Algebraic geometry for geometric modeling

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SIAM AG17
Atlanta, Georgia, USA
August 1, 2017

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Applied algebraic geometry in the old days:

EU Training networks

- GAIA – Application of approximate algebraic geometry in industrial computer aided geometry
- GAIA II – Intersection algorithms for geometry based IT-applications using approximate algebraic methods
- SAGA – ShApes, Geometry, and Algebra

and Centre of Mathematics for Applications (University of Oslo)

Algebraic geometry for geometric modeling?

Ron Goldman: "The main contribution of algebraic geometry to geometric modeling is *insight*, not computation."

His concern is that geometric modeling is interested in *metric* properties of curves and surfaces, whereas algebraic geometry is concerned with *affine and projective invariants*.

Algebraic geometry can provide *constructive tools* for computation. Geometric modeling needs *effective numerical methods*.

Classical projective geometry

Classical geometry was real and Euclidean. To get solutions to polynomial equations, imaginary numbers were introduced, and projective geometry in order to have stability of intersections.

Complex projective geometry is much easier to work with than affine real geometry.

For computers: the field $\mathbb{Q}$, for pictures the field $\mathbb{R}$, for proving theorems the field $\mathbb{C}$.

Projective invariants

Let $X \subset \mathbb{P}^n$ be a nonsingular projective algebraic variety.
The *Chern classes* of X are *topological* invariants.
The *Euler–Poincaré characteristic* $\chi(\mathcal{O}_X)$ is a *birational* invariant.

Example

If X is a surface, with Chern classes $c_1(X)$ and $c_2(X)$, then Noether's formula holds:

$$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

If $X' \to X$ is the blowing up of a point, then
$c_1(X')^2 = c_1(X)^2 - 1$ and $c_2(X') = c_2(X) + 1$, so
$\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X)$.

Polar varieties and Chern classes

Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension $m - k + 2$, where $m = \dim X$.

The *kth polar variety* of of $X \subset \mathbb{P}^n$ with respect to L is

$$M_k(L) := \{P \in X \mid \dim(T_P X \cap L) \geq k - 1\}.$$

Its class is $[M_k] = c_k(\mathcal{P}^1_X(1)) \cap [X]$, hence we get the Todd–Eger relation

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} c_i(X) \cap [X], \tag{1}$$

where $c_i(X)$ are the Chern classes of the tangent bundle of X and $h = c_1(\mathcal{O}_X(1))$ is the class of a hyperplane.

Singular varieties

Conversely, the Chern classes are expressed in terms of the polar varieties:

$$c_k(X) \cap [X] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap [M_i].$$

This works also for *singular* varieties: replace M_k by the closure of $M_k|_{X_{ns}}$ to get the *Chern–Mather classes* $c_k^M(X)$.

Let $\pi : X \to X' \subset \mathbb{P}^{n'}$ be a (suitably generic) linear projection:

$$c_k^M(X') = \pi_*(c_k^M(X)).$$

The Chern–Mather classes and the polar classes are *projective* invariants (the Chern–Schwartz–MacPherson classes are *topological* invariants).

Geometric interpretation: the contour curve

Let $X = Z(F) \subset \mathbb{P}^3$ be a surface, $P = (p_0 : \cdots : p_3) \in \mathbb{P}^3$.
The first polar variety of X wrt P is the intersection of X with its first polar:

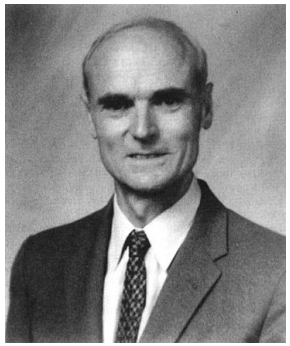
$$M_1(P) = X \cap Z(\sum p_i F_i) = Z(F, \sum p_i F_i),$$

and $M_1(P) \subset X$ is the *contour curve* on X under the projection $X \to \mathbb{P}^2$ from P.

If X is smooth, of degree d, and P is general, then $\deg M_1 = d(d-1)$, and the projection of M_1 has $d(d-1)(d-2)$ cusps.

cf. MS Algebraic Vision yesterday!

Affine space as Euclidean space



Coxeter: "Kepler's invention of points at infinity made it possible to regard the projective plane as the affine plane plus the line at infinity. A converse relationship was suggested by Poncelet (1822) and von Staudt (1847): regard the affine plane as the projective plane minus an arbitrary line ℓ, and then regard the Euclidean plane as the affine plane with a special rule for associating pairs of points on ℓ (in "perpendicular directions")."

Affine space + notion of perpendicularity = "Euclidean space"

Euclidean normal bundle

Let $X \subset \mathbb{P}^n$ be a variety of dimension m. Define the *Euclidean normal bundle* $\mathcal{E}$ with respect to a non-degenerate quadric Q in the hyperplane H_∞.

