. Both Wachspress and MV coordinates have been generalized to higher dimensions.

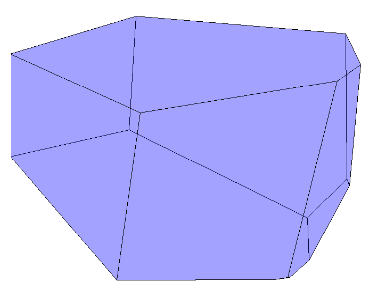

Fig. 12 Simple, convex polyhedron.

7.1 Wachspress coordinates in 3-D

Warren [32] generalized the coordinates of Wachspress to simple convex polyhedra: convex polyhedra in which all vertices have three incident faces. In [33], Warren et al. derived the same coordinates in a different way (avoiding the so-called 'adjoint'), generalizing (7) as follows. Let $P \subset \mathbb{R}^3$ be a simple convex polyhedron, with faces F and vertices V. For each face $f \in F$, let $\mathbf{n}_f \in \mathbb{R}^3$ denote its unit outward normal, and for any $\mathbf{x} \in P$, let $h_f(\mathbf{x})$ denote the perpendicular distance of $\mathbf{x}$ to f, which can be expressed as the scalar product

$$h_f(\mathbf{x}) = (\mathbf{v} - \mathbf{x}) \cdot \mathbf{n}_f,$$

for any vertex $\mathbf{v} \in V$ belonging to f. For each vertex $\mathbf{v} \in V$, let f_1, f_2, f_3 be the three faces incident to $\mathbf{v}$, and for $\mathbf{x} \in P$, let

$$w_{\mathbf{v}}(\mathbf{x}) = \frac{\det(\mathbf{n}_{f_1}, \mathbf{n}_{f_2}, \mathbf{n}_{f_3})}{h_{f_1}(\mathbf{x}) h_{f_2}(\mathbf{x}) h_{f_3}(\mathbf{x})}, \qquad (22)$$

where it is understood that f_1, f_2, f_3 are ordered such that the determinant in the numerator is positive. Here, for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$,

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Thus the ordering of f_1, f_2, f_3 must be anticlockwise around $\mathbf{v}$, seen from outside P. In this way, $w_{\mathbf{v}}(\mathbf{x}) > 0$, and it was shown in [33] that the functions

$$\phi_{\mathbf{v}}(\mathbf{x}) := \frac{w_{\mathbf{v}}(\mathbf{x})}{\sum_{\mathbf{u} \in V} w_{\mathbf{u}}(\mathbf{x})} \qquad (23)$$

are barycentric coordinates for $\mathbf{x} \in P$ in the sense that

$$\sum_{\mathbf{v} \in V} \phi_{\mathbf{v}}(\mathbf{x}) = 1, \qquad \sum_{\mathbf{v} \in V} \phi_{\mathbf{v}}(\mathbf{x}) \mathbf{v} = \mathbf{x}. \qquad (24)$$

To deal with non-simple polyhedra, it was suggested in [33] that one might decompose a non-simple vertex into simple ones by perturbing its adjacent facets. Later, Ju et al. [15] found a cleaner solution, using properties of the so-called *polar dual*. With respect to each $\mathbf{x}$ in a general convex polyhedron $P \subset \mathbb{R}^3$, there is a dual polyhedron,

$$\tilde{P}_{\mathbf{x}} := \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \leq 1, \mathbf{z} \in P\}.$$

It contains the origin $\mathbf{y} = 0$, and its vertices are the endpoints of the vectors

$$\mathbf{p}_f(\mathbf{x}) := \frac{\mathbf{n}_f}{h_f(\mathbf{x})}, \qquad f \in F,$$

when placed at the origin. Suppose that a vertex $\mathbf{v} \in V$ has k incident faces, $f_1, \ldots, f_k$, for some $k \geq 3$, where we again assume they are ordered in some anticlockwise fashion around $\mathbf{v}$, as seen from outside P. The endpoints of the k vectors $\mathbf{p}_{f_1}(\mathbf{x}), \ldots \mathbf{p}_{f_k}(\mathbf{x})$ form a k-sided polygon. This polygon is the face of $\tilde{P}_{\mathbf{x}}$, dual to the vertex $\mathbf{v}$ of P. This face and the origin in $\mathbb{R}^3$ form a polygonal pyramid, $Q_{\mathbf{v}} \subset \tilde{P}_{\mathbf{x}}$. It was shown in [15] that if we define

$$w_{\mathbf{v}}(\mathbf{x}) = \mathrm{vol}(Q_{\mathbf{v}}),$$

then the functions $\phi_{\mathbf{v}}$ in (23) are again barycentric coordinates. In practice we could triangulate the face dual to $\mathbf{v}$ by connecting the endpoint of $\mathbf{p}_{f_1}(\mathbf{x})$ to the endpoints of all the other $\mathbf{p}_{f_i}(\mathbf{x})$, and so compute $\mathrm{vol}(Q_{\mathbf{v}})$ as a sum of volumes of tetrahedra. Thus, we could let

$$w_{\mathbf{v}}(\mathbf{x}) = \sum_{i=2}^{k-1} \det(\mathbf{p}_{f_1}(\mathbf{x}), \mathbf{p}_{f_i}(\mathbf{x}), \mathbf{p}_{f_{i+1}}(\mathbf{x})). \tag{25}$$

Some matlab code for evaluating these coordinates and their gradients can be found in [6].

7.2 MV coordinates in 3-D

MV coordinates were generalized to three dimensions in [8] and [16], the basic idea being to replace integration over the unit circle, as in Section 4, by integration over the unit sphere.

