



THE NORWEGIAN ASTRONOMY AND ASTROPHYSICS
OLYMPIAD COMMITTEE

IOAA NOTES

Part 1: Celestial Mechanics

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1 Gravitation

NB: This handout assumes you are comfortable with basic rotation that you learn in physics. Key words are: centripetal force/acceleration, orbital radius, linear velocity and angular velocity

Before we start to work with any celestial mechanics, we need to gain appreciation for what makes it all possible - namely gravitation. Gravitation arises from an intrinsic property of all matter that we all know as mass. This means that gravitational attraction is present between all matter, but only on a larger, astronomical scale does it have the noticeable effects.

1.1 Force

It has been shown experimentally that the vector describing the gravitational force between two bodies with masses M and m is given as:

$$\vec{F} = -G \frac{Mm}{r^2} \hat{r}, \quad (1)$$

where $\hat{r}$ denotes the unit vector pointing in the direction of the shortest distance between the gravitational centres. Often, in problems, you will assume that the centre of gravity is the same as the centre of mass of a body. In reality, however, this is not true, since the strength of gravitation varies with distance - this leads to the Empire State Building's centre of gravity actually being 1 m lower than its centre of mass!

Angular Frequency of Rotating Object

Assuming we have an object of mass m orbiting an object of mass M in a perfectly circular orbit of radius r , find the angular frequency of the rotation.

Knowing that $v = \omega r$, and that the centripetal force is given as $F_{\text{centripetal}} = m \frac{v^2}{r}$ via Newton's Second Law where the acceleration is $a_{\text{centripetal}} = \frac{v^2}{r}$, we can equate this with the gravitational force between the two objects which is the only one contributing, and write:

$$G \frac{Mm}{r^2} = m\omega^2 r \implies \boxed{\omega = \frac{GM}{r^3}}$$

1.2 Potential Energy

When you do your physics problems, you draw the gravitational force pointing straight down. This is, in fact, the gravitational force pointing in towards the centre of the Earth, attracting whichever object the force is applied on. This means that the object accelerates when it is being pulled by a big body to the centre of gravity.

With acceleration, we know that we must gain kinetic energy. This means that we must be losing potential energy. In astronomy, we take the energy of an object infinitely far away to be 0. Because of the previous assertion that the body loses potential energy as it moves closer to the centre of gravity, we actually get **negative** potential energy the closer the body gets! This very counterintuitive result is summed up in this equation:

$$E_p = -G \frac{Mm}{r} \quad (2)$$

We provide the proof using calculus here for those interested, but it is not necessary to be completely familiar with why this is the case - just remember that in astronomy, the reference point is set to be infinitely far away, where we have the maximum potential energy of 0!

Proof of Gravitational Potential Energy

The work done by a force is defined as the integral of the force over the path it is working on.

$$W_{AB} = \int_{r_A}^{r_B} \vec{F} \cdot d\vec{r}$$

Notice that we have a dot product between the force vector and the radius vector. Since these are opposite to each other (also called antiparallel), we get a negative sign in front of the integral and using (1), we can write:

$$\begin{aligned} W_{AB} &= - \int_{r_A}^{r_B} G \frac{Mm}{r^2} dr \\ &= -GMm \int_{r_A}^{r_B} \frac{1}{r^2} dr = \\ &= -GMm \left(-\frac{1}{r} \right) \Big|_{r_A}^{r_B} = \\ &= GMm \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \end{aligned}$$

Now, we know that the work done by a force must be equal to the change in potential energy. The force is bringing an object from infinitely far away with $r = \infty$ to $r = r$, and we define the potential energy in the infinity condition as 0.

Now, if we take r_B to be infinitely large, as in the force bringing an object from infinitely far away, so that $r_B = \infty$, and have r_A just be an arbitrary distance between the two objects the force is working between and denote it r , we can write:

$$\begin{aligned} \Delta E_p &= E_{p,B} - E_{p,A} = -W_{AB} \\ E_{p,B} = 0 \wedge E_{p,A} = E_p &\implies -E_p = -W_{AB} \\ E_p &= GMm \left(\frac{1}{\infty} - \frac{1}{r} \right) = -G \frac{Mm}{r} \end{aligned}$$