Use the polarity in $H_\infty \cong \mathbb{P}^{n-1}$ induced by Q to define *Euclidean normal spaces* at each smooth point $P \in X \setminus H_\infty$:

$$N_P X = \langle P, (T_P X \cap H_\infty)^\perp \rangle$$

Then $\mathcal{E} = \mathcal{N}_X(1)^\vee \oplus \mathcal{O}_X(1)$, where $\mathcal{N}_X$ is the conormal bundle (or the "Mather–Nash conormal", if the variety is singular).

Reciprocal polar varieties

Define *reciprocal* polar varieties

$$M_k(L)^\perp := \{P \in X | N_P X \cap L \neq \emptyset\},$$

for L linear space of codimension $n - m + k$. Then

$$[M_k^\perp] = s_k(\mathcal{E}) \cap [X] = \sum_{j=0}^{k} c_j(\mathcal{P}_X^1(1)) c_1(\mathcal{O}_X(1))^{k-j} \cap [X].$$

In particular,

$$\deg[M_m^\perp] = \sum_{j=0}^{m} \deg[M_j].$$

This is the degree of the *end point map* and also the *Euclidean distance degree*.

MS Euclidean distance degree – Thursday!

Building blocks for modeling

Making pots before the wheel was invented?

Piecewise linear

Natural quadrics

Parameterized surfaces

Most algebraic surfaces used in geometric modeling are *rational*, meaning that they can be obtained as the image of a map

$$(s,t) \mapsto (p_1(s,t), p_2(s,t), p_3(s,t)) \in \mathbb{R}^3$$

for (s,t) in some plane domain $D \subset \mathbb{R}^2$, where the p_i are polynomials or rational functions with coefficients in $\mathbb{Q}$.

Typically D is either the *triangle* with vertices $(0,0), (1,0), (0,1)$ or the *square* with vertices $(0,0), (1,0), (0,1), (1,1)$.

Projective "triangular" surfaces

Let $M_d = \{m_0, \ldots, m_N\}$ be the set of all monomials of degree d in three variables.

The image of the map $\mathbb{P}^2 \to \mathbb{P}^N$ given by

$$(s : t : u) \mapsto (m_0(s,t,u) : \cdots : m_N(s,t,u))$$

is the *Veronese surface* of degree d^2.

Any triangular parameterized surface in $\mathbb{P}^3$ is a linear projection of this surface from a center $L \subset \mathbb{P}^N$. Algebraic geometry gives *insight* into the properties of such projections.

The Steiner surface ($d = 2$)

There are 6 different *real* Steiner surfaces.

A. $x^2y^2 + y^2z^2 + z^2x^2 - xyz = 0$

B. $x^2y^2 - y^2z^2 + z^2x^2 - xyz = 0$

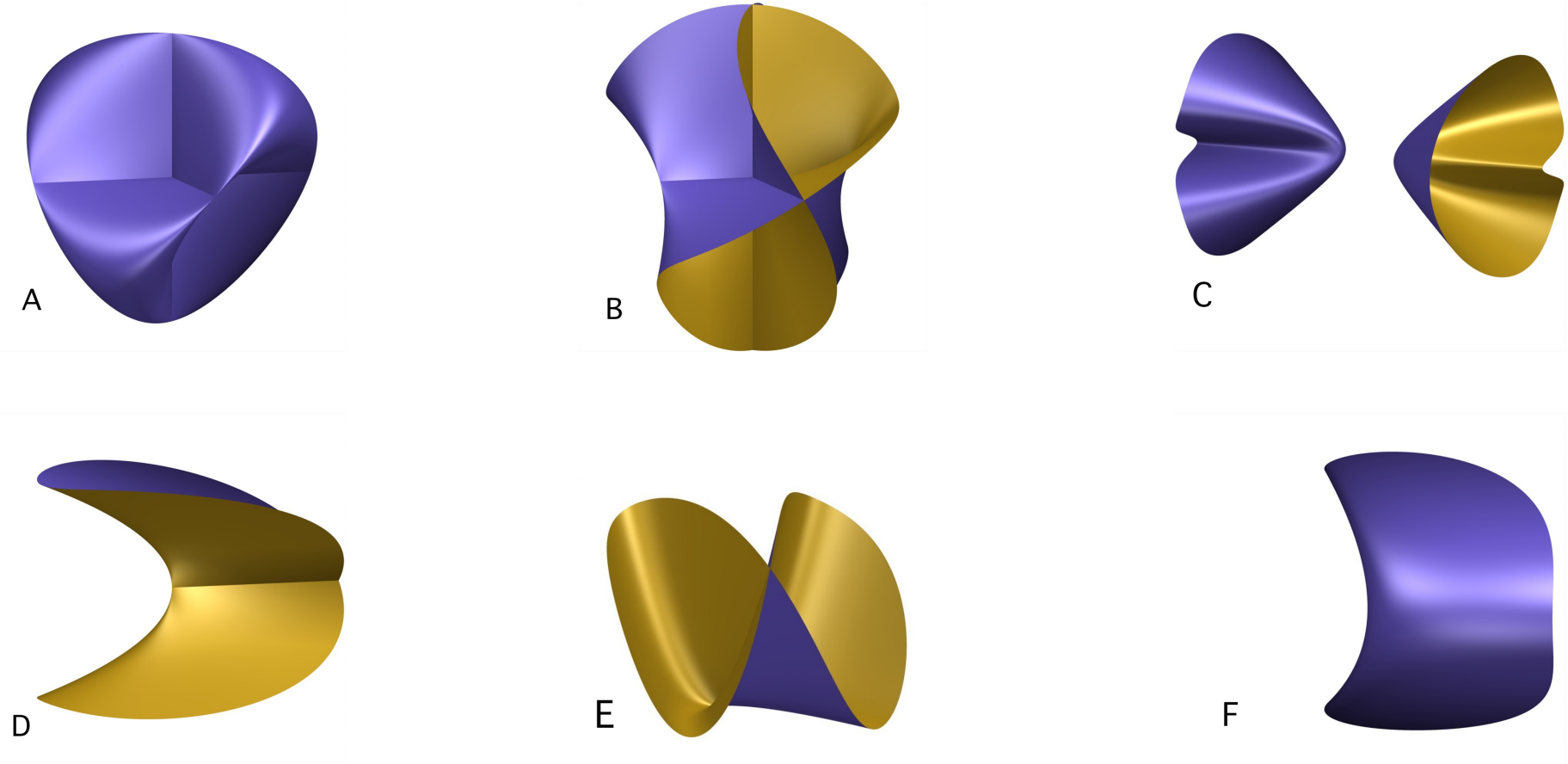
C. $xyz^2 + xy - x^2 - z^4 - 2z^2 - 1 = 0$

