Enriques diagrams and equisingular strata of families of curves

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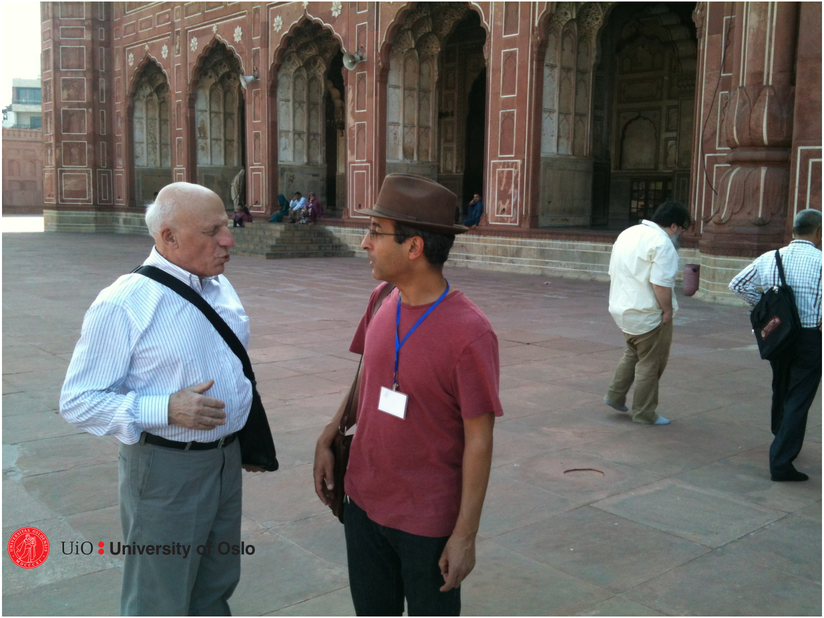
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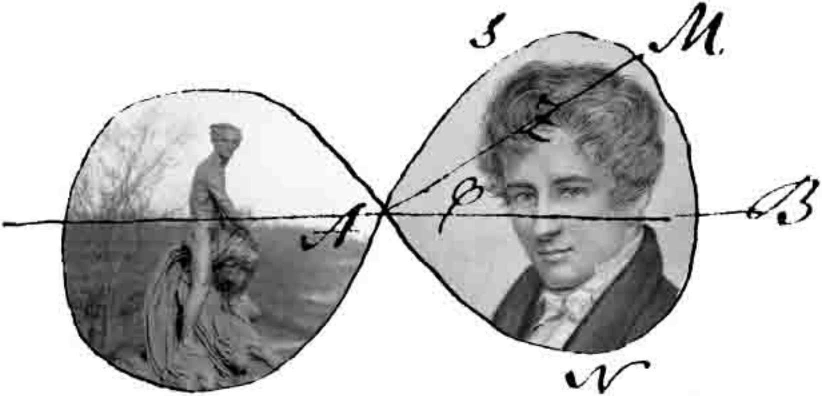
Planar curve singularities

A planar curve singularity $(C, 0)$, given by $f(x, y) = 0$, and its normalization $n\colon C' \to C$.

Many numerical invariants associated to $(C, 0)$:

- multiplicity m: $f \in \mathfrak{m}^m$, $f \notin \mathfrak{m}^{m+1}$
- delta invariant $\delta = \dim n_*\mathcal{O}_{C'}/\mathcal{O}_C$
- Milnor number $\mu = \dim k[[x, y]]/(f_x, f_y)$
- Tjurina number $\tau = \dim k[[x, y]]/(f, f_x, f_y)$
- number of branches $r = \#n^{-1}(0)$

Milnor's formula: $\mu = 2\delta - r + 1$

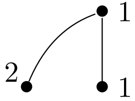

S
M
φ
A
B
N

Resolution by blowing ups

A singular point on a curve on a smooth surface: $0 \in C \subset S$

By a succession of blow-ups we get a *good embedded resolution* $\nu \colon S' \to S$: the strict transform C' is smooth and $\nu^{-1}(0)_{\mathrm{red}} = C' \cup \bigcup_i E_i$ is a normal crossing divisor.

