



THE NORWEGIAN ASTRONOMY AND ASTROPHYSICS
OLYMPIAD COMMITTEE

IOAA NOTES

Part 2: Spherical Geometry and Coordinate Systems

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1 Basic Spherical Geometry

NB: This handout assumes you are comfortable with basic trigonometry

This is probably the most encompassing part where a lot needs to be covered - so far, you've been used to 2D geometry and trigonometry, but now we go up a dimension. Contrary to what some internet skeptics will have you believe, our Earth is a sphere, and with that come a lot of extremely cool (and complex!) properties! This handout will take you on a journey through basic spherical trigonometry and then go on to our little blue sphere and show you coordinate systems, timekeeping and other phenomena that will give you an extremely good basis to do observation!

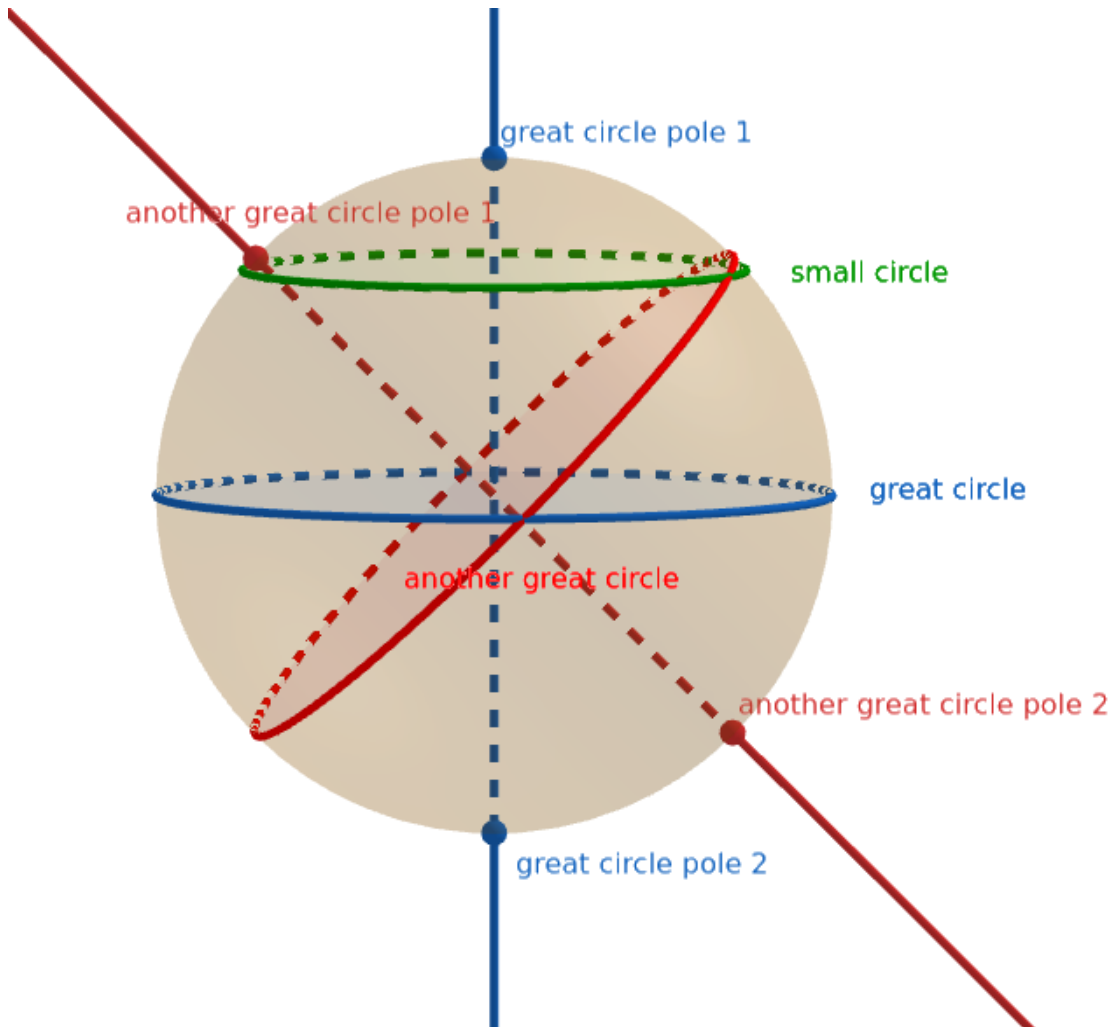


Figure 1: A basic sphere with marked great circles, their corresponding poles, and a small circle, drawn in Geogebra

As you can see in Figure 1, the basics of spherical geometry are none other than circles! We differ between great circles, whose diameter is the diameter of the sphere, and small circles that are any other circles on the sphere that can't be defined as great.

Great circles also have poles! Poles are intersections between the sphere surface and the line normal to the circle surface. If we were to apply this analogy to the Earth, the Equator would be a great circle, and the North and South poles would be its poles!

