

Polar varieties revisited

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Workshop on
Computer Algebra and Polynomials
Linz, November 28, 2013



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Introduction

The theory of polars and polar varieties has played an important role in the quest for understanding and classifying projective varieties. Their use in the definition of projective invariants is the very basis for the geometric approach to the theory of characteristic classes, such as Todd classes and Chern classes.



Applications

Polar varieties have been applied to study

- ▶ singularities (Lê–Teissier, Merle, ...)
- ▶ the topology of real affine varieties (Giusti, Heinz et al., Safey El Din–Schost)
- ▶ real solutions of polynomial equations (Giusti, Heinz, et al.)
- ▶ complexity questions (Bürgisser–Lotz)
- ▶ foliations (Soares, ...)
- ▶ focal loci and caustics of reflection (Catanese, Trifogli, Josse–Pène)
- ▶ Euclidean distance degree (Sturmfels et al.)



Today I will survey the classical theory of polar varieties and the less classical of reciprocal polar varieties. I will take a look at three applications:

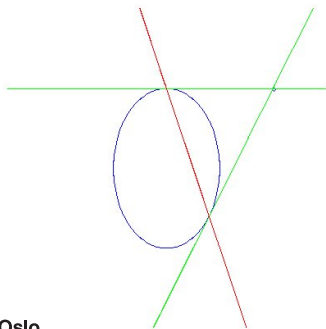
- ▶ finding points on real components
- ▶ focal loci and caustics of reflection
- ▶ Euclidean distance degree

The work on polars and reciprocal polars of *real* projective and affine algebraic varieties is joint work with [Heidi Mork](#) (“Real algebraic curves and surfaces,” PhD Thesis 2011, University of Oslo). The pictures are made by her, using SURF and Maple.



Pole and polars

Let Q be a conic section. Let P be a point in the plane. There are two tangents to Q passing through P . The **polar** of P is the line joining the two points of tangency. Conversely, if L is a line, it intersects the conic in two points. The **pole** of L is the intersection of the tangents to Q at these two points.



More generally, a quadric hypersurface Q in \mathbb{P}^n given by a quadratic form q , sets up a polarity between points and hyperplanes:

$$P = (b_0 : \dots : b_n) \mapsto H : \sum b_i \frac{\partial q}{\partial X_i} = 0$$

The **polar** hyperplane P^\perp of P is the linear span of the points on Q such that the tangent hyperplane at that point contains P .

If H is a hyperplane, its **pole** H^\perp is the intersection of the tangent hyperplanes to Q at the points of intersection with H .

If the quadric is $q = \sum X_i^2$, then the polar of

$$P = (b_0 : \dots : b_n)$$

is the hyperplane

$$P^\perp : b_0 X_0 + \dots + b_n X_n = 0.$$



Grassmann and Schubert varieties

Let $k = \mathbb{R}$ or \mathbb{C} . Let $\mathbb{G}(m, n)$ denote the **Grassmann variety** of $(m + 1)$ -spaces in k^{n+1} , or equivalently, of m -dimensional linear subspaces of \mathbb{P}_k^n .

Let

$$\mathcal{L}: L_0 \subset L_1 \subset \cdots \subset L_m \subset \mathbb{P}_k^n$$

be a flag of linear subspaces, with $\dim L_i = a_i$.

The **Schubert variety** $\Omega(\mathcal{L})$ is defined by

$$\Omega(\mathcal{L}) := \{W \in \mathbb{G}(m, n) \mid \dim W \cap L_i \geq i, 0 \leq i \leq m\}.$$

The class of $\Omega(\mathcal{L})$ depends only on the a_i . Write

$$\Omega(\mathcal{L}) = \Omega(a_0, \dots, a_m).$$



Example

- $m = 1, n = 3$: $\mathbb{G}(1, 3) = \text{lines in } \mathbb{P}^3$.

$\Omega(1, 3) = \text{lines meeting a given line}$

$\Omega(0, 3) = \text{lines through a given point}$

$\Omega(1, 2) = \text{lines in a given plane}$

$\Omega(0, 2) = \text{lines in a plane through a point in the plane}$

- $m = 2, n = 5$: $\mathbb{G}(2, 5) = \text{planes in } \mathbb{P}^5$.