Another result worth noting for this, regardless of whether or not you followed the calculus proof, is:

$$F = -\frac{dE_p}{dr} \quad (3)$$

Now, we can define a so-called gravitational potential, which is the gravitational potential energy per mass:

$$V = \frac{E_p}{m} = -\frac{GM}{r} \quad (4)$$

Why would this be important? Let's look at what this value is for the Earth, given that its radius is $r \approx 6371 \cdot 10^3$ m, and its mass $M \approx 5.972 \cdot 10^{24}$ kg with $G = 6.67430 \cdot 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$:

$$V_{\oplus} \approx -\frac{6.67430 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24}}{(6371 \cdot 10^3)^2} \frac{\text{N}}{\text{kg}} \approx -9.81997 \frac{\text{N}}{\text{kg}}$$

Looks familiar, doesn't it? :) We define the gravitational field strength using the gravitational potential as:

$$g = -\frac{dV}{dr} \quad (5)$$

2 The Shell Theorems

This might be an unknown term to you, but you have actually used them many times before, maybe even without being aware! Proving the theorems took Newton almost a decade, and since you only have till summer to prepare for the competition, we can call it a day with just presenting the two theorems:

- Theorem 1: A uniformly dense spherical shell attracts an external particle as if all the mass of the shell was concentrated at its centre.
- Theorem 2: A uniformly dense spherical shell exerts 0 gravitational force on a particle located inside of the shell

Let's see an example where the shell theorems are of use:

Tunnel Through Earth

Assume we managed to dig a straight tunnel through a uniform-density spherical Earth along its diameter. How will a particle with mass m behave when dropped through the tunnel?

The second shell theorem tells us that not the entirety of the Earth will be exerting a gravitational force on the mass, but only the part inside of a sphere with radius r , r being the distance of the mass from the centre of the Earth.

The first shell theorem tells us that all the mass affecting the small object is concentrated at the very centre of the sphere. We can therefore write:

$$\begin{aligned}\frac{M}{M_{\oplus}} &= \frac{\frac{4}{3}\pi r^3 \cdot \rho}{\frac{4}{3}\pi R_{\oplus}^3 \cdot \rho} \implies M = M_{\oplus} \frac{r^3}{R_{\oplus}^3} \\ F &= G \frac{mM}{r^2} = G \frac{m}{r^2} \cdot M_{\oplus} \frac{r^3}{R_{\oplus}^3} \\ F &= G \underbrace{\frac{mM_{\oplus}}{R_{\oplus}^3}}_{\text{constant!}} \cdot r\end{aligned}$$

Notice the constant term, given that all of these are unchanged, which makes sense. Calling this constant k , we can write $F = k \cdot r$, which is exactly Hooke's Law for springs! This means that the small mass will exhibit oscillatory movement as if affected by a spring with a constant $k = G \frac{mM_{\oplus}}{R_{\oplus}^3}$.

3 Introduction to Orbits

Finally, we're well acquainted enough with gravity to see and analyse how exactly it makes orbits possible, and what kind of orbits that entails!

In some problems, you might be asked to find the so-called "escape-velocity" or "circular velocity". The circular velocity might already be familiar to you - it is found by equating the force of gravity with the centripetal force for a perfectly circular orbit, and denotes the velocity needed to get a projectile to move in a circular orbit about a mass M .

$$m \cdot \frac{v_{\text{circular}}^2}{r} = G \frac{Mm}{r^2} \implies v_{\text{circular}} = \sqrt{\frac{GM}{r}}$$

The escape velocity, however, is the velocity needed to give to a projectile to launch it so that it escapes the gravitational pull of the massive object. The gravitational pull is escaped only in infinity, when both the velocity of the object as well as its gravitational potential energy are 0. Using conservation of energy, we can write:

$$-G \frac{Mm}{r} + \frac{1}{2} m v_{\text{escape}}^2 = 0 \implies v_{\text{escape}} = \sqrt{\frac{2GM}{r}}$$

You may notice that both of these velocities are completely independent of the mass of the projectile!