In spherical geometry, we also have triangles, but these are a little bit different to what you might be used to! A so-called spherical triangle is found at the intersection of three great circles. In Figure 2, you can see the spherical triangle $\triangle ABC$.

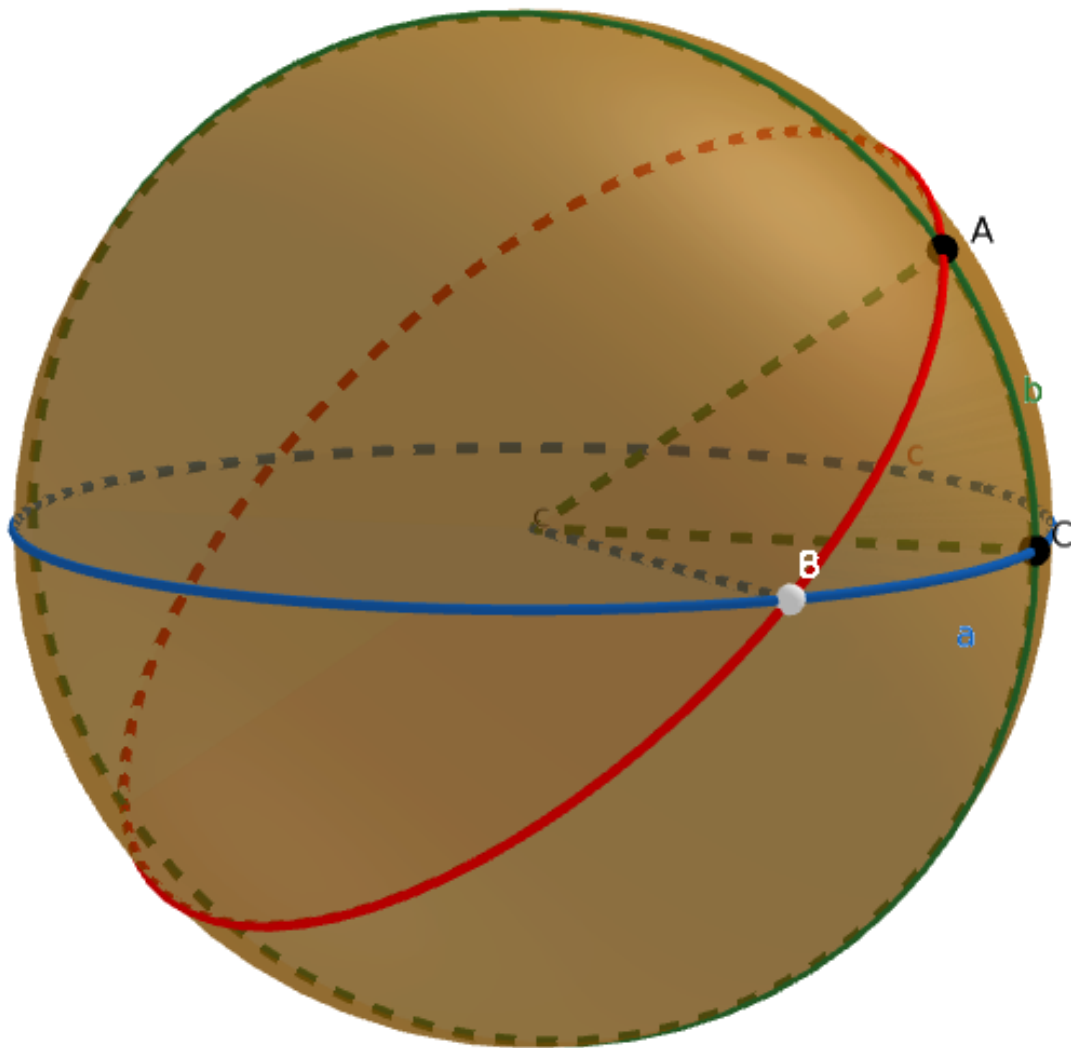


Figure 2: A spherical triangle $\triangle ABC$ constructed by intersecting three great circles.

Contrary to what you are used to from 2D geometry, spherical triangles have the fun property that the sum of their internal angles is greater than 180° ! We have isolated the spherical triangle $\triangle ABC$ in Figure 3. With spherical triangles, we use capital letters to denote the angles, and small letters to denote the arcs that make up the sides of a spherical triangle.

This important identity can be written as:

$$A + B + C > 180^\circ \quad (1)$$

Through (1), we can define what's called the spherical excess as the difference between the sum of the angles in a spherical triangles and 180° - in other words:

$$E = (A + B + C) - 180^\circ \quad (2)$$

This takes us to an amazing property of spherical triangles where the area can be expressed via the spherical excess simply enough as:

$$A = Er^2 \quad (3)$$

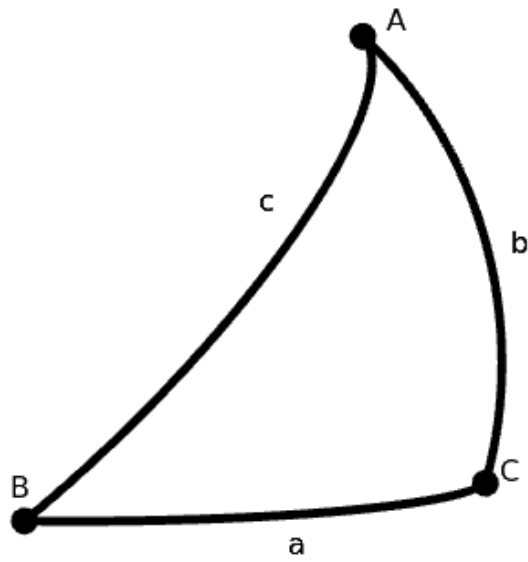


Figure 3: The isolated spherical triangle

The side lengths are often not expressed in units of length, but rather as angles themselves, just like arcs of a circle in 2D (which is what they essentially are).

2 Spherical Geometry of the Earth

For purposes of geometric analyses, we will ignore what we already know of the Earth being shaped like an oblate spheroid, and regard it as a perfect sphere. As such, the great circle whose poles determine the axis of rotation of the planet is the Equator. Any small circle parallel to the equator is called a parallel of latitude - in Figure 4, this is the red small circle.

Any great semicircle connecting the poles is a meridian. Measuring along the meridian, the latitude ϕ is defined as the angle measured from the Equator to the observer. The longitude λ is the angle measured from the Greenwich meridian (say, the green great semicircle) to the local meridian. Conventions vary, so if you look more into it, you will see people use both positive eastwards and positive westwards, but the more common one is positive eastwards, and that's what we're adopting here as well.

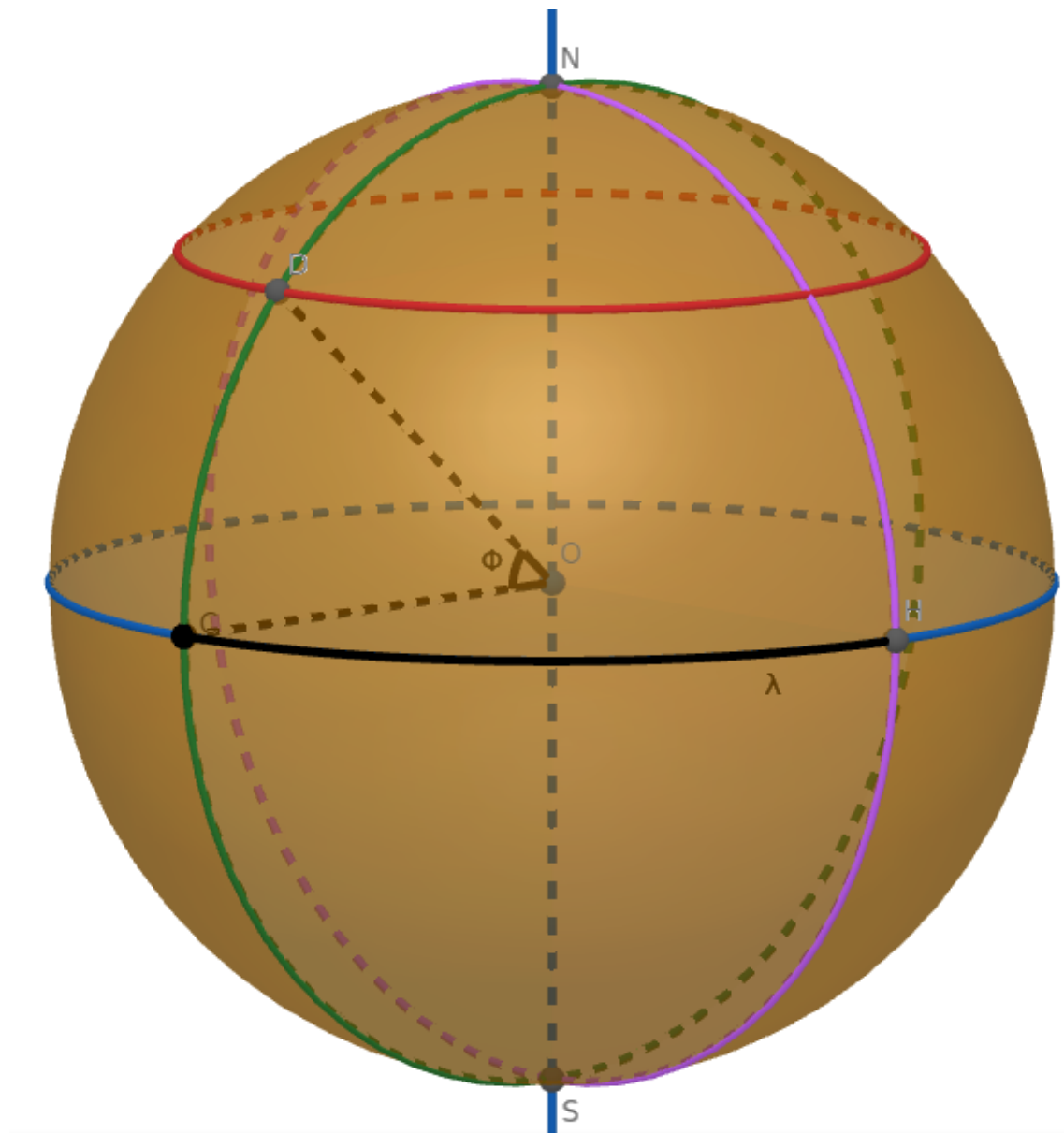


Figure 4: A spherical model of the Earth.