D. $xz^2 - y^2 + z^4 = 0$

E. $x^4 + y^2 + z^2 - 2x^2y - 2x^2z + 2yz - 4yz = 0$

F. $y^2 + 2yz^2 + z^4 - x = 0$

A: Steiner's Roman surface. Three real double lines meeting in a triple point. Each line has two real pinchpoints ($d = 4$, $\epsilon = 3$, $t = 1$, $\nu_2 = 6$).
B: Three real double lines meeting in a triple point. One line has two real pinch points ($d = 4$, $\epsilon = 3$, $t = 1$, $\nu_2 = 2$).
C: One real double line. The line has one real pinch point ($d = 4$, $\epsilon = 1$, $\nu_2 = 1$).
D: One simple and one double double lines meeting in a triple point. The simple line has two real pinch points ($d = 4$, $\epsilon = 2$, $t = 1$, $\nu_2 = 2$).
E: Somewhat similar to D.
F: One threefold double line containing a triple point.


A
B
C
D
E
F

Projective "tensor" surfaces

Let $M_{a,b} = \{m_0, \ldots, m_{N'}\}$ be the set of all monomials in four variables, of bidegree (a, b). Then the image of the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{N'}$ given by

$$(s : t) \times (u : v) \mapsto (m_0(s : t; u : v) : \cdots : m_N(s : t; u : v))$$

is the *Segre surface* of degree $2ab$.

Any "tensor" surface in $\mathbb{P}^3$ is the linear projection of a Segre surface from some center L.

For computability, a and b should be small!

Example: $a = 1$, $b = 2$

A tensor surface of bidegree $(1, 2)$ in 3-space is the projection of the rational balanced normal scroll $X \subset \mathbb{P}^5$ of type $(2, 2)$ and degree 4 from a line L.

A classification is obtained by the position of L with respect to X, its tangent spaces, its osculating spaces, etc.

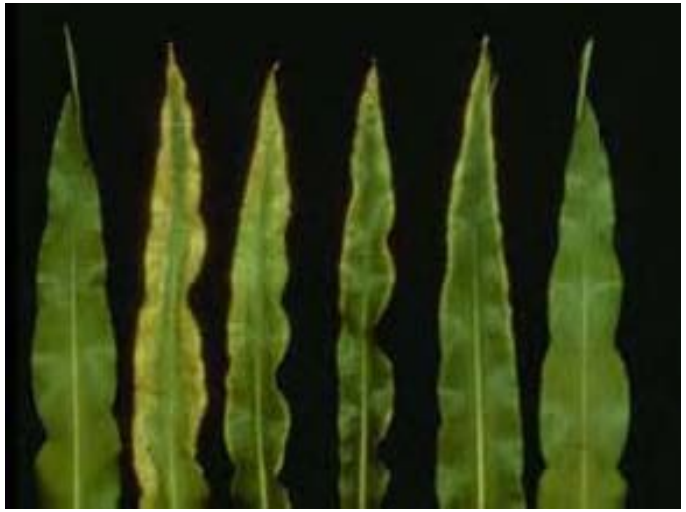
A general projection has a twisted cubic as its double curve, with 4 pinch points and no triple points.

