

Vortex-wave interactions in Bose-Einstein condensates

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Samandrag

Me undersøker vekselverknad mellom kvervlar, og mellom kvervlar og bølger i veikt vekselverkande Bose-Einsteinkondensat i 2D. Me ser spesielt på kollisjonar mellom dipolar, og mellom ein enkelt kvervel og ei enkeltstående bølge. Desse prosessane er sentrale i avtakande kvanteturbulens. I dipolkollisjonane ser me på energioverføring frå den inkompressible kinetiske energien og den kompressible kinetiske energien. Me finn ut at når dipolane utsleitar einannan, går denne energioverføringa føre seg via *kvanteenergien*. Når avstanden mellom kvervlane er for stor til at dei kan utslette einannan, blir energien overført frå dei inkompressible modane til dei kompressible modane som følgje av akselerasjonen til kvervlane. For kollisjonen mellom den enkeltstående bølga og kvervelen oppdagar me at bølga splittar seg opp og dannar mellombels *kvasidefektar*. Desse gjennomgår eit topologisk byte med kvervelen, der kvervelen utsleittast av den eine kvasidefekten medan den andre kvasidefekten omdannast til ein stabil kvervel. Til slutt undersøker me avtakande kvanteturbulens i 2D direkte. Me brukar utslettingsraten til kvervlane og energispektra til dei ulike energiformane til å analysere turbulensen.

Abstract

We study vortex-vortex interactions and vortex-wave interactions in 2D weakly interacting Bose-Einstein condensates. In particular we focus on head-on collisions between dipoles and collisions between a solitary wave and a vortex. These processes are central constituents of decaying quantum turbulence. For the dipole-dipole collisions we study the energy transfer from the incompressible kinetic energy to compressible kinetic energy and find that during the annihilation event this transfer is mediated by the *quantum energy*. Above the threshold for vortex annihilation there is a transfer of energy from the incompressible modes to the compressible modes due to the acceleration of the vortices. For the collision between the solitary wave and the vortex we found that the wave splits up, forming temporary *quasi-defects*, which undergo a topological exchange with the vortex. The vortex is annihilated by one of the quasi-defects, and the remaining quasi-defect develops a stable core and becomes a stable defect. Finally we study 2D quantum turbulence directly, using the vortex decay rate and the energy spectrum associated to the different energy modes for analysis.

Takk

Eg vil først og fremst takke Luiza for begeistring, stort engasjement og tolmod. Eg er takksam for å ha fått høve til å utforske kvantekverlar med deg, og at du har vert open for å bruke tid på både dei simple, dei overambisiøse og dei gode spørsmåla. Det har bidratt til å gjere arbeidet meiningsfullt og kjekt. Takk til Jonas for gode tilbakemeldingar og klartenkte svar på alt mogleg rart.

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Chapter 1

Introduction

Since the first experimental realisation of a Bose-Einstein condensate (BEC) in 1995 [1] atomic BECs have become a prime platform for studying superfluidity and quantum turbulence [2]. In superfluids, vorticity is quantised and localised at discrete vortices. Quantum turbulence is characterised by the collective interactions of these quantum vortices. Since vorticity is quantised the vortices can only disappear by colliding with a vortex with the opposite rotational flow, an anti-vortex, or through leaving the system. In classical fluids vorticity varies continuously, and the kinetic energy gradually dissipates due to viscosity. Viscous dissipation is not present in quantum turbulence.

At finite temperature dissipation in BECs emerges through the interaction of the condensate with the thermal cloud, which is out of condensate states. In this case vortices and anti-vortices attract and annihilate with each other [3]. Within the Gross-Pitaevskii theory, the strength of this effect is set through the phenomenological damping coefficient, γ , which represents the dissipative effect on the condensate from interaction with the thermal cloud [2], i.e. $T > 0$ corresponds to $\gamma > 0$. At finite temperature then, vortex decay can be attributed primarily to a two-vortex process where a vortex and nearby anti-vortex attract each other and annihilate.