Consider first the case that $P \subset \mathbb{R}^3$ is a convex polyhedron with triangular faces (though it does not need to be simple). Fix $\mathbf{x} \in P$ and consider the radial projection of the boundary of P onto the unit sphere centred at $\mathbf{x}$. A vertex $\mathbf{v} \in V$ is projected to the point (unit vector) $\mathbf{e}_{\mathbf{v}} := (\mathbf{v} - \mathbf{x})/|\mathbf{v} - \mathbf{x}|$. A face $f \in F$ is projected to a spherical triangle $f_{\mathbf{x}}$ whose vertices are $\mathbf{e}_{\mathbf{v}}$, $\mathbf{v} \in V_f$, where $V_f \subset V$ denotes the set of (three) vertices of f. Let $\mathbf{I}_f$ denote the (vector-valued) integral of its unit normals,

$$\mathbf{I}_f := \int_{f_{\mathbf{x}}} \mathbf{n}(\mathbf{y})\, d\mathbf{y}.$$

Since the three vectors $\mathbf{e}_{\mathbf{v}}$, $\mathbf{v} \in V_f$, are linearly independent, there are three unique weights $w_{\mathbf{v},f} > 0$ such that

$$\mathbf{I}_f = \sum_{\mathbf{v} \in V_f} w_{\mathbf{v},f} \mathbf{e}_{\mathbf{v}}. \tag{26}$$

The weights can be found as ratios of 3×3 determinants from Cramer's rule. Since the integral of all unit normals of the unit sphere is zero, and letting $F_{\mathbf{v}} \subset F$ denote the set of faces that are incident on the vertex $\mathbf{v}$, we find, by switching summations, that

$$0 = \sum_{f \in F} \mathbf{I}_f = \sum_{f \in F} \sum_{\mathbf{v} \in V_f} w_{\mathbf{v},f} \mathbf{e}_{\mathbf{v}} = \sum_{\mathbf{v} \in V} \sum_{f \in F_{\mathbf{v}}} w_{\mathbf{v},f} \mathbf{e}_{\mathbf{v}},$$

and so the functions

$$w_{\mathbf{v}} := \sum_{f \in F_{\mathbf{v}}} \frac{w_{\mathbf{v},f}}{|\mathbf{v}-\mathbf{x}|}, \tag{27}$$

satisfy

$$\sum_{\mathbf{v}\in V} w_{\mathbf{v}}(\mathbf{x})(\mathbf{v}-\mathbf{x}) = 0.$$

It follows that the functions $\phi_{\mathbf{v}}$ given by (23) with $w_{\mathbf{v}}$ given by (27) are barycentric coordinates, i.e., they are positive in P and satisfy (24).

It remains to find the integral $\mathbf{I}_f$ in terms of the points $\mathbf{v} \in V_f$ and $\mathbf{x}$. We follow the observation made in [8]. The spherical triangle $f_{\mathbf{x}}$ and the point $\mathbf{x}$ form a wedge of the solid unit sphere centred at $\mathbf{x}$. Since the integral of all unit normals over this wedge is zero, the integral $\mathbf{I}_f$ is minus the sum of the integrals over the three planar faces of the wedge. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the vertices of f in anticlockwise order, and let $\mathbf{e}_i = \mathbf{e}_{\mathbf{v}_i}$. For $i = 1,2,3$, the i-th side of the wedge is the sector of the unit circle formed by the two unit vectors $\mathbf{e}_i$ and $\mathbf{e}_{i+1}$, with the cyclic notation $\mathbf{v}_{i+3} := \mathbf{v}_i$. If $\beta_i \in (0,\pi)$ is the angle between $\mathbf{e}_i$ and $\mathbf{e}_{i+1}$ then the area of the sector is $\beta_i/2$, and hence

$$\mathbf{I}_f = \frac{1}{2}\sum_{i=1}^{3} \beta_i \mathbf{m}_i, \tag{28}$$

where

$$\mathbf{m}_i = \frac{\mathbf{e}_i \times \mathbf{e}_{i+1}}{|\mathbf{e}_i \times \mathbf{e}_{i+1}|}.$$

Equating this with (26) gives

$$w_{\mathbf{v}_i,f} = \frac{1}{2}\sum_{j=1}^{3} \beta_j \frac{\mathbf{m}_j \cdot \mathbf{m}_{i+1}}{\mathbf{e}_i \cdot \mathbf{m}_{i+1}}.$$

These 3-D MV coordinates were used for surface deformation in [16] when the surface is represented as a dense triangular mesh. Some contour plots of the coordinate functions can be found in [8].

For a polyhedron with faces having arbitrary numbers of vertices, the same approach can be applied, but there is no longer uniqueness. Suppose $f \in F$ is a face with $k \geq 3$ vertices. The integral $\mathbf{I}_f$ is again well-defined, and can be computed as the sum of k terms, generalizing (28). However, there is no unique choice of the local weights $w_{\mathbf{v},f}$ in (26) for $k > 3$, since there are k of these. Langer et al. [17] proposed using a certain type of spherical polygonal MV coordinates to determine the $w_{\mathbf{v},f}$, but other choices are possible.

8 Final remarks

We have not covered here other kinds of generalized barycentric coordinates, and related coordinates, which include Sibson's natural neighbour coordinates [24], Sukumar's maximum entropy coordinates [25], Gordon and Wixom coordinates [12], spherical barycentric coordinates [17], harmonic coordinates [14], Green coordinates [21], Poisson coordinates [18], and others. A more general survey paper is being planned in which some of these other coordinates will be included.

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