Application: modeling corn leaves (Thi Ha Lê)

Use Bézier surface tensor patches $[0,1]^2 \to \mathbb{R}^3$:

$$B(t_1,t_2) = \frac{\sum_{(i,j)} w_{(i,j)} p_{(i,j)} F_{(i,j)}(t_1,t_2)}{\sum_{(i,j)} w_{(i,j)} F_{(i,j)}(t_1,t_2)},$$

where $F_{(i,j)}(t_1,t_2) = \binom{2}{i} t_1^i (2-t_1)^{2-i} \binom{1}{j} t_2^j (1-t_2)^{1-j}$.



General and special projections – singularities

Generic projections of the d-uple Veronese surface has

- a double curve of degree $\epsilon = \binom{d}{2}(d^2 + d - 3)$
- $\nu_2 = 6(d-1)^2$ pinch points
- $t = \frac{1}{6}(d^6 + 9d^3 + 44d^2) - 2d^4 - 12d + 5$ triple points

Generic projections of the bidegree (a, b) Segre surface has

- a double curve of degree $\epsilon = 2ab(ab - 2) + a + b$
- $\nu_2 = 4(3ab - 2(a + b) + 1)$ pinch points
- $t = \frac{1}{3}4ab(a^2b^2 + 11) - 8a^2b^2 + 2ab(a + b) - 8(a + b) + 4$ triple points

Insight: these numbers are upper bounds in the real case.

Toric surfaces

Let $\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^2$ be a lattice point configuration, $\mathbb{K}^{*2} \to \mathbb{P}^N_{\mathbb{K}}$ the corresponding toric embedding, and $X_{\mathcal{A}}$ the closure of the image.

Let $\mathcal{A}'$ be a lattice point configuration obtained from $\mathcal{A}$ by removing $N-3$ points.Then the toric embedding $X_{\mathcal{A}'} \subset \mathbb{P}^3$ is the (toric) linear projection of $X_{\mathcal{A}}$ with center equal to the linear span of the "removed points".

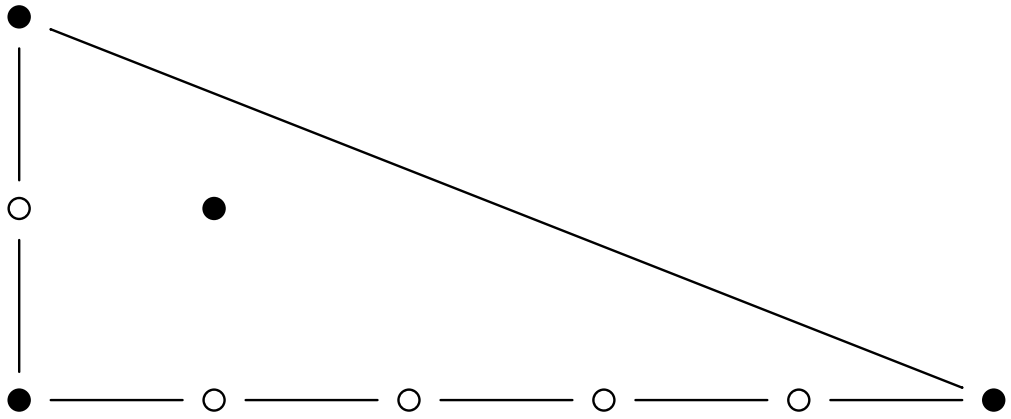
Any toric surface in 3-space is a (special) linear projection of a Veronese or Segre surface.

Example

Take $m, n \in \mathbb{N}$, with $\gcd(m,n) = 1$, $m \geq 3$, $m \geq n \geq 2$,and $\mathcal{A} = \{(0,0),(m,0),(1,1),(0,n)\}$. Then $(u,v) \mapsto (1 : u^m : uv : v^n)$ gives the surface

$$X_{\mathcal{A}} = Z(x^n z^m w^{mn-m-n} - y^{mn}) \subset \mathbb{P}^3.$$

The surface has three singular lines: $Z(x,y), Z(y,z), Z(y,w)$.



Real picture for $m = 5$, $n = 2$

In the affine space $z = 1$:

In the affine space $x = 1$:

Toric patches (R. Krasauskas)

A *toric surface patch* associated with a lattice polygon $\Delta \subset \mathbb{R}^2$ is a piece of an algebraic surface parameterized by the rational map $B_\Delta : \Delta \to \mathbb{R}^3$ given by

$$B_\Delta(t_1, t_2) = \frac{\sum_{a \in \widehat{\Delta}} w_a p_a F_a(t_1, t_2)}{\sum_{a \in \widehat{\Delta}} w_a F_a(t_1, t_2)},$$

where $\widehat{\Delta} = \Delta \cap \mathbb{Z}^2$, $p_a \in \mathbb{R}^3$ are control points, $w_a > 0$ are weights, and $F_a(t_1, t_2) = c_a h_1(t_1, t_2)^{h_1(a)} \cdots h_r(t_1, t_2)^{h_r(a)}$, where h_i are linear forms defining the r edges of Δ.