Now, why are these velocities relevant at all? Believe it or not, they define the shapes of orbits, meaning that if you know the velocity of a small object orbiting around a bigger one, you can say something about the shape of the orbit! The orbit is:

1. circular if $v = \sqrt{\frac{GM}{r}}$, and the total energy is $E = -G \frac{mM}{2r}$
2. elliptic if $v_{\text{circular}} < v < v_{\text{escape}}$, and the total energy is $E = -G \frac{mM}{2a}$
3. parabolic/hyperbolic if $v_{\text{escape}} < v$, and the total energy is $E = 0$

3.1 Properties of Elliptical Orbits

You're probably almost an expert on circular orbits, but they are very boring! In reality, many orbits are elliptical and there are many more different properties to ellipses as compared to circles. We will first look at a geometric understanding of ellipses.

An ellipse is drawn in Geogebra using the equation $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ and shown in Figure 1. A general equation for an ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b \quad (6)$$

With the ellipse equation above, the semi-major axis is a , the semi-minor axis is b , and the ellipse is centred in (h, k) . You can think of the semi-axes as the "radii" of the ellipse - the semi-major is longer, and the semi-minor is shorter. The area of an ellipse is $A = \pi ab$ (don't even ask about the circumference, that's beyond what we need (and my paygrade :D) - let Matt Parker explain it to you at <https://www.youtube.com/watch?v=5nW3nJhBHL0>).

For any point P on the ellipse, the focus points (foci) of the ellipse are defined such that $\overline{PF_1} + \overline{PF_2} = \text{const.}$ The foci are along the semi-major axis in the coordinates $(0, -\sqrt{a^2 - b^2})$ and $(0, \sqrt{a^2 - b^2})$ for an ellipse centred in the origin.

An ellipse is quantified by its eccentricity, in other words, how far away from a perfect circle it is. A perfect circle has eccentricity 0, and the eccentricity is given by the following formula:

$$e = \sqrt{1 - \frac{b^2}{a^2}} \quad (7)$$

The maximum value for the eccentricity is 1, and that is technically an infinitely straight line. We don't call it a straight line, though, rather a "degenerate ellipse".