The resolution can be encoded in a diagram, the *Enriques diagram*.

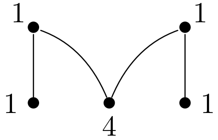


An ordinary cusp
$f(x,y) = y^2 - x^3$
$\nu^{-1}(0)_{\mathrm{red}} = C' \cup E_1 \cup E_2 \cup E_3$,
where E_3 intersects each of C', E_1 and E_2 in one point.
No other intersections.

Enriques diagrams

An *Enriques diagram* is a finite directed graph with no loops, and with assigned weights to the vertices.

There are three types of vertices: roots, free vertices and satellites.



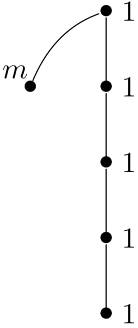


Figure: The Enriques diagram $\mathbf{M}_{m,p}$, with $m \geq p = 5$. For $m = 5$, $f(x, y) = y^5 - x^6$.

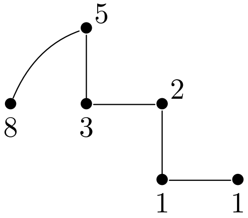


Figure: The Enriques diagram corresponding to the Fibonacci singularity $f(x, y) = y^8 - x^{13}$.

Numerical characters

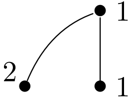
Let $\mathbf{D}$ be a diagram with one root R. Define

- $\delta(\mathbf{D}) := \sum_{V\in\mathbf{D}} \binom{m_V}{2}$
- $r(\mathbf{D}) := \sum_{V\in\mathbf{D}}(m_V - \sum_{W \text{ prox } V} m_W)$
- $\mu(\mathbf{D}) := 2\delta\mathbf{D}) - r(\mathbf{D}) + 1$

Define for any $\mathbf{D}$

- $\dim(\mathbf{D}) := \mathrm{rts}(\mathbf{D}) + \mathrm{frs}(\mathbf{D})$
- $\deg(\mathbf{D}) := \sum_{V\in\mathbf{D}} \binom{m_V+1}{2}$
- $\mathrm{cod}(\mathbf{D}) := \deg(\mathbf{D}) - \dim(\mathbf{D})$

The $\mathbf{A}_2$ singularity (ordinary cusp)



$$\delta(\mathbf{D}) = 1$$
$$r(\mathbf{D}) = 2 - 1 - 1 + 1 - 1 + 1 = 1$$
$$\mu(\mathbf{D}) = 2$$
$$\dim(\mathbf{D}) = 1 + 2 = 3$$
$$\deg(\mathbf{D}) = \binom{3}{2} + \binom{2}{2} + \binom{2}{2} = 5$$
$$\operatorname{cod}(\mathbf{D}) = 5 - 3 = 2.$$

Complete ideals

Let $\nu\colon S' \to S$ be a good embedded resolution of $0 \in C \subset S$.

Let $\mathbf{D}$ be the associated Enriques diagram.

Each vertex $V \in \mathbf{D}$ corresponds to an infinitely near point of 0.

Set $E := \sum_{V \in \mathbf{D}} m_V E_V$, where E_V is the total transform of the exceptional divisor coming from blowing up the point corresponding to V.

The ideal $\mathcal{I} := \nu_* \mathcal{O}_{S'}(-E)$ is *complete* (integrally closed).

Enriques–Hoskin–Deligne–Casas: $\dim \mathcal{O}_S/\mathcal{I} = \deg(\mathbf{D})$

The diagram $\boldsymbol{D}$ *can be recovered from* $\mathcal{I}$*.*

Hilbert schemes

Let $\mathbf{D}$ be a diagram, $d := \deg(\mathbf{D})$. Set

$$H(\mathbf{D}) := \{\mathcal{I} \subset \mathcal{O}_S \mid \mathcal{I} \text{ has diagram } \mathbf{D}\} \subset \mathrm{Hilb}_S^d$$

Proposition

$H(\boldsymbol{D})$ is locally closed, smooth and irreducible, of dimension $\dim(\boldsymbol{D})$.