Monoid surfaces

A monoid surface in $\mathbb{P}^3$ of degree d is a surface with a singular point of multiplicity $d-1$.

Let $Z(f_{d-1}), Z(f_d) \subset \mathbb{P}^2$ be plane curves, with no common components and no common singular points.

The *monoid surface* $Z(f_{d-1}x_0 + f_d) \subset \mathbb{P}^3$ has a rational parameterization $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$

$$(a_1 : a_1 : a_3) \mapsto (f_d(a) : f_{d-1}(a)a_1 : f_{d-1}(a)a_2 : f_{d-1}(a)a_3).$$

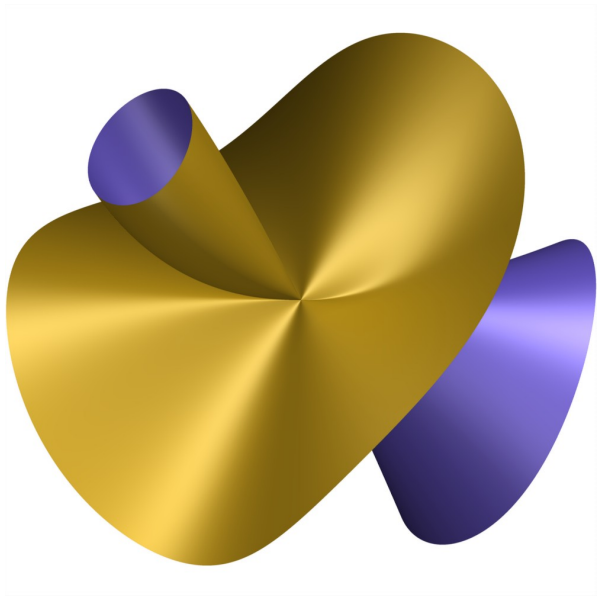
The point $O := (1:0:0:0)$ has multiplicity $d-1$. All other singular points are contained in lines on the surface through O, corresponding to the base points $Z(f_{d-1}) \cap Z(f_d)$ of the parameterization.

Applications

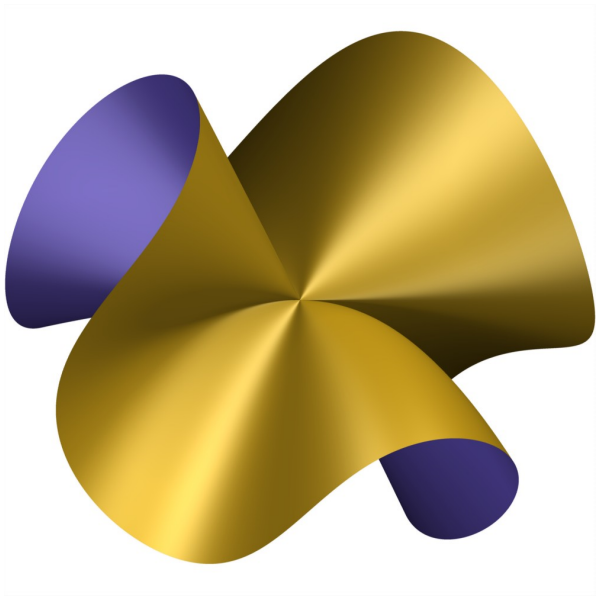
Polynomials defining a monoid of a given degree are "sparse" compared to all polynomials of that degree. Therefore they have been used in approximate implicitization problems.

Real quartic monoid surfaces

Classify by the form of the projective tangent cone $Z(f_3)$ at O. There are nine cases, giving nine strata in the space of all quartic monoids, which have been described.