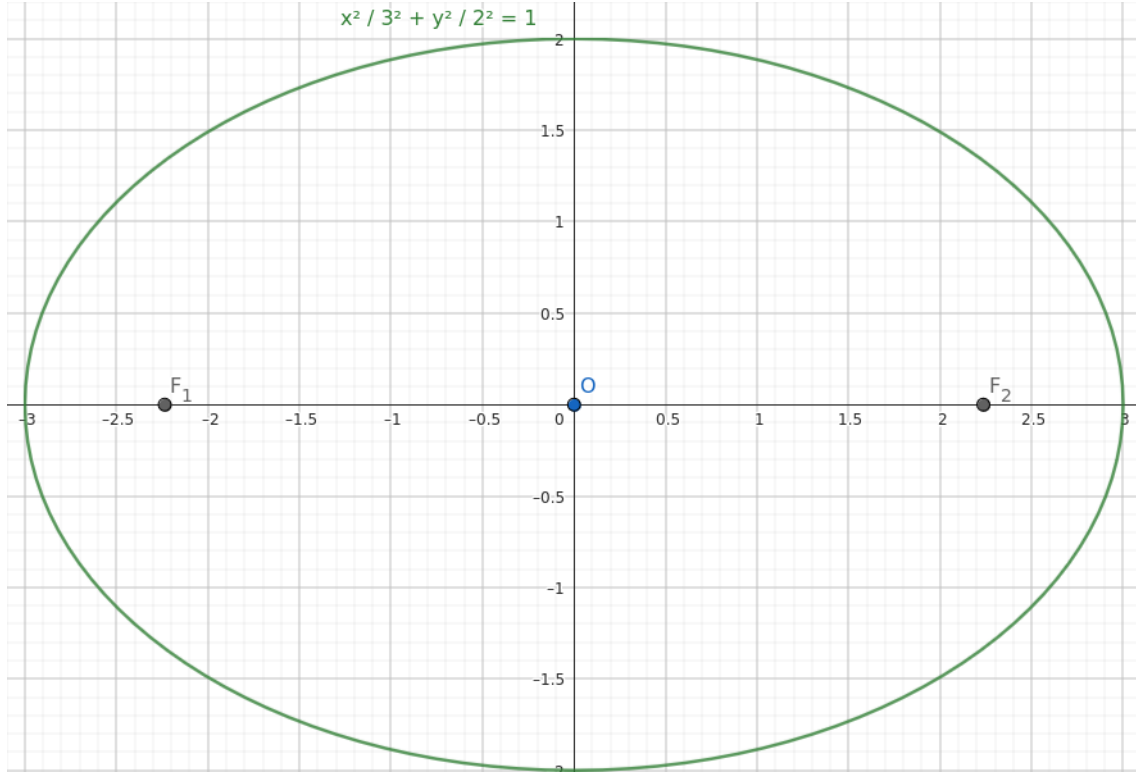


Figure 1: Geogebra ellipse centred in origin with $a = 3$, $b = 2$

We now have the necessary basis to describe orbits. An orbiting body will follow an elliptical path, and the body it orbits around will be in a focus point. The focus of an elliptical orbit will technically be in the centre of mass of the system, but for star-planet systems, the masses of stars are much larger than the masses of planets that the centre of mass coincides reasonably with the star in question. For binary star systems, we have two massive stars, so the stars won't be in the foci, but the centre of mass of the system will be the focus of the orbiting-

The shortest distance from the orbiting body to the focus point is called the periapsis. If the body that is being orbited around is the Earth, the periapsis and apoapsis are called the perigee and apogee (from Greek "ge", meaning Earth), and if it's the Sun, perihelion and aphelion (from Greek "helios", meaning Sun). In any case, they're given by:

$$\text{periapsis} = r_{\min} = a(1 - e) \qquad \text{apoapsis} = r_{\max} = a(1 + e) \qquad (8)$$

The velocity in an elliptical orbit is described by the vis-viva equation, which has the following form:

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right) \qquad (9)$$

Vis-Viva Proof

$$\begin{aligned}
E_{\text{total}} &= -G \frac{Mm}{2a} \wedge E_{\text{total}} = \frac{1}{2}mv^2 - G \frac{mM}{r} \\
-G \frac{Mm}{2a} &= \frac{1}{2}mv^2 - G \frac{mM}{r} \\
GMm \left(-\frac{1}{2a} + \frac{1}{r} \right) &= \frac{1}{2}mv^2 \\
v^2 &= 2GM \left(\frac{1}{r} - \frac{1}{2a} \right) \\
v^2 &= GM \left(\frac{2}{r} - \frac{1}{a} \right)
\end{aligned}$$

This is a very powerful formula that you can use straight up if you know it, and it will greatly simplify a lot of elliptical orbital mechanics problems.

Another useful identity for orbital mechanics is the Virial Theorem that gives a relationship between the average kinetic and average potential energies as:

$$\langle E_k \rangle = -\frac{1}{2} \langle E_p \rangle \quad (10)$$

Remember that this is for average energies, and not instantaneous, so we can only see what happens with the energies in the long run by using this theorem! Since the total energy is the sum of the kinetic and the potential energies, for "the long run", we can write:

$$\begin{aligned}
E_{\text{total}} &= \langle E_k \rangle + \langle E_p \rangle = \\
&= -\frac{1}{2} \langle E_p \rangle + \langle E_p \rangle = \\
&= \frac{1}{2} \langle E_p \rangle = \\
&= -\langle E_k \rangle
\end{aligned}$$

4 Kepler's Laws

Kepler's three laws state the following:

1. **Law of Orbits:** The orbit of every planet is an ellipse with the Sun being one of the foci.
2. **Law of Areas:** As planets move, they sweep through elliptical arcs of equal area during equal amounts of time
3. **Law of Periods:** As the radius of orbit changes, the period of orbit changes according to the proportion $T^2 \propto a^3$, a being the semi-major axis. The constant of proportionality is $\frac{4\pi^2}{GM}$.

We have already talked about the first law in the section about elliptical orbits, so here we'll dive a bit deeper into the second and third laws.

For the second law, let's look at a small body in an elliptical orbit. In a very small time dt , the body goes a very small angle $d\theta$. Even though we have an elliptical orbit, the distance covered is so small that the area covered can be approximated by a triangle. If the distance to one of the foci is given as r , the area of the triangle is:

$$\begin{aligned}\frac{dA}{dt} &= \frac{\frac{1}{2}(rd\theta) \cdot r}{dt} = \\ &= \frac{1}{2}r^2 \frac{d\theta}{dt} = \\ &= \frac{1}{2}r^2\omega\end{aligned}$$

Object Falling into Star

A small object m is r away from a star with mass M and falling directly into it. If the mass of the small object is much smaller than the mass of the star, how long will it take for the object to fall into the star?

