Discriminants, polytopes and toric geometry

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Resultants

Eliminate, eliminate, eliminate
Eliminate the eliminators of elimination theory.

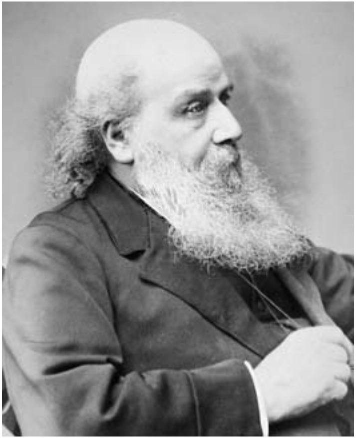
S. S. Abhyankar (1970)

Question: For which $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ do

$$f(x) = a_m x^m + \cdots + a_0 \text{ and } g(x) = b_n x^n + \cdots + b_0$$

have a common root?

James Joseph Sylvester (1814–1897)



The *Sylvester matrix* is the $(m+n)\times(m+n)$-matrix

$$\begin{pmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & \cdots \\ 0 & a_m & a_{m-1} & a_{m-2} & \cdots \\ \vdots & & & \vdots & \\ b_n & b_{n-1} & b_{n-2} & \cdots & \cdots \\ 0 & b_n & b_{n-1} & b_{n-2} & \cdots \\ \vdots & & & \vdots & \end{pmatrix}$$

The *resultant* is the determinant of this matrix.

A student of Sylvester: Florence Nightingale (1820-1910)

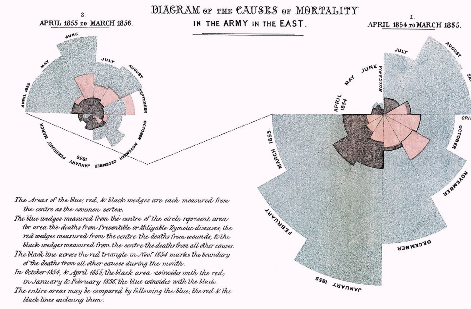
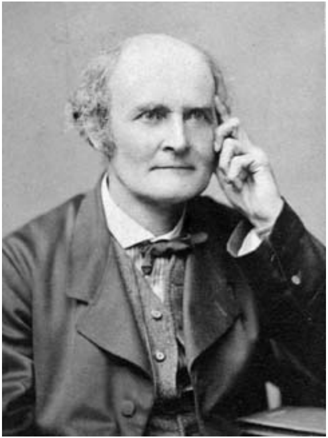


Figure: Diagram of the Causes of Mortality in the Army in the East

Arthur Cayley (1821–1895)



Set

$$h(x, y) := f(x) + yg(x).$$

If α is a common root of f and g, then

$$(\alpha, -\frac{f_x(\alpha)}{g_x(\alpha)})$$

is a common zero of h, h_x, h_y.

The Cayley trick

Consider

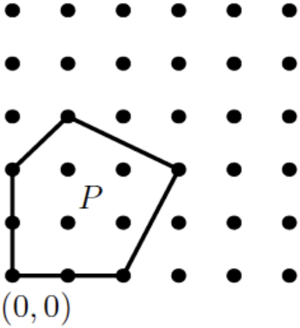
$$h(x_1, \dots, x_k, y_1, \dots, y_k) := \\ f_0(x_1, \dots, x_k) + y_1 f_1(x_1, \dots, x_k) + y_k f_k(x_1, \dots, x_k).$$

The *discriminant* $\Delta(h)$ of h is obtained by eliminating the x_i's and y_i's from the $2k+1$ equations

$$h = 0, \partial h / \partial x_i = 0, \partial h / \partial y_i = f_i = 0.$$

Hence $\Delta(h) \sim \mathrm{Res}(f_0, \dots, f_k)$.

Convex lattice polytopes



Cayley polytopes

Let $P_0, \ldots, P_k \subset \mathbb{R}^{n-k}$ be convex lattice polytopes, and $e_0, \ldots, e_k$ are the vertices of $\Delta_k \subset \mathbb{R}^k$.

The polytope

$$P = \mathrm{Conv}\{e_0 \times P_0, \ldots, e_k \times P_k\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n,$$

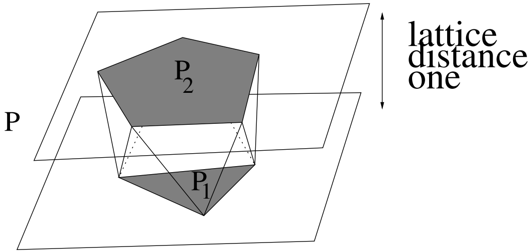
is called a *Cayley polytope*.