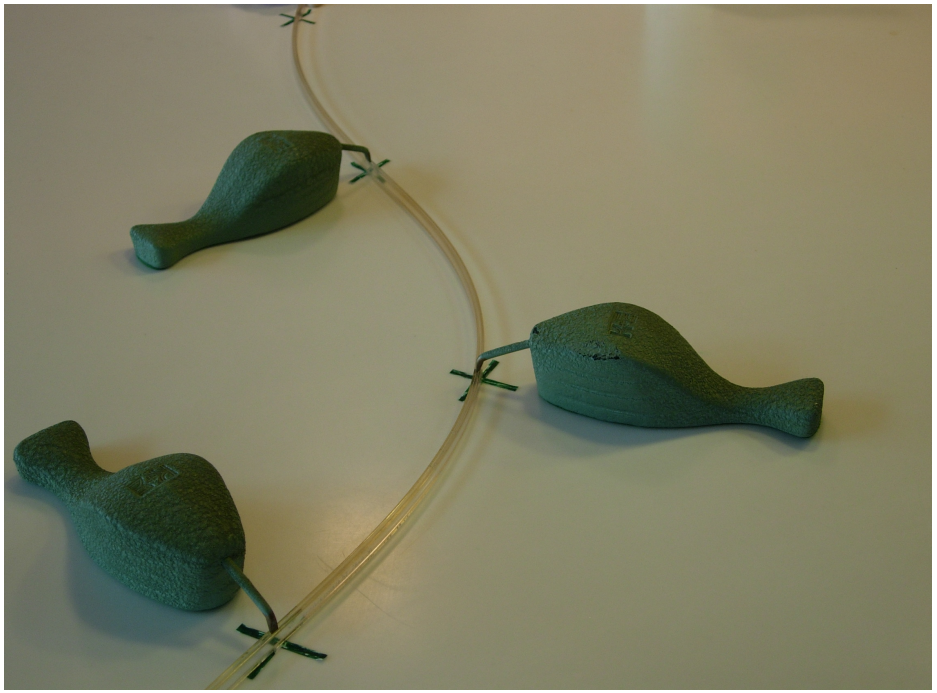
$Z(x^3 + y^3 + 5xyz - z^3(x+y))$



$Z(x^3 + y^3 + 5xyz - z^3(x-y))$

Algebraic splines

A draftsman's spline:



Algebraic spline rings

Let $\Delta \subset \mathbb{R}^d$ be a (pure) d-dimensional simplicial complex.

Let $C^r(\Delta)$ denote the set of piecewise polynomial functions (algebraic splines) on Δ of smoothness r.

$C^r(\Delta)$ is a ring under the usual pointwise addition and multiplication.

The (global) polynomial functions $\mathbb{R}[x_1, \ldots, x_d]$ are r-smooth for any r, so

$$C^\infty(\Delta) := \mathbb{R}[x_1, \ldots, x_d] \subset C^r(\Delta),$$

for any r.

The vector spaces $C_k^r(\Delta)$

Let $C_k^r(\Delta) \subset C^r(\Delta)$ consist of splines of degree $\leq k$.

These subsets are vector spaces over $\mathbb{R}$.

Standard problems, that many people have worked on, are

- to determine (upper and lower bounds for) $\dim_{\mathbb{R}} C_k^r(\Delta)$,
- to construct a basis for $C_k^r(\Delta)$.

But also: to determine the structure of the rings $C^r(\Delta)$ and their geometric interpretation.

Classical Stanley–Reisner rings

Let $\Delta \subset \mathbb{R}^d$ be a simplicial complex, with vertices $\{v_1, \ldots, v_n\}$.

The *Stanley–Reisner ring*, or face ring, of Δ is the ring

$$A_\Delta := \mathbb{R}[Y_1, \ldots, Y_n]/I_\Delta,$$

where I_Δ is the monomial ideal generated by the products $Y_{i_1} \cdots Y_{i_j}$ such that $\{v_{i_1}, \ldots, v_{i_j}\}$ is not a face of Δ.

It is known [Bruns–Gubeladze] that if two simplicial complexes have isomorphic Stanley–Reisner rings, then they are themselves isomorphic.

Identify Y_i with the *Courant function* $Y_i(v_j) = \delta_{ij}$ extended by linearity. Then

$$\textstyle\sum_i Y_i = 1,$$

and

$$A_\Delta/(\textstyle\sum_i Y_i - 1) = C^0(\Delta)$$

is the ring of (continuous) splines on Δ.

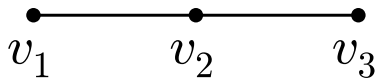
Call this ring the *affine* Stanley–Reisner ring of Δ. Then

$$\mathrm{Spec}(C^0(\Delta)) = \mathrm{Spec}(A_\Delta) \cap Z(\textstyle\sum_i Y_i - 1) \subset \mathbb{A}^n_{\mathbb{R}} = \mathbb{R}^n,$$

and the points that have non-negative coordinates, give a *model* of Δ.

Example 1: $d = 1$

Let Δ be a one-dimensional simplicial complex with three vertices $v_1, v_2, v_3 \in \mathbb{R}$, and assume $v_1 < v_2 < v_3$.



We have

$$C^0(\Delta) = A_\Delta/(\textstyle\sum_{i=1}^3 Y_i - 1) = \mathbb{R}[Y_1, Y_2, Y_3]/(Y_1Y_3, Y_1+Y_2+Y_3-1)$$

$$\mathrm{Spec}(C^0(\Delta)) = Z(Y_1, Y_2 + Y_3 - 1) \cup Z(Y_3, Y_1 + Y_2 - 1) \subset \mathbb{R}^3$$

The segments of these two lines contained in the positive octant mimic the two 1-faces of Δ, and they intersect transversally.

Trivial splines

Let $v_1, \ldots, v_n$ be the vertices of $\Delta \subset \mathbb{R}^d$.

Write $v_i = (v_{i1}, \ldots, v_{id}) \in \mathbb{R}^d$.