The path is a straight line, but as we saw before, in orbital mechanics, a straight line isn't a straight line, but a degenerate ellipse. If it's an ellipse, Kepler works, so let's apply Kepler's third law! The semi-major axis of this ellipse is half of the distance to the star, or $\frac{r}{2}$.

$$\begin{aligned}\frac{T^2}{a^3} &= \frac{4\pi^2}{GM} \\ T &= \sqrt{a^3 \cdot \frac{4\pi^2}{GM}} = \\ &= \sqrt{\frac{r^3}{8} \cdot \frac{4\pi^2}{GM}} = \\ &= \frac{2\pi}{\sqrt{8}} \sqrt{\frac{r^3}{GM}} = \\ &= \frac{\pi}{\sqrt{2}} \sqrt{\frac{r^3}{GM}}\end{aligned}$$

Now, the orbital period is technically double the time the small object takes to fall into the star, so:

$$t = \frac{T}{2} = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{r^3}{GM}}$$

5 Lagrange Points

A very good practical use of orbital mechanics is in spacecraft. You might have previously read something along the lines of "The James Webb space telescope is located in L2". In this section, we'll hopefully explain what you need to understand what that sentence means, why these Lagrange points are useful and where they come from!

When we speak about Lagrange points, we often refer to these points as defined by the Earth-Sun system, but any and all two-body systems have Lagrange points. These are five points where the gravitational forces from the two big bodies as well as the centrifugal force of the entire rotating system balance out to give regions of enhanced attraction and repulsion, and they are numbered and named L_1 , L_2 , L_3 , L_4 , and (you guessed it) L_5 .

The physics and mathematics needed to find these points are outside of the scope of what you need to know, but you should know that L_1 , L_2 and L_3 lie on the line connecting the two big bodies, and that L_4 and L_5 are placed such that they make two equilateral triangles with the massive bodies, like shown in Figure 2.