We write

$$P = P_0 \star \cdots \star P_k$$

A Cayley polytope is "hollow": it has no interior lattice points.

An example



The codegree and degree of a polytope

$$\operatorname{codeg}(P) = \min\{m \,|\, mP \text{ has interior lattice points}\}.$$

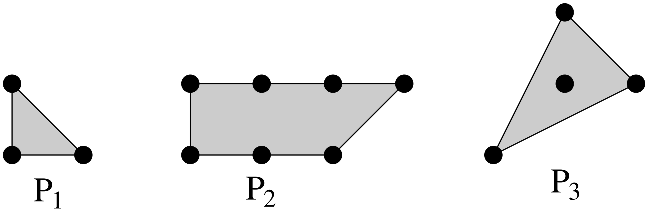
$$\deg(P) = n + 1 - \operatorname{codeg}(P)$$

Example (1)

$$\operatorname{codeg}(\Delta_n) = n + 1 \text{ and } \operatorname{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil.$$

Example (2)

$$P = P_0 \star \cdots \star P_k \text{ implies } \operatorname{codeg}(P) \geq k + 1.$$



$$\operatorname{codeg}(P_1) = 3 \qquad \operatorname{codeg}(P_2) = 2 \qquad \operatorname{codeg}(P_3) = 1$$

The Cayley polytope conjecture

Question (Batyrev–Nill): Is there an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Answer (Haase–Nill–Payne): Yes, and $N(d) \leq (d^2 + 19d - 4)/2$

Question: Is $N(d)$ linear in d?

Answer (Dickenstein–Di Rocco–P.): Yes, $N(d) = 2d + 1$ (if P is smooth and $\mathbb{Q}$-normal).

Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.

Theorem (Dickenstein, Di Rocco, P., Nill)

Let P be a smooth lattice polytope of dimension n. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = P_0 \star \cdots \star P_k$ *is a smooth Cayley polytope with* $k+1 = \operatorname{codeg}(P)$ *and* $k > \frac{n}{2}$.

(3) P *is defective, with defect* $2k - n > 0$.

The proof is essentially algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

Lattice polytopes and toric embeddings

The polytope P_0:

$\bullet$ —— $\bullet$ —— $\bullet$

corresponds to the toric embedding $\mathbb{C}^* \to \mathbb{P}^2$ given by $x \mapsto (1 : x : x^2)$; its closure X_{P_0} is a conic.

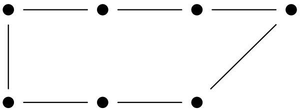
The polytope P_1:

$\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$

corresponds to the toric embedding $\mathbb{C}^* \to \mathbb{P}^3$ given by $x \mapsto (1 : x : x^2 : x^3)$; its closure X_{P_1} is a twisted cubic curve.

The Cayley sum

The polytope $P = P_0 \star P_1$:



corresponds to the embedding

$$(\mathbb{C}^*)^2 \to \mathbb{P}^6$$

given by

$$(x, y) \mapsto (1 : x : x^2 : y : xy : x^2y : x^3y);$$

its closure X_P is a rational normal scroll of type $(2, 3)$.

Hyperplane sections and discriminants

$P = P_0 \star \cdots \star P_k$ gives $X_P \subseteq \mathbb{P}^N$.

A hyperplane section of X_P:

$$h(x_1, \ldots, x_{n-k}, y_1, \ldots, y_k) := f_0 + y_1 f_1 + \cdots + y_k f_k = 0,$$

($f_i = 0$ is a hyperplane section of X_{P_i}) is singular if
$h = \partial h / \partial x_i = \partial h / \partial y_j = 0$.

Generalize the Cayley trick:

$$\mathrm{Res}(f_0(x_1, \ldots, x_{n-k}), \ldots, f_k(x_1, \ldots, x_{n-k})) \sim \Delta(h).$$

Discriminants and dual varieties

If $k \leq n-k$, then $\Delta(h)$ is a polynomial in the coefficients of h, and defines a hypersurface: the *dual variety* $X_P^\vee \subseteq (\mathbb{P}^N)^\vee$ of X_P.

If $k > n-k$, the system $f_0 = \cdots = f_k = 0$ has more equations than variables. Hence "the discriminant" of h is of more than one polynomial, and the dual variety is not a hypersurface.

A variety X is called *defective* if its dual variety $X^\vee$ is not a hypersurface. A polytope P is defective if X_P is defective.

The *defect* of a defective variety X is the positive integer $\operatorname{codim} X^\vee - 1$.

We conclude: If $k > n-k$, then the Cayley polytope $P = P_0 \star \cdots \star P_k$ is defective.