Set

$$H_j := v_{1j} Y_1 + \ldots + v_{nj} Y_n.$$

Then H_j is equal to the jth coordinate function on $\Delta \subset \mathbb{R}^d$.

So

$$\mathbb{R}[H_1, \ldots, H_d] \subset C^r(\Delta)$$

is the subring of *trivial splines*.

Example 1: $d = 1$

Now

$$H := v_1 Y_1 + v_2 Y_2 + v_3 Y_3$$

is the trivial spline: $H(x) = x$ for any point $x \in |\Delta|$,

and $\mathbb{R}[H] \subseteq C^r(\Delta)$, for any $r \geq 0$.

Computations show that that H and Y_1^{r+1} generate the subring $C^r(\Delta)$ of $C^0(\Delta)$.

Define

$$\varphi : \mathbb{R}[y_1, y_2, y_3] \to \mathbb{R}[Y_1, Y_2, Y_3]/(Y_1 Y_3, Y_1 + Y_2 + Y_3 - 1)$$

$\varphi(y_1) = (v_1 - v_2)^{r+1} Y_1^{r+1}$,
$\varphi(y_2) = 1 - (H - v_2) = 1 - (v_1 - v_2) Y_1 - (v_3 - v_2) Y_3$,
$\varphi(y_3) = (v_3 - v_2)^{r+1} Y_3^{r+1}$.

Then

$$\varphi(\mathbb{R}[y_1, y_2, y_3]) = C^r(\Delta)$$

and

$$\operatorname{Ker}\varphi = (y_1 y_3, y_1 + y_3 - (1 - y_2)^{r+1}).$$

Hence

$$C^r(\Delta) \cong \mathbb{R}[y_1, y_2, y_3]/(y_1 y_3, y_1 + y_3 - (1 - y_2)^{r+1}),$$

and

$$\operatorname{Spec}(C^r(\Delta)) = Z(y_1, y_3 - (1 - y_2)^{r+1}) \cup Z(y_3, y_1 - (1 - y_2)^{r+1}).$$

For $r \geq 1$, both curves have the y_2-axis as tangent at their point of intersection (the origin), and the tangent intersects each curve with multiplicity $r + 1$.

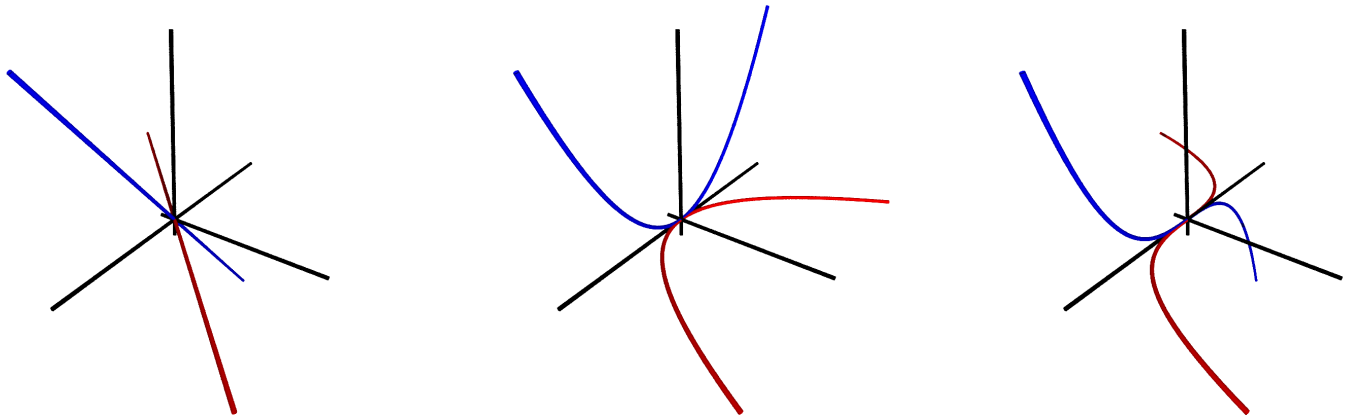
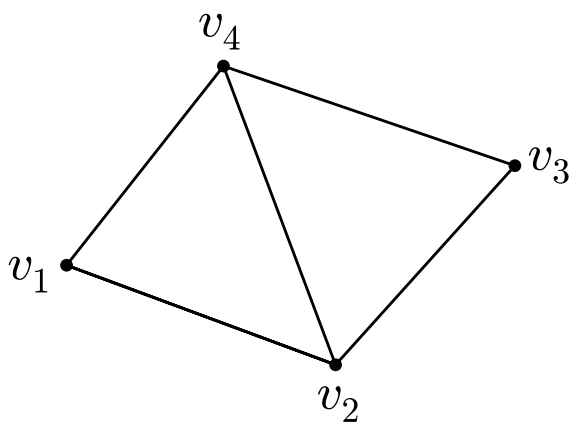


Figure: $\mathrm{Spec}(C^r(\Delta))$ for $r = 0, 1, 2$ in Example 1.