If two observers are located on the same parallel (e.g. both are somewhere along the red small

circle), then the distance between them is called the departure and is given by:

$$d = \Delta\lambda \cos \phi \quad (4)$$

The proof for this is simple: if we look at a cross-section of the Earth, we can notice that it's a circle, just like in Figure 5. The radius of any small circle is therefore $r \cos \phi$, and the rest is straightforward to show.

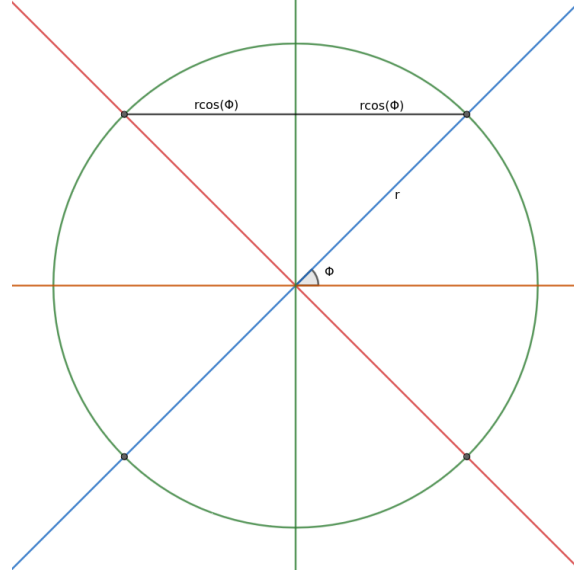


Figure 5: Cross-section view of Earth to find radius of small circle given latitude

The spherical properties of Earth are used for navigational purposes, where we define the unit nautical mile as the arc length of 1 arcmin. Just like with actual hours, minutes, and seconds, we have that:

$$1^\circ(\text{deg}) = 60'(\text{arcmin}) = 3600''(\text{arcsec}) \quad (5)$$

3 Spherical Trigonometry

Now, since we have triangles on a sphere, we also have trigonometry we can apply to the spherical triangle, just like normal trigonometry on a $2D$ one! Remember that we technically have 6 angles, since we regard the arcs comprising the spherical triangle as angles themselves!

The two most basic identities are the sine and the cosine laws for spherical triangles. For any spherical triangle $\triangle ABC$ like the one in Figure 3, the sine law states:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad (6)$$

and the cosine law:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad (7)$$

or

$$\cos A = \cos B \cos C + \sin B \sin C \cos a \quad (8)$$

These can actually be expanded to 10 different equations, but those are a bother to memorise! Luckily, the genius that was John Napier made a mnemonic that reduces needing to remember 10 things to needing to remember 3. The first and foremost part of Napier's Rules is the circle shown in Figure 6. This works only on spherical triangles where one of the angles is 90° !

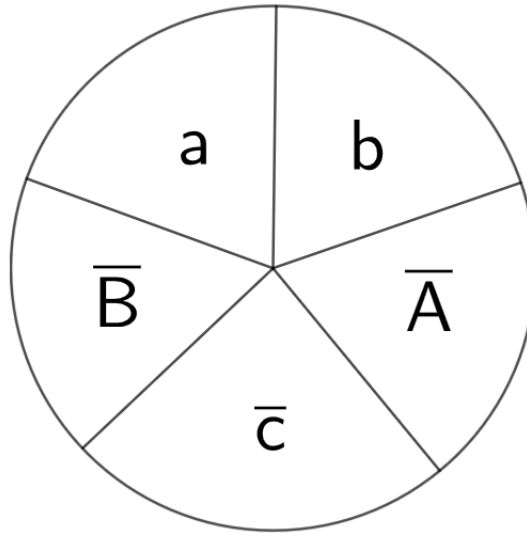


Figure 6: Napier's Circle

Now, say that we want to find the side a . It can be expressed either as $\sin a = \tan \bar{B} \cdot \tan b$ or $\sin a = \cos \bar{A} \cdot \cos \bar{c}$. The rules are:

1. The sine of an angle in the circle is equal to the product of the tangents of its adjacents in Napier's circle ($\sin \text{angle} = \tan \text{adjacent}_1 \cdot \tan \text{adjacent}_2$).
2. The sine of an angle in the circle is equal to the product of the cosines of its opposites in Napier's circle ($\sin \text{angle} = \cos \text{opposite}_1 \cdot \cos \text{opposite}_2$).

The lines above some of the angles and sides indicate that the angles in question are complementary to the angles represented by the letters. In other words, for any angle X , the following identity holds:

$$\bar{X} + X = 90^\circ \quad (9)$$

This is an incredibly powerful tool for finding unknowns in spherical trigonometry!