$\Omega(1, 4, 5) = \text{planes meeting a given line}$

$\Omega(2, 4, 5) = \text{planes meeting a given plane}$



The Gauss map

- Projective variety $X \subset \mathbb{P}^n$, $\dim X = m$. The Gauss map¹ is

$$\gamma: X \dashrightarrow \mathbb{G}(m, n); P \mapsto T_P$$

T_P = the projective tangent space to X at P .

The Gauss map is finite and birational (Zak, 1987).

- Affine variety $X \subset \mathbb{A}^n$, $\dim X = m$. The Gauss map is

$$\gamma: X \dashrightarrow \mathbb{G}(m-1, n-1); P \mapsto t_P$$

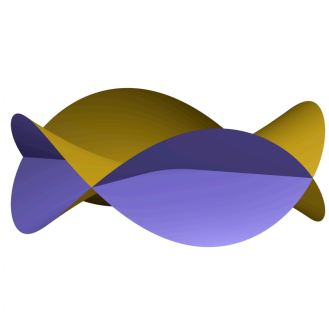
t_P = the affine tangent space to X at P (considered as a subspace of k^n).

¹The most ingenious thing Gauss did, according to W. Hsiang.



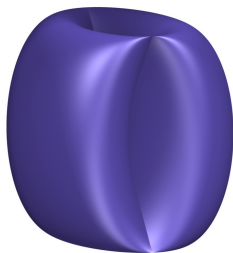
Example (The double pillow)

Figure: Picture in xyw -space ($z = 1$) of
$$x^4 - 2x^2y^2 + y^4 - 2x^2w^2 - 2y^2w^2 - 16z^2w^2 + w^4 = 0$$



The *double pillow* is the image of the Gauss map (the dual surface).

Figure: $16x^2 - y^4 + 2y^2z^2 - 8x^2y^2 - z^4 - 8x^2z^2 - 16x^4 = 0$



Polar varieties

The **polar varieties** of $X \subset \mathbb{P}^n$ are the inverse images

$$P(\mathcal{L}) := \gamma^{-1}\Omega(\mathcal{L})$$

of the Schubert varieties via the Gauss map.

Example

- ▶ $X \subset \mathbb{P}^3$ is a curve ($m=1$). Then $P(1,3)$ is the set of points $P \in X$ such that T_P meets a given line, i.e., the ramification points of the projection map $X \rightarrow \mathbb{P}^1$.
- ▶ $X \subset \mathbb{P}^5$ is a surface ($m=2$). Then $P(1,4,5)$ is the ramification locus of the projection map $X \rightarrow \mathbb{P}^3$ with center a line L_0 .



Projective invariants and characteristic classes

The polar *classes* are *projective invariants* (invariant under linear projections and under sections by linear spaces).

Certain combinations of the polar classes are intrinsic invariants, depending only on the variety, not on the projective embedding, hence define *characteristic classes* (Severi, Todd, Eger):

$$c_i(X) = \sum_{j=0}^i (-1)^{i-j} \binom{m+1-i+j}{j} [M_j] h^j,$$

where $M_j = \gamma^{-1} \Omega(\mathcal{L})$ for $\Omega(\mathcal{L})$ a first special Schubert cycle with $\dim L_{j-1} = j + n - m - 2$, $m = \dim X$, and h is the class of a hyperplane section.



Example

Let $X \subset \mathbb{P}^5$ be a surface. Set $\mu_0 = \deg X$, $\mu_1 = \deg P(2, 4, 5)$, $\mu_2 = \deg P(2, 3, 5)$, $\nu_2 = \deg P(1, 4, 5)$.

Around 1871 M. Noether showed that

$$\mu_2 - 6\mu_1 + 9\mu_0 + \nu_2$$

is an invariant and Zeuthen discovered that

$$\mu_2 - 2\mu_1 + 3\mu_0$$

also is an invariant. Indeed,

$$\mu_2 - 6\mu_1 + 9\mu_0 + \nu_2 = c_1(X)^2 \text{ and } \mu_2 - 2\mu_1 + 3\mu_0 = c_2(X).$$



Singular varieties

The theory of polar varieties can be extended to the case of singular varieties: replace X by the set of smooth points and take closures.

One can then use the Todd–Eger relations to define characteristic classes for singular varieties (Chern–Mather classes). These are not topologically well behaved; the solution is to use the Euler obstruction to define Chern–Schwarz–MacPherson classes.

Moreover, one can find formulas for the degrees of the polar varieties in terms of the degree of the variety and the degrees of the singular loci.