The degree of the dual variety

Gelfand–Kapranov–Zelevinski: If X_P is smooth,

$$\deg X_P^\vee = \sum_{F \subseteq P} (-1)^{\operatorname{codim} F} (\dim F + 1) \mathrm{Vol}_{\mathbb{Z}}(F).$$

The degree of the dual variety X_P is a polynomial in the Chern classes of X_P and the line bundle L_P giving the toric embedding.

The Chern classes of X_P and L_P can be expressed combinatorially. For example

$c_n(T_{X_P}) = \#$ vertices of P
$c_1(L_P)^n = \mathrm{Vol}_{\mathbb{Z}}(P)$
$c_i(T_{X_P}) c_1(L_P)^{n-i} = \sum_{\operatorname{codim} F_i = i} \mathrm{Vol}_{\mathbb{Z}}(F_i)$.

kth order dual varieties

$$
\begin{aligned}
X^{(k)} &= \overline{\{H \in \mathbb{P}^{m\vee} \mid H \text{ is tangent to } X \text{ to the order } k\}} \\
&= \overline{\{H \in \mathbb{P}^{m\vee} \mid H \supseteq \mathbb{T}^k_{X,x} \text{ for some } x \in X_{\text{smooth}}\}}.
\end{aligned}
$$

$\mathbb{T}^k_{X,x} = k$th osculating space to X at x.

$\dim \mathbb{T}^k_{X,x} \leq \binom{n+k}{k} - 1$, $n = \dim X$.

$X^{(1)} = X^\vee$ and $X^{(k-1)} \supseteq X^{(k)}$

Expected dimension of $X^{(k)} = n + m - \binom{n+k}{k}$.

X is *k-defective* if $\dim X^{(k)} < n + m - \binom{n+k}{k}$.

Toric threefolds

Theorem (Dickenstein–Di Rocco–P.)

$(X,P) = (X_P, L_P)$ *smooth, 2-regular toric threefold embedding is* 2*-defective if and only if* $(X,L) = (\mathbb{P}^3, \mathcal{O}(2))$. *Moreover:*

(1) $\deg X^{(2)} = 120$ *if* $(X,L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$

(2) $\deg X^{(2)} = 6(8(a+b+c)-7)$ *if* $(X,L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi)$, *where* ξ *denotes the tautological line bundle,*

(3) *In all other cases,* $\deg X^{(2)} = 62V - 57F + 28E - 8v + 58V_1 + 51F_1 + 20E_1$, *where* V, F, E *(resp.* V_1, F_1, E_1*) denote the (lattice) volume, area of facets, length of edges of* P *(resp. the adjoint polytope* $\mathrm{Conv}(\mathrm{int} P)$*), and* $v = \#\{$*vertices of* $P\}$.

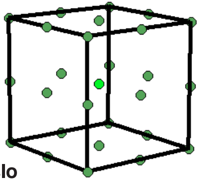
Example

If P is a cube with edge lengths 2, then
$(X_P, L_P) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2,2))$.

$V = 3!8 = 48$, $F = 6 \cdot 2 \cdot 4 = 48$, $E = 12 \cdot 2 = 24$, $v = 8$.

$V_1 = F_1 = E_1 = 0$ ($\mathrm{int}(P) = \{(1,1,1)\}$ is a point)

$$\deg X^{(2)} = 62V - 57F + 28E - 8v = 848.$$



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k-selfdual toric varieties (joint with A. Dickenstein)

$\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ a lattice point configuration, and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric embedding.

Form the matrix A by adding a row of 1's to the matrix $(a_0|\cdots|a_N)$. Denote by $\mathbf{v}_0 = (1, \ldots, 1), \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^{N+1}$ the row vectors of A.

For any $\alpha \in \mathbb{N}^{n+1}$, denote by $\mathbf{v}_\alpha \in \mathbb{Z}^{N+1}$ the vector obtained as the coordinatewise product of α_0 times the row vector $\mathbf{v}_0$ times $\ldots$ times α_n times the row vector $\mathbf{v}_n$.

For any k, we define the matrix $A^{(k)}$ as follows. Order the vectors $\{\mathbf{v}_\alpha : |\alpha| \leq k\}$ (for instance, by degree and then by lexicographic order with $0 > 1 > \cdots > n$), and let $A^{(k)}$ be the $\binom{n+k}{k} \times (N+1)$ integer matrix with these rows.