Example 2: $d = 2$

Consider a two-dimensional simplicial complex Δ, with four vertices $v_1, \ldots, v_4 \in \mathbb{R}^2$ (no three on a line), $v_i = (v_{i1}, v_{i2})$, and $\{v_1, v_3\}$ as the only non-face.



Then

$$C^0(\Delta) = \mathbb{R}[Y_1, Y_2, Y_3, Y_4]/(Y_1Y_3, Y_1 + Y_2 + Y_3 + Y_4 - 1),$$

where the Y_i are the Courant functions and

$$H_j := v_{1j}Y_1 + v_{2j}Y_2 + v_{3j}Y_3 + v_{4j}Y_4, \text{ for } j = 1, 2$$

are the trivial splines.

Observe that $Y_1^{r+1}, Y_3^{r+1} \in C^r(\Delta)$.

We can deduce the following linear relation

$$\frac{H_1 - v_{21}}{v_{41} - v_{21}} - \frac{H_2 - v_{22}}{v_{42} - v_{22}} =$$
$$\left(\frac{v_{11} - v_{21}}{v_{41} - v_{21}} - \frac{v_{12} - v_{22}}{v_{42} - v_{22}}\right)Y_1 + \left(\frac{v_{31} - v_{21}}{v_{41} - v_{21}} - \frac{v_{32} - v_{22}}{v_{42} - v_{22}}\right)Y_3.$$

$$C^r(\Delta) = \varphi_r(\mathbb{R}[y_1, y_2, y_3, y_4]),$$

where

$$\varphi_r : \mathbb{R}[y_1, y_2, y_3, y_4] \to \mathbb{R}[Y_1, Y_2, Y_3, Y_4]/(Y_1 Y_3, \textstyle\sum_i Y_i - 1),$$

$$\varphi_r(y_1) = \left(\tfrac{v_{11}-v_{21}}{v_{41}-v_{21}} - \tfrac{v_{12}-v_{22}}{v_{42}-v_{22}}\right)^{r+1} Y_1^{r+1}$$

$$\varphi_r(y_2) = \tfrac{H_1-v_{21}}{v_{41}-v_{21}}$$

$$\varphi_r(y_3) = \left(\tfrac{v_{31}-v_{21}}{v_{41}-v_{21}} - \tfrac{v_{32}-v_{22}}{v_{42}-v_{22}}\right)^{r+1} Y_3^{r+1}$$

$$\varphi_r(y_4) = \tfrac{H_2-v_{22}}{v_{42}-v_{22}}$$

$\operatorname{Ker}\varphi_r = (y_1y_3, y_1 + y_3 - (y_2 - y_4)^{r+1})$, so

$$C^r(\Delta) \cong \mathbb{R}[y_1, y_2, y_3, y_4]/(y_1y_3, y_1 + y_3 - (y_2 - y_4)^{r+1}).$$

Hence

$$\operatorname{Spec}(C^r(\Delta)) = \\ Z(y_1, y_3 - (y_2 - y_4)^{r+1}) \cup Z(y_3, y_1 - (y_2 - y_4)^{r+1}).$$

The intersection of these surfaces is the line $Z(y_1, y_3, y_2 - y_4)$. The plane $Z(y_1, y_3)$ is the tangent plane to both surfaces at all points of their line of intersection. The intersection of this tangent plane and each surface is the line, with multiplicity $r + 1$.

The local spline ring conjecture

Let Δ be a (general) d-dimensional simplicial complex consisting of two d-simplices intersecting in a $(d-1)$-simplex. Then we can realize $\mathrm{Spec}(C^r(\Delta)) \subset \mathbb{R}^{d+2}$ as the union of two smooth d-dimensional varieties V_1 and V_2 intersecting along a linear $(d-1)$-dimensional space L, such that V_1 and V_2 have the same d-dimensional linear space T as tangent space at each point of L and such that V_i and T have order of contact $r+1$ at each point of L.

For a proof, and generalizations:

Nelly Villamizar's talk tomorrow

MS Multivariate Splines and Algebraic Geometry

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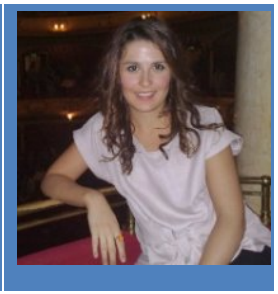
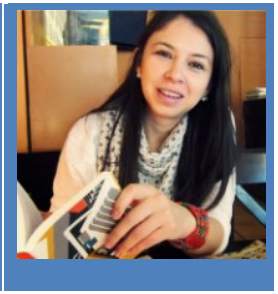
Thanks to my CMA, GAIA, and SAGA students!

			
Cathrine Tegnander	Wenche Oxholm	Margrethe Naalsund	Dana Skrbo
Heidi Mork	Pål Hermunn Johansen	Georg Muntingh	Nelly Villamizar
			

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