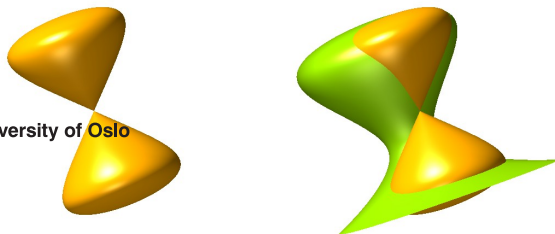
Polars

If $X \subset \mathbb{P}^n$ is a *hypersurface*, defined by a homogeneous polynomial $F = 0$, then its *polar* with respect to a point $P = (b_0 : \dots : b_n)$ is the hypersurface $\sum b_i \frac{\partial F}{\partial X_i} = 0$. The intersection of the variety with the polar is the polar variety $M_1 = P(0, 2, \dots, n)$ (with $L_0 = P$).

Figure: The sextic surface $x^2 + y^2 - z^2 + z^6 = 0$ and its polar w.r.t. the point $(1 : 1 : 1 : 1)$, $6z^5 + 4x^2 + 4y^2 - 4z^2 + 2x + 2y - 2z = 0$. The polar variety is the intersection of the two surfaces.



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Finding real points

Let $X \subset \mathbb{R}^n$ be a real algebraic variety. Then X may have many connected components. How to find a way of determining a point on each connected component?

Bank, Giusti, Heintz, Mbakop, and Pardo proved that the polar varieties of *real, affine, non-singular, compact* varieties contain points on each connected component of the variety. Related work has also been done by Safey El Din and Schost.

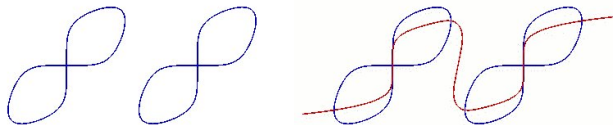
What if the variety is singular? Can one find *nonsingular* points on each component?



Singular curves

Let $X \subset \mathbb{R}^2$ be a *compact* curve. We proved: if X has only ordinary multiple points as singularities, then the polar variety contains a non-singular point of each connected component.

Figure: A sextic and its polar.



There exist compact singular real affine plane curves such that no polar variety contains a point from each connected component, e.g. this sextic with eight cusps.:



The image of the affine Gauss map in $\mathbb{P}^1_{\mathbb{R}}$ is the union of two disjoint intervals.



Reciprocal polar varieties

Let $\mathcal{L} : L_0 \subset L_1 \subset \cdots \subset L_{n-1}$ be a flag in \mathbb{P}^n , where $L_{n-1} = H_\infty$ is the hyperplane at infinity. We then get the polar flag with respect to Q :

$$\mathcal{L}^\perp : L_{n-1}^\perp \subset L_{n-2}^\perp \subset \cdots \subset L_1^\perp \subset L_0^\perp$$

Definition

The i -th reciprocal polar variety $W_{L_{i+p-1}}^\perp(V)$, $1 \leq i \leq n-p$, of a variety X with respect to the flag \mathcal{L} , is defined to be the Zariski closure of the set

$$\{P \in X_{\text{sm}} \setminus L_{i+p-1}^\perp \mid T_P X \not\subset \langle P, L_{i+p-1}^\perp \rangle^\perp\}$$



Hypersurfaces

When $X \subset \mathbb{P}^n$ is a hypersurface, the reciprocal polar variety is

$$W_{L_{n-1}}^\perp(X) = \overline{\{P \in X_{\text{sm}} \mid T_P X \supset \langle P, L_{n-1}^\perp \rangle^\perp\}}.$$

Note that

$$T_P X \supseteq \langle P, L_{n-1}^\perp \rangle^\perp \Leftrightarrow T_P X^\perp \in \langle P, L_{n-1}^\perp \rangle.$$

The point $T_P X^\perp$ is on the line $\langle P, L_{n-1}^\perp \rangle$ if and only if the point L_{n-1}^\perp is on the line $\langle P, T_P X^\perp \rangle$. So the reciprocal polar variety is

$$W_{L_{n-1}}^\perp(X) = \overline{\{P \in X_{\text{sm}} \mid L_{n-1}^\perp \in \langle P, T_P X^\perp \rangle\}}.$$



Reciprocal polars for hypersurfaces

Let $Q: q = 0$ be a quadric and $X = Z(F)$ be a hypersurface.