Example

$\dim H(\mathbf{A}_1) = 2$, in fact $H(\mathbf{A}_1) \cong S$.

Deformation space

Let $(C, 0)$ be a singularity, given by $f(x, y) = 0$, with diagram $\mathbf{D}$. The tangent space to the versal deformation space is $B = \mathcal{O}_{C,0}/(f, f_x, f_y)$.

$\dim B = \tau$, the Tjurina number. Note: $\tau \leq \mu$, and equality holds for quasi-homogeneous singularities.

Let $B_{\text{es}} \subset B$ denote the (topological) equisingular locus. We have $\operatorname{cod}(B_{\text{es}}, B) = \operatorname{cod}(\mathbf{D})$.

Example

If $f(x, y) = xy(x^2 - y^2)$ is an ordinary quadruple point, then $\tau = \mu = 9$ and $\operatorname{cod}(B_{\text{es}}, B) = \dim B - \dim B_{\text{es}} = 9 - 1 = 8 = \operatorname{cod}(\mathbf{D}) = \deg(\mathbf{D}) - \dim(\mathbf{D}) = 10 - 2$.

Arbitrarily near points

Let $\pi\colon F \to Y$ be a family of surfaces, $F' \to F \times_Y F$ the blow up of the diagonal, and $\pi'\colon F' \to F$ the composition of the blow up with the second projection .

Define recursively $\pi^{(i)}\colon F^{(i)} \to F^{(i-1)}$ in the same way. Let $E^{(i)}$ denote the exceptional divisor.

Theorem

Let $\boldsymbol{D}$ be an ordered (unweighted) Enriques diagram on $n+1$ vertices. The functor of sequences of arbitrarily near T-points of F/Y with diagram $\boldsymbol{D}$ is represented by a subscheme $F(\boldsymbol{D}) \subset F^{(n)}$, which is Y-smooth with irreducible geometric fibers of dimension $\dim \boldsymbol{D}$.

Relative Hilbert schemes

Let $\mathbf{D}$ be an Enriques diagram with $\deg(\mathbf{D}) = d$. As in the case of a single surface S, define $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$.

Theorem

There is a natural bijective map

$$\Psi\colon Q(\boldsymbol{D}) := F(\boldsymbol{D}) / \operatorname{Aut}(\boldsymbol{D}) \to H(\boldsymbol{D}),$$

which is an isomorphism in characteristic 0.

The map Ψ can be purely inseparable in characteristic $p > 0$, e.g. for $\mathbf{D} = \mathbf{M}_{m,p}$ (Tyomkin).

The equisingular stratification

Let $D \subset F \to Y$ be a family of curves on a family of surfaces.

For a given Enriques diagram $\mathbf{D}$ with $\deg(\mathbf{D}) = d$, let $Y(\mathbf{D})$ parameterize the fibers of D that have singularities of type $\mathbf{D}$. The expected codimension of $Y(\mathbf{D})$ is $\operatorname{cod}(\mathbf{D})$.

Problem: *Determine the class of the cycle $[\overline{Y(\boldsymbol{D})}]$ in terms of the Chern classes of F/Y and D.*

To define the cycle, we set

$$G(\mathbf{D}) := \mathrm{Hilb}^d_{D/Y} \times_{\mathrm{Hilb}^d_{F/Y}} H(\mathbf{D})$$

and define $Y(\mathbf{D})$ as the image of the map $G(\mathbf{D}) \to Y$.

Applications to curve counting

A family $D \subset F/Y$ is said to be *r-generic* if for every $\mathbf{D}$ and every $y \in Y(\mathbf{D})$, we have

$$\operatorname{cod}(Y(\mathbf{D}), Y) \leq \min\{r+1, \operatorname{cod}(\mathbf{D})\}$$

Construct a derived family $D' \subset F'/Y'$, where $Y' =$ the set of points of D that are singular in a fibre of D/Y, a fiber of $F' \to Y'$ is the blowup of a fiber of F/Y at a point of Y', and $D' = D - 2E$.