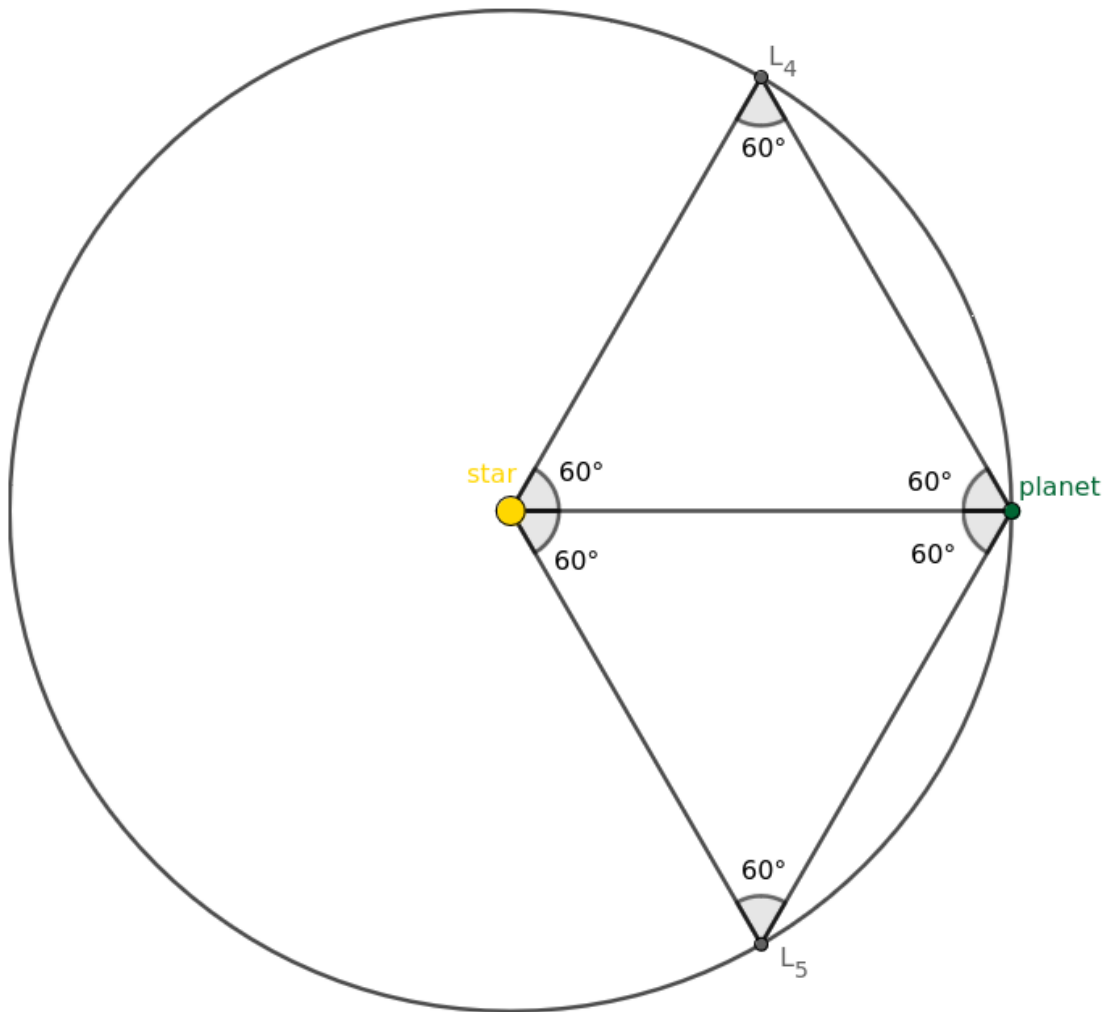


Figure 2: Graphical representation of relationship of points L_4 and L_5 compared to the massive bodies

Figure 4 shows what we call a contour map, where each closed line represents a region along which the gravitational potential energy is equal. You can think of this contour map just like the contour map of a mountain that you might have seen in geography, shown in Figure 3! The places where

the contours intersect are points L_1 , L_2 and L_3 , while L_4 and L_5 are placed where the contour is at its largest.

Because of this, we can further conclude that L_1 , L_2 and L_3 are unstable Lagrange points. An unstable point is one where we can put a small mass in equilibrium, but any small perturbation would send it flying away. Contrary to this, L_4 and L_5 are stable points, meaning that whatever small object we put there will keep at those points.



Figure 3: A contour map of a mountain.