4 Celestial Coordinate Systems

The basis of locating anything in space is in coordinates, whether it be here on Earth or far away in space. When we do astronomical observations on our night sky, we don't place celestial objects in a Cartesian xy coordinate system like you're used to - rather, we use two different coordinate systems: horizontal (alt-az) and equatorial. In the observational part, you will see how telescopes are mounted differently to follow both of these.

These coordinate systems are based on the concept of the celestial sphere. We as the observers are inside of this sphere, and celestial objects are placed on its surface. Since we have a sphere, it is only natural to describe the coordinates in terms of angles. We also know that since the Earth rotates, we seemingly observe that the celestial objects move from east to west. Instead of having the earth rotate when we talk about coordinates, we say that the celestial sphere is rotating instead.

4.1 Horizontal Coordinate System

In the horizontal coordinate system, the observer is the origin, or the centre of the celestial sphere. The coordinates we have in this system are called altitude and azimuth, and the system is often referred to as alt-az.

A sketch of a basic alt-az system is shown in Figure 7. Any great semicircle arc connecting the zenith to the Equator is called a vertical. In Figure 7, an example of a vertical is the semicircle arc connecting the zenith to the red point, and going through the star.

The Equator plane is referred to as the horizon - an observer can see all celestial objects above the horizon, and no celestial objects below the horizon. The cardinal directions are just like you know them from everyday speak - north, south, east and west. Two more points you should be aware of are zenith, which is the point on the celestial sphere directly above the observer, and nadir, which is the point on the celestial sphere directly below the observer (under the horizon).

As stated before, any celestial object (say, a star) is on the surface of the celestial sphere and can be described by two parameters: altitude and azimuth. Think of altitude as a sort of height, i.e. how far above the horizon a star is. With that definition, the altitude is the angle spanned by the vertical from the horizon to the star. Altitude is positive above the horizon and negative below - this means that if you see a negative altitude, that object is not visible for the observer in question!

The azimuth is the angle measured from the cardinal North to the point where the vertical going through the star intersects the horizon (from North to the red point in Figure 7). The convention used at IOAA is positive eastwards, but sometimes the organisers might give another convention, so read and follow along with what convention is used. If you are unsure, always write your assumptions down!

In this coordinate system, we define the culmination as the point where the star reaches its maximum altitude. This system is good for observation, but it is difficult to generalise it, since it is observer-dependent. Both of the parameters for the star coordinates vary with time - just like the Sun, any star will rise in the east, culminate and set in the west. By definition, we say that a star that crosses the horizon culminates twice - once above the horizon and once below!

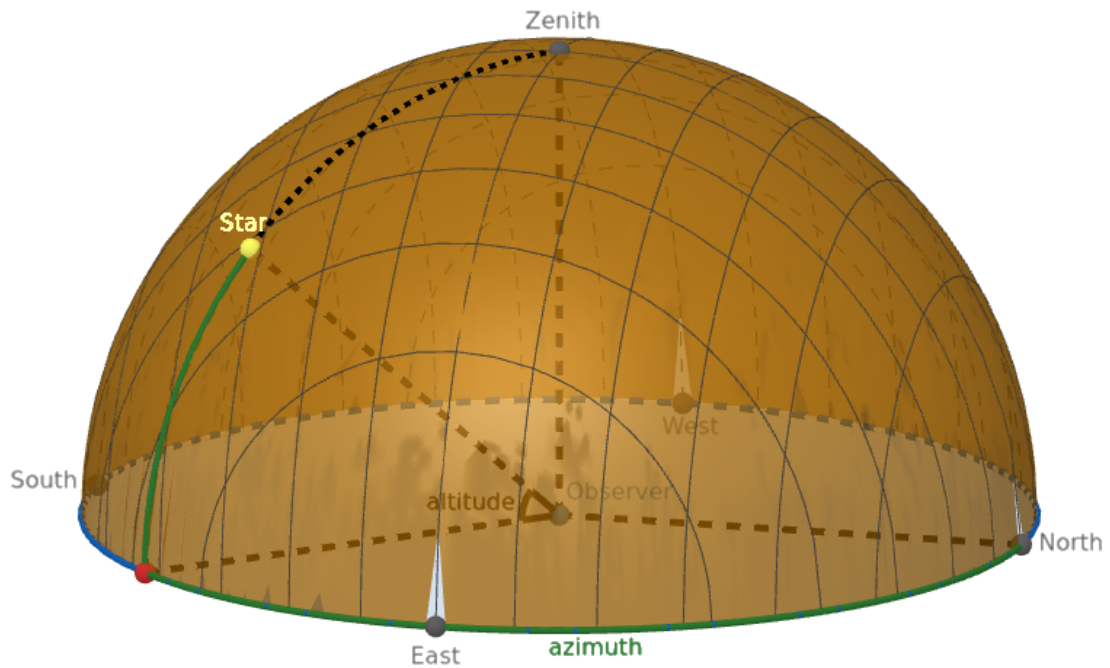


Figure 7: A sketch of the horizontal coordinate system

4.2 Equatorial Coordinate System

With this coordinate system, the origin is placed in the centre of our planet, meaning that it is completely independent on who is observing where. The celestial sphere can be described by the so-called north celestial pole and south celestial pole, as well as the celestial equator, which all match up with their Earth analogues.