Proposition

If $D \subset F/Y$ is r-generic, then $D' \subset F'/Y'$ is $(r-1)$-generic.

This allows us to prove formulas e.g. for the classes $[\overline{Y(r\mathbf{A}_1)}]$.

Curves with singularities of given type

Given an Enriques diagram $\mathbf{D} = \mathbf{D}_1, \ldots, \mathbf{D}_n$, how many curves in a given linear system $|\mathcal{L}|$ have singularities of these types (and pass through the required number of points)?

Theorem

(Li–Tzeng, Rennemo) *There exists a universal polynomial P of degree n in four variables such that this number is equal to P evaluated in the Chern numbers of S and $\mathcal{L}$.*

Proof.

(Rennemo (2013)) The number is given by the degree of the class $c_m(\mathcal{L}^{[d]}) \cap \overline{H(\mathbf{D})}$, where $m = \dim H(\mathbf{D})$, $d = \deg(\mathbf{D})$, and $\mathcal{L}^{[d]} = q_* p^* \mathcal{L}$ with $p \colon \mathcal{Z} \to S$, $q \colon \mathcal{Z} \to S^{[d]}$ are the projections from the incidence scheme $\mathcal{Z} \subset S \times S^{[d]}$.

□

Cuspidal curves

For a cusp (unibranch singularity) the Enriques diagram can be replaced by the multiplicity sequence.

Many open questions:

- How many cusps can a plane cuspidal curve have? Tono: $\leq (21g+17)/2$. Conjecture: ≤ 4 for $g=0$.
- Coolidge–Nagata: Any rational plane cuspidal curve can be transformed to a line via a sequence of Cremona transformations.
- Orevkov–Chéniot: For a plane rational cuspidal curve, $\sum \overline{M} \leq 3(\deg C - 3) + \dim \mathrm{Stab}_{PGL(3)}(C)$.

T. K. Moe (2013): A cuspidal curve on a Hirzebruch surface can have at most $(21g+29)/2$ cusps.

Fibonacci cusps

Let φ_k denote kth Fibonacci number:

$$\varphi_0 = 0, \varphi_1 = 1, \varphi_2 = 1, \varphi_3 = 2, \varphi_4 = 3, \varphi_5 = 5, \dots$$

The Fibonacci singularity $\mathbf{F}_k$: $y^{\varphi_k} - x^{\varphi_{k+1}} = 0$ has the following numerical characters:

- $\dim(\mathbf{F}_k) = 3$
- $\deg(\mathbf{F}_k) = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 1$
- $\mathrm{cod}(\mathbf{F}_k) = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 4$
- $\delta(\mathbf{F}_k) = (\varphi_k - 1)(\varphi_{k+1} - 1)/2$
- $\mu(\mathbf{F}_k) = 2\delta(\mathbf{F}_k) = (\varphi_k - 1)(\varphi_{k+1} - 1)$
- $\overline{M} := \mathrm{cod}(\mathbf{F}_k) - \delta(\mathbf{F}_k) = \varphi_{k+2} - 4$

Fibonacci curves

The rational plane curve $C_k : y^{\varphi_k} z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0$ has two cusps: one Fibonacci cusp and one “semi”-Fibonacci cusp: $z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0$.

- C_k is toric, and (hence) self-dual (the cusps are interchanged under the Gauss map).
- C_k is “maximally inflected” (i.e., all cusps and flexes are real — here there are no flexes).
- The sum of the $\overline{M}$-numbers is (for $k \geq 4$)

$$\varphi_{k+2} - 4 + \varphi_{k+1} - 4 + \varphi_{k-1} = 3(\varphi_{k+1} - 3) + 1$$

cf. Orevkov– Chéniots conjecture:

$$\sum \overline{M} \leq 3(\deg C - 3) + \dim \mathrm{Stab}_{PGL(3)}(C)$$

Thank you for your attention!