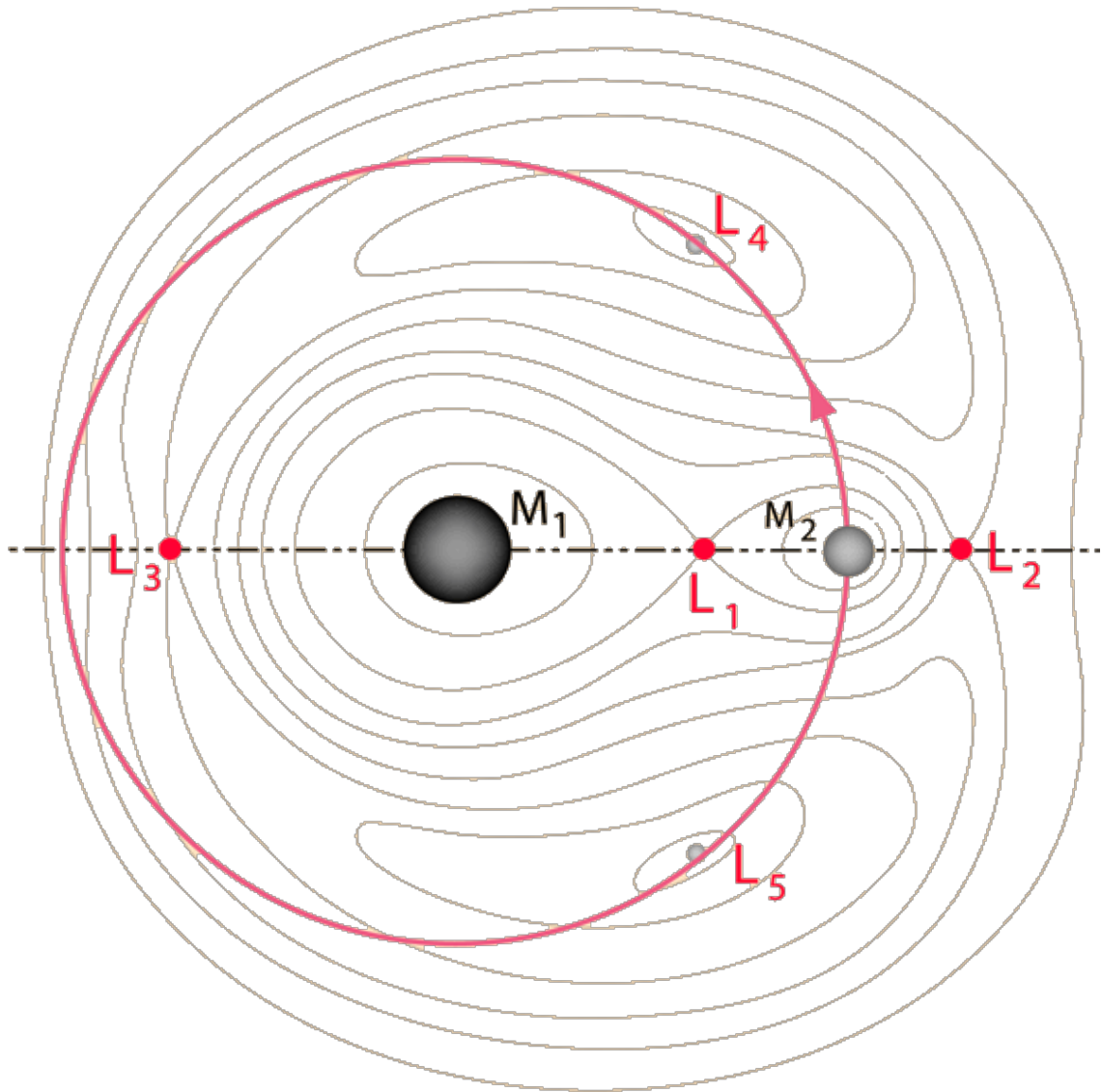


Figure 4: A contour map of gravitational potential.

Now we know where the Lagrange points are and why they're there, so we can finally see why they're useful:

- L_1 is extremely useful - whatever we put there will constantly be between the Sun and the Earth, and it is therefore perfect for solar observations. SOHO (Solar and Heliospheric Observatory) was launched and placed in L_1 in 1995 and gives us valuable data on the sun.
- L_2 is ideal for deep space observations and is where the James Webb Space Telescope is located. It's close to Earth, so we can easily communicate with spacecraft there, and since it's in Earth's shadow, the interference from the Sun is substantially reduced.
- L_3 is always behind the Sun, so it's very inaccessible and we currently have little use for it.
- L_4 and L_5 are stable, and a lot of space junk accumulates there. Other celestial objects in L_4 and L_5 are called trojans, after Achilles, Agamemnon and Hector, the three asteroids in L_4 and L_5 of the Jupiter-Sun system. No trojan satellites exist in the Earth-Sun L_4 and L_5

The reason we place probes and telescopes in L_2 rather than L_4 or L_5 for deep space observation is to minimise the signal noise from the Sun. Recall that L_2 is an unstable Lagrange point, so the way satellites are "kept" at L_2 is by small course corrections that don't utilise that much energy. This holds for L_1 as well, only that we usually put solar observation missions there.

6 Stable Satellite Orbits

As you probably know, smaller objects that orbit planets are called satellites. Some planets have multiple satellites that orbit the planet at large radii. This means that there is a range of radii which the satellite orbits can have and still be in a stable orbit around the planet. If we have a host star that is massive in comparison to the planet in question, this range is determined by the Roche limit (lower range) and Hill sphere radius (upper range).

The Hill sphere radius is the region where the gravity of the planet dominates. If we look at Figure 4, the planet's gravity dominates up until L_1 and L_2 .

The Roche limit is defined as how close a satellite can come to its host planet without disintegrating. It's speculated that the debris that Saturn's rings comprise of might come from a past satellite that went past its Roche limit and disintegrated.

If the radius of the satellite's orbit lies between the Roche limit and the Hill sphere radius, we can say that the satellite has a stable orbit. Remember that any two-body system technically orbits its centre of mass! Usually, one of the bodies is massive enough that the centre of mass of the entire system coincides with the centre of mass of the massive body, but watch out for systems that have masses in the same order of magnitude!