The *reciprocal polar variety* of X with respect to

$L_{n-1}^\perp = (b_0 : \cdots : b_n)$ is the locus on X given by the vanishing of the maximal minors of the determinant of

$$\begin{pmatrix} b_0 & \cdots & b_n \\ \frac{\partial q}{\partial X_0} & \cdots & \frac{\partial q}{\partial X_n} \\ \frac{\partial f}{\partial X_0} & \cdots & \frac{\partial f}{\partial X_n} \end{pmatrix}.$$

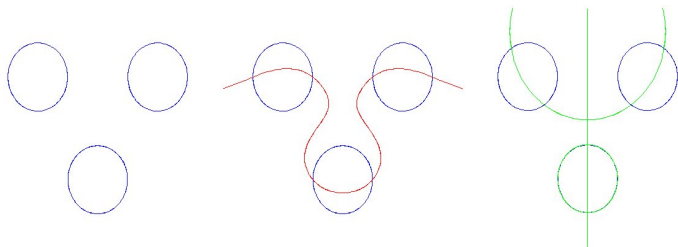
Taking $n = 2$, $L_1 : X_0 = 0$ gives the *affine reciprocal polar*

$$\frac{\partial q}{\partial X_2} \frac{\partial f}{\partial X_1} - \frac{\partial q}{\partial X_1} \frac{\partial f}{\partial X_2} = 0,$$

of the same degree as X .

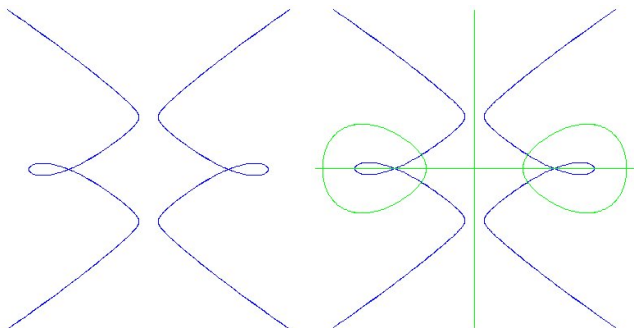


Figure: A sextic curve with its polar and its reciprocal polar.



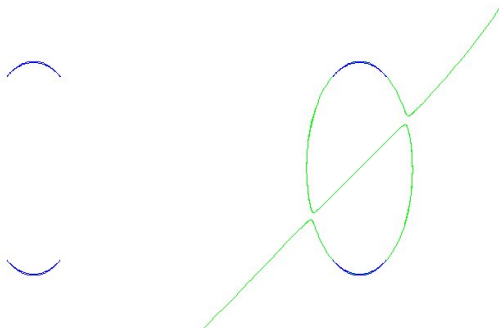
Replacing the polar with the reciprocal polar, we need not assume the curve is compact.

Figure: A non-compact affine real sextic curve with its reciprocal polar.



If the curve has arbitrary singularities, the result is no longer true.

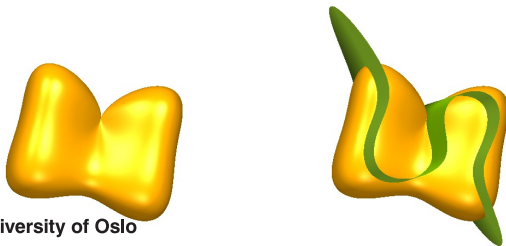
Figure: A compact sextic curve with four cusps and its reciprocal polar.



Singular surfaces

Mork also studied surfaces and found similar results as for curves.

Figure: The quartic surface $x^3 + y^2 - z^2 + z^4 + y^4 + x^4 = 0$ has an A_2^- -singularity. Its polar with respect to the point $(1 : 0 : 1 : 0)$, $4x^3 + 4z^3 + 3x^2 - 2z = 0$. The polar variety is the intersection of the two surfaces.



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Figure: The quartic surface $x^4 + x^3w + y^2w^2 + z^2w^2 = 0$ has an A_2^+ -singularity. Its polar with respect to the point $(1 : 0 : 1 : 0)$, $4x^3 + 3x^2w + 2zw^2 = 0$. The polar variety is the intersection of the two surfaces.