Figure 8 shows an observer, just like in Figure 7, but instead of the sphere being centred at them, the observer is now standing inside of the celestial sphere. Notice how the North and South Celestial Poles don't coincide with zenith and nadir.

The parameters for the star's coordinates are declination, which does not change with time, and the hour angle, which does. The declination is measured along the star's meridian from the celestial equator, with positive being north of the celestial equator. For any given celestial object, this parameter is the same for all observers, no matter where they are.

The hour angle is measured westwards from the observer meridian, which is defined as the circle arc connecting the cardinal North, Zenith, and cardinal South, and it is dependent on both time and the observer's latitude. What is special about the hour angle is that it "resets" - it always increases with time from 0^h to $24^h = 360^\circ$, after which it resets back to 0^h . In other words, we always take it to be positive. If you do find a negative hour angle, just add 24^h .

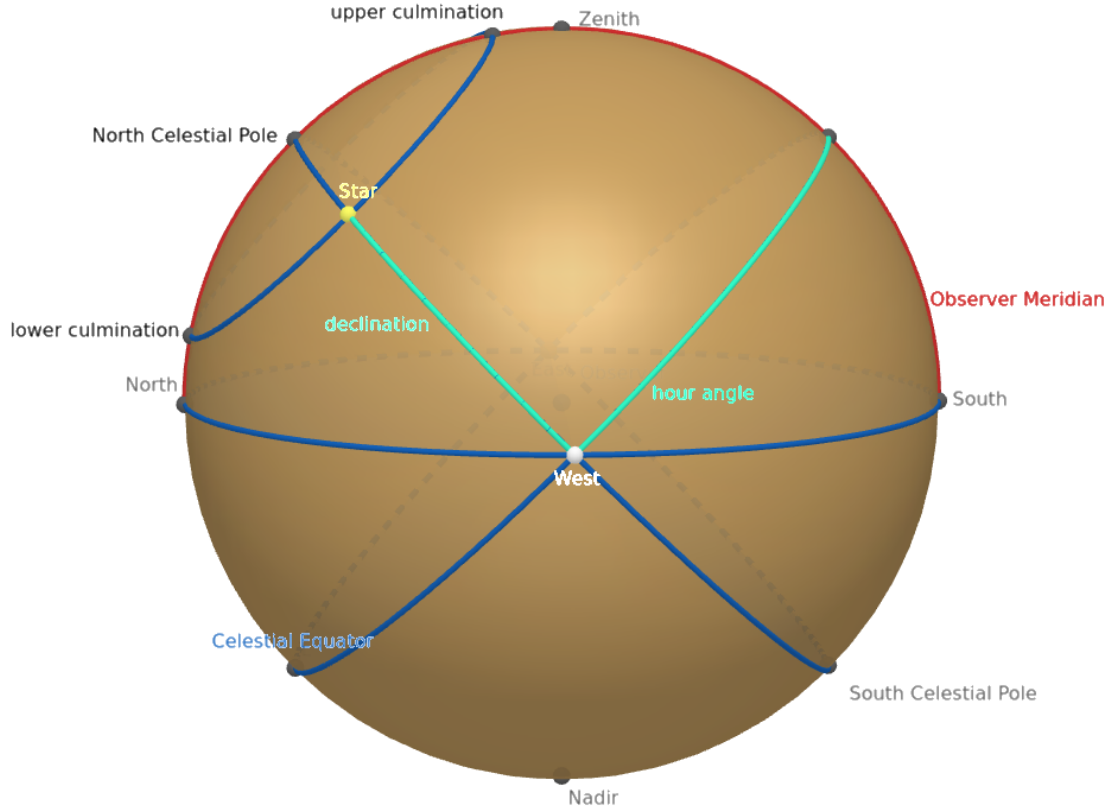


Figure 8: A sketch of the equatorial coordinate system for an observer on the Northern Hemisphere.

It is not ideal to have a time- and position varying coordinate such as the hour angle, so an alternative that is used more is called right ascension. To be able to define right ascension, we need an understanding of what the ecliptic is. As we know well, the Earth orbits the Sun, and revolves around its own rotational axis. These two rotational planes are not in alignment, with the Earth's self-rotation axis being tilted 23.5° . The ecliptic is, then, the apparent path the Sun follows as seen from the Earth.

Now we have the tools we need to define right ascension - its reference point is what we call the "vernal equinox". The vernal equinox is the point where the ecliptic plane crosses the celestial sphere equator. Measuring from the vernal equinox eastwards along the equatorial plane gives us the right ascension angle. This is now completely independent of the observer's meridian, and therefore the observer's position, and it is also fixed in time!

4.2.1 Culmination

The culmination points are defined as the points where the star's meridian crosses the observer's meridian. We differ between upper culmination and lower culmination, depending on what altitude the star has on culmination. No matter the culmination, the hour angle is 0, while the azimuth is either 0° or 180° .

A given star can culminate either north of zenith, or south of zenith. If it culminates north of zenith, the observer is located at latitude ϕ , and the star has declination δ , we have:

$$a_{\max} = 90^\circ + \phi - \delta \quad (10)$$

$$a_{\min} = \delta + \phi - 90^\circ \quad (11)$$

If it culminates south of zenith, the altitude is simply $180^\circ - a_{\text{north}}$:

$$a_{\text{max}} = 90^\circ - \phi + \delta \quad (12)$$

$$a_{\text{min}} = 3\delta - \phi + 90^\circ \quad (13)$$

The specific star marked in Figure 8 is also circumpolar, meaning that it never sets below the horizon. Mathematically speaking, this means that $a_{\text{min}} > 0$. Non-circumpolar stars still culminate twice, only with one of the culminations having negative altitude.

For circumpolar stars that culminate north of zenith, we can derive the following expression:

$$\begin{aligned} a_{\text{max}} - a_{\text{min}} &= (90^\circ + \phi - \delta) - (\delta + \phi - 90^\circ) \\ a_{\text{max}} - a_{\text{min}} &= -2\delta + 180^\circ \end{aligned}$$

This is for stars that culminate north of zenith and it gives:

$$\delta = \frac{1}{2} (a_{\text{min}} - a_{\text{max}} + 180^\circ) \quad (14)$$

We can do an equivalent derivation for circumpolar stars that culminate south of zenith, for which we find:

$$\delta = \frac{1}{2} (a_{\text{min}} - a_{\text{max}}) \quad (15)$$

In both cases, we can see that the declination of the star is completely independent on latitude, meaning that if you know approximately where a certain star culminates, you can find its declination by performing observations!

4.3 Ecliptic Coordinate System

The ecliptic coordinate system is described by ecliptic longitude λ and ecliptic latitude β . Just like right ascension, they are measured from the vernal equinox, with the longitude going along the ecliptic, and the latitude going from the ecliptic plane to the ecliptic north pole.

Since all planets in the Solar System orbit the Sun in approximately the same plane, this system is very useful for representing positions of objects in the Solar System with constant parameters.

4.4 Galactic Coordinate System

The galactic coordinate system is a system with its origin in the Sun. The galactic equator is then the intersection of the celestial sphere that has the Sun in its centre and the plane made by the disk at the centre of our galaxy. It's useful to give constant coordinates for objects in the Milky Way.

The following table provides coordinate conversions between the different systems. Proving these is outside of the scope of this paper, but it is often expected that one is familiar with these formulae at the IOAA.

To / From	Horizontal	Equatorial
Horizontal	/	$\sin a = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$ $\cos A = \frac{\sin \delta - \sin a \sin \phi}{\cos a \cos \phi}$
Equatorial	$\sin \delta = \sin \phi \sin a + \cos \phi \cos a \cos A$ $\cos H = \frac{\sin a - \sin \delta \sin \phi}{\cos \delta \cos \phi}$	/
Ecliptic	/	$\sin \beta = \cos \epsilon \sin \delta - \sin \epsilon \cos \delta \sin \alpha$ $\sin \lambda = \frac{\sin \delta - \sin \beta \cos \epsilon}{\cos \beta \sin \epsilon}$
Galactic	/	$\sin b = \sin \delta_G \sin \delta - \cos \delta_G \cos \delta \cos (\alpha - \alpha_G)$ $\cos (\theta - l) = \frac{\sin \delta - \sin b \sin \delta_G}{\cos b \cos \delta_G}$

5 Concept of Time

The word sidereal (pronounced si-de-real, not side-real) is often seen in problem sets and when talking about astronomy. It comes from the Latin word "sidus", meaning "star", so stars are used as reference for time. More concretely, a sidereal day is the period between two passages of the vernal equinox across the observer meridian. On Earth, the sidereal day is approximately 4 min shorter than the 24^h day you are used to.

The day-to-day definition of a day (heh) is what we in astronomy call a solar day, meaning that the Sun is used as reference. Analogously to the sidereal day, a solar day is the period between two passages of the Sun across the observer meridian. The reason for the difference between a solar day and a sidereal day is that the Earth orbits the Sun and therefore needs to compensate a little by turning around its own axis so that the Sun comes into the same position across the observer meridian again.

With a sidereal day, the hour angle of the vernal equinox goes from 0 to 24^h, and we call this hour angle the local sidereal time. For any object X (so not necessarily the vernal equinox), we can write:

$$\text{local_sidereal_time} = LST = H(X) + RA(X) \quad (16)$$

and knowing that by definition $RA(\text{vernal equinox}) = 0$, we know that $LST = H(\text{vernal equinox})$. Since LST depends on the hour angle, it is also location-specific.