Rational normal curve

Take $\mathcal{A} = \{0, \ldots, d\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \end{pmatrix},$$

and

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 1 & 4 & 9 & \cdots & d^2 \\ 0 & 1 & 8 & 27 & \cdots & d^3 \end{pmatrix}.$$

Note that

$$A^{(3)} \cong \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 3 & \cdots & \binom{d}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{d}{3} \end{pmatrix}.$$

The case $k = 2$

Denote by $\mathbf{v}_i * \mathbf{v}_j \in \mathbb{Z}^{m+1}$ the vector given by the coordinatewise product of these vectors. Define the $\binom{n+2}{2} \times (m+1)$-matrix

$$A^{(2)} = \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{v}_1 * \mathbf{v}_1 \\ \mathbf{v}_1 * \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} * \mathbf{v}_n \\ \mathbf{v}_n * \mathbf{v}_n \end{pmatrix},$$

$\mathbf{v}_i * \mathbf{v}_j$, $1 \leq i \leq j \leq n$. Then, $\mathbb{P}(\mathrm{Rowspan}(A^{(2)})) = \mathbb{T}^2_{X_{\mathcal{A}},\mathbf{1}}$ describes the second osculating space of $X_{\mathcal{A}}$ at the point $\mathbf{1}$.

Non-pyramidal configurations

The configuration $\mathcal{A}$ is *non-pyramidal* (nap) if the configuration of columns in A is not a pyramid (i.e., no basis vector e_i of $\mathbb{R}^{N+1}$ lies in the rowspan of the matrix).

The configuration $\mathcal{A}$ is $k$nap if the configuration of columns in $A^{(k)}$ is not a pyramid.

Example

A is a pyramid:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$

Characterization of k-self dual configurations

$X_{\mathcal{A}}$ is *k-selfdual* if $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$ for some $\phi\colon \mathbb{P}^N \cong (\mathbb{P}^N)^\vee$.

Theorem (Dickenstein–P.)

(1) *$X_{\mathcal{A}}$ is k-selfdual if and only if* $\dim X_{\mathcal{A}} = \dim X_{\mathcal{A}}^{(k)}$ *and $\mathcal{A}$ is knap.*

(2) *If $\mathcal{A}$ is knap and* $\dim \operatorname{Ker} A^{(k)} = 1$, *then $X_{\mathcal{A}}$ is k-selfdual.*

(3) *If $\mathcal{A}$ is knap and k-selfdual, and* $\dim \operatorname{Ker} A^{(k)} = r > 1$, *then $\mathcal{A}$ is r-Cayley.*

The proof generalizes [Bourel–Dickenstein–Rittatore] ($k = 1$).

A surface in $\mathbb{P}^3$

$$\mathcal{A} = \{(0,0),(1,0),(1,1),(0,2)\}$$

gives

$$X_{\mathcal{A}} : (x,y) \mapsto (1 : x : xy : y^2)$$

and

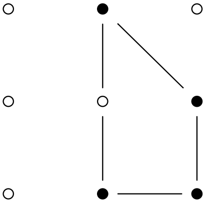
$$X_{\mathcal{A}}^{\vee} \cong X_{\mathcal{A}^{\vee}} : (x,y) \mapsto (-y^2 : 2x^{-1}y^2 : -2x^{-1}y : 1),$$

with

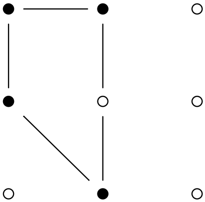
$$\mathcal{A}^{\vee} = \{(0,2),(-1,2),(-1,1),(0,0)\}.$$

This surface is self dual.

The corresponding polytopes



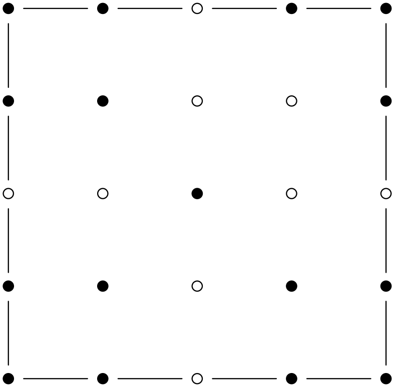
$\mathcal{A}$



$\mathcal{A}^\vee$

Example

This square is an example of a 4-selfdual smooth surface which is not centrally symmetric.



Connections with number theory

Any non-trivial linear relation between the rows of $A^{(k)}$ corresponds to a polynomial of degree k that vanish on $\mathcal{A}$.

Example

Consider three quadrics $Q_1, Q_2, Q_3 \in \mathbb{Z}[x_1, x_2, x_3]$ with

$$Q_1 \cap Q_2 \cap Q_3 = \{a_0, \ldots, a_7\} \subset \mathbb{Z}^3 \subset \mathbb{R}^3.$$

Then $\mathcal{A} = \{a_0 \ldots, a_7\}$ gives a 2-selfdual threefold:

The rank of $A^{(2)}$ is $10 - 3 = 7$, which is one less than the maximal rank.

Such constructions give an interesting connection to diophantine theory: polynomials with many integer solutions.

Thank you for your attention!

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