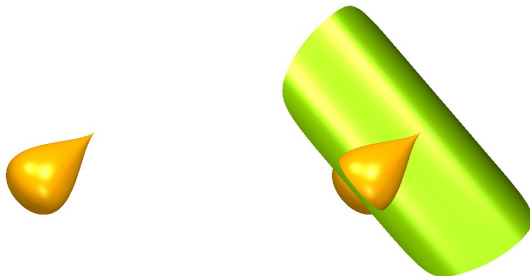
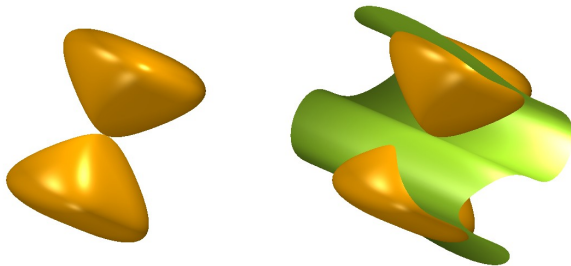


Figure: The sextic surface $z^6 + x^4 + y^2 - z^2 = 0$ has an A_3^- -singularity. Its polar with respect to the point $(1 : 0 : 1 : 0)$, $6z^5 + 4x^3 - 2z = 0$. The polar variety is the intersection of the two surfaces.



Euclidean normal bundle (Catanese – Trifogli)

Consider $X \subset \mathbb{P}^n$, fix a hyperplane $H_\infty \subset \mathbb{P}^n$ at infinity and a smooth quadric Q in H_∞ . Use the polarity in H_∞ induced by Q to define Euclidean normal spaces at each point $x \in X$:

$$N_P X = \langle P, (T_P X \cap H_\infty)^\perp \rangle$$

The normal spaces are the fibers of the projective bundle $\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n \rightarrow X$, where $\mathcal{E} = \mathcal{N}_{X/\mathbb{P}^n}^\vee(-1) \oplus \mathcal{O}_X(1)$.



The Euclidean endpoint map

Let $p: \mathbb{P}(\mathcal{E}) \rightarrow X$, and let $q: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$ denote the projection on the second factor – the endpoint map.

Let $A \in \mathbb{P}^n \setminus H_\infty$. Then $p(q^{-1}(A))$ is a reciprocal polar variety:

$$p(q^{-1}(A)) = \{P \in X \mid A \in \langle P, (T_P X \cap H_\infty)^\perp \rangle\}$$

so the degree of q is the degree of the reciprocal polar variety.

Example

Assume $X \subset \mathbb{P}^2$ is a (general) plane curve of degree d .

The *reciprocal* polar variety is the intersection of the curve with its reciprocal polar, which has degree d , so q has degree d^2 .



Euclidean distance degree (Sturmfels et al.)

The (general) *Euclidean distance degree* is the degree of the map $q: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$. Hence

$$\text{ED deg } X = p_* c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^n = s_m(\mathcal{E}),$$

where $m = \dim X$. We can compute:

$$s(\mathcal{E}) = s(\mathcal{N}_{X/\mathbb{P}^n}^\vee(-1))s(\mathcal{O}_X(1)) = c(\mathcal{P}_X^1(1))c(\mathcal{O}_X(-1))^{-1}$$

We conclude:

$$\text{ED deg } X = \sum_{i=0}^m \mu_i,$$

where μ_i is the degree of the i th polar variety $[M_i]$ of X .



The focal locus (or ED discriminant)

The focal locus Σ_X is the branch locus of the map q .

It is the image of the subscheme given by the ideal $F^0(\Omega_{\mathbb{P}(\mathcal{E})/\mathbb{P}^n}^1)$, so its class is

$$[\Sigma_X] = q_*(c_1(\Omega_{\mathbb{P}(\mathcal{E})}^1) - q^*c_1(\Omega_{\mathbb{P}^n}^1)).$$

Example

$X \subset \mathbb{P}^2$ is a (general) plane curve of degree d . Then the focal locus is the *evolute* (or caustic) of X . Its degree is

$$\deg \Sigma_X = (c_1(\Omega_{\mathbb{P}(\mathcal{E})}^1) - q^*c_1(\Omega_{\mathbb{P}^2}^1))c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = 3d(d-1).$$





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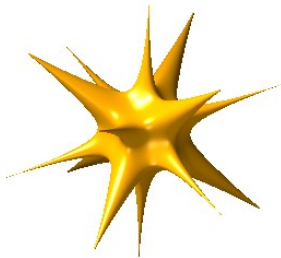


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