At zero temperature however, there is no mechanism of energy dissipation, and no central forces between vortices. The two-vortex decay process is not an available path for vortex annihilation [4] in this case. Still, the vortex number decays over time. Previous studies have identified a four-vortex process to be the dominant decay mechanism at $T = 0$ [4, 3]. In this process, the energy of a vortex anti-vortex pair (dipole) is transferred to a third vortex. If the energy of the dipole becomes low enough it collapses to form a solitary wave [5, 4, 3]. Despite the phase slips of the vortices having vanished, the solitary wave might later split up again forming vortices. The last step of the annihilation process then, is a collision between the solitary wave and a single vortex which disperses the energy of the solitary wave irreversibly into sound waves. It is possible however for other processes with more or less than four vortices to participate in the vortex decay [6], and the prevalence of different mechanisms depends both on trap geometry, the vortex density and the temperature of the condensate.

The details of the isolated annihilation events influence the long term evolution of the turbulent state. If the incompressible kinetic energy of the vortices is not transferred to other forms of energy during the annihilation event the incompressible kinetic energy per vortex in the system will increase. The incompressible kinetic energy of the vortices is dictated by their configuration, and vice versa. As the incompressible kinetic energy per vortex increases the vortices are forced into a high energy configuration. The extreme case of such a configuration are Onsager vortices, where all the like signed vortices

are jumbled together into clusters of like signed vortices [7]. This state minimises screening between opposite signed vortices, and amplifies the superfluid velocity around the clusters. 2D decaying quantum turbulence at $T = 0$ has been observed to generate clusters [8, 9, 4].

Atomic BECs are highly compressible, and vortices coexist with wave excitations. In particular, solitary waves are a form of compression waves which play a central role in vortex annihilation and collision events [5, 4]. Recently, Kanai and Zhang [6] found that the presence of sound waves influences the dominant decay mechanism in 2D decaying quantum turbulence. Furthermore the transfer of energy from incompressible to compressible modes during annihilation events, or through other mechanisms has a direct impact on the incompressible kinetic energy per vortex, which again dictates vortex configuration. The role of sound waves and compressible effects have been acknowledged in previous studies of vortex dynamics [10, 11] and 2D decaying quantum turbulence [12, 13, 14, 9, 3, 8], however they are still poorly understood.

This thesis is a theoretical and computational study of the dynamical patterns of collision events between vortices with themselves and with solitary waves. The overarching goal of this thesis is to better understand these collision events, their role in decaying quantum turbulence at zero temperature, and the role of compressible effects during these collision events. For this, we focus on a detailed study of collision events between dipoles, and between a vortex and a solitary wave.

The remaining part of the thesis is structured as follows: In Chapter 2, we begin with a brief introduction to Bose Einstein condensation and the Gross-Pitaevskii theory for the weakly interacting Bose gases. We will present Landau's criterion for superfluidity and show that atomic BECs are superfluid below a critical velocity. After this we will introduce quantum vortices and discuss quantum turbulence in analogy to classical turbulence. In Chapter 3, we will present the pseudo-vorticity field of the condensate as a way of tracking vortices and derive their equation of motion. Using this approach, we arrive at the point vortex model (PVM) in the zero-core approximation. We derive analytically the scaling laws of the kinetic energy spectrum for an isolated vortex. This is useful for comparison with the energy spectrum induced by many vortices. Numerical methods used to solve the GPE and the PVM are detailed in Chapter 4.

With these tools in place, we will address the main goal of this thesis in Chapter 5. We start by showing that the *isolated* dipole velocity derived from the GPE at $T = 0$ agrees with that obtained from the PVM and follows the theoretical scaling law. Namely that the dipole speed decreases inversely proportional to the dipole separation. Next, we study temporal evolutions and the transfer between different types of kinetic energy during a dipole-dipole collision for different initial dipole separations. We observe numerically that there is a critical initial separation below which this collision results in the pairwise annihilation of the four vortices. Above this critical threshold, they collide through partner exchanges which results in phonon radiation. By looking at the transfer of energy between vortices and waves, we find that the energy associated to quantum pressure plays an important role in mediated transport from the incompressible kinetic energy to the compressible kinetic energy.

We also investigate the collision of a solitary wave with a single vortex. We describe this process in detail and find a phenomena that has not been previously reported, to our knowledge. Before the collision, two phase slips form inside the wave due to interaction with the vortex. We call them quasi-defects. During the collision, one of the quasi-defects annihilates the vortex to form a wave, while the other one turns into a vortex. The collision takes place as a topological exchange. After the exchange the new vortex is

shifted from the position of the original vortex, and slowly drifts back towards the initial position of the original vortex. Finally, we present the results from a set of simulations of 2D decaying quantum a uniform condensate and in a disk trap at different damping coefficients, γ . We look the vortex decay rate and structure