The solar days we have talked about so far are mean solar days that last 24^h. The reason for using mean solar days is clear once we look at an apparent solar day. An apparent solar day is the definition we mentioned before, for which an hour of passed time makes the hour angle of the Sun increase by an hour (huh, lots of hours, keep track!), but the problem is that because of the inclination of Earth's orbit, an apparent solar day is not constant.

To make calculations easier, we use a fictitious dynamical mean Sun that moves along the ecliptic so that the inclination effects are eliminated. Remember that the real Sun moves along the celestial equator, so because of orbital eccentricity, if we use the ecliptic as reference, it will move in a non-constant manner. If we plot the differences in right ascensions of both the dynamical mean sun and the true sun, we will see the effect of eccentricity of Earth's orbit. In Figure 9, this is the red dashed line. The blue dashed line represents the obliquity effect from the fact that Earth's rotational axis is tilted.

Summing both these effects gives the black line that we call the equation of time. If you take long exposure photographs of the Sun every day, it will trace an 8-like shape that we call the analemma. The north-south component of the analemma is due to this equation of time, while the east-west component is due to the declination of the Sun.

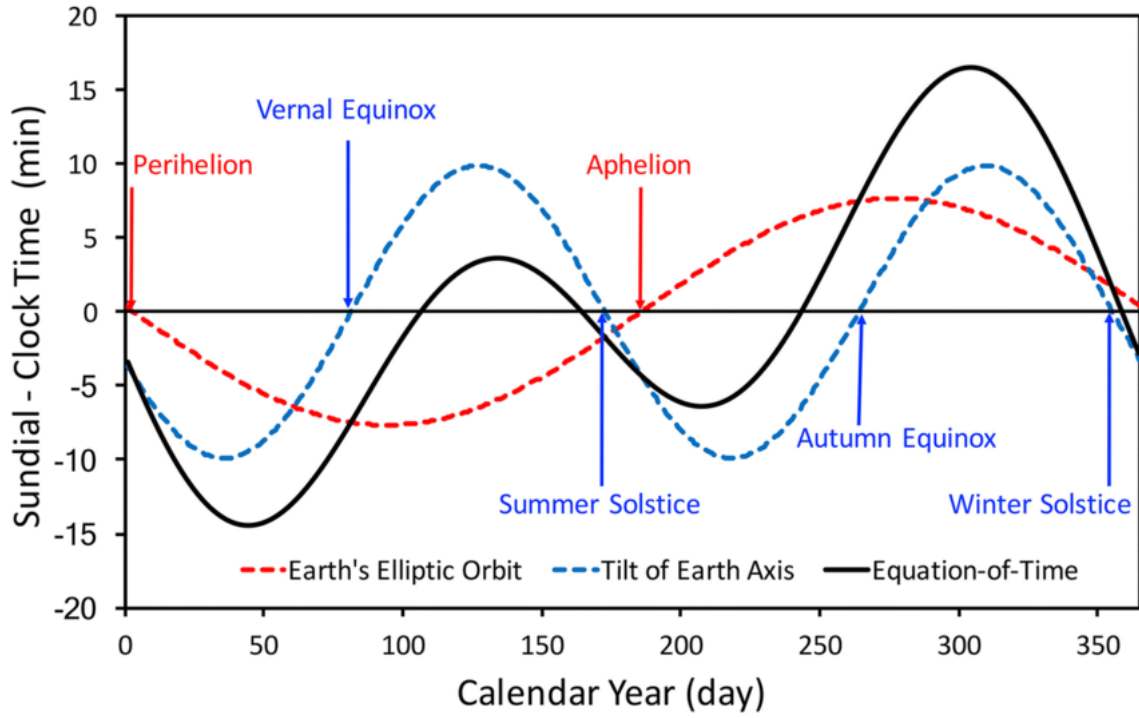


Figure 9: Equation of time along with its components

The modern calendar we use today in the Western world is the Gregorian calendar. Before this, the Julian calendar was dominant. Even though we have gone past the Julian calendar for timekeeping, we still use the Julian Date Number System. This is because when the shift from Julian to Gregorian happened, it was much more difficult to compare lengths of times between observations and still remain unambiguous as to what is going on.

The Julian Date Number System counts days from the mean noon of January 1st 4713 BC (the beginning of the Julian period). You will see satellite orbit data expressed in the Modified Julian Date Number System, for which the zero point is taken at mean noon November 17th, 1858 (has to do with computer memory limitations in the 50s and 60s). The modified Julian date is therefore:

$$JD_{\text{modified}} = JD - 2\,400\,000.5 \quad (17)$$

where the .5 means that the modified Julian day starts from midnight so as to have night time observations fall on the same day.

Finally, we have the Heliocentric Julian day that compensates for the extra length light from stars has to travel due to Earth's orbit. The Heliocentric Julian Day is the timing we would get if we were located in the centre of our Sun and it is given by:

$$HJD = JD - 0.0058^d \cos \beta \cos (\lambda_{\odot} - \lambda_s) \quad (18)$$

with β being the ecliptic latitude of the source, $\lambda_{\odot}$ the solar longitude, and λ_s the ecliptic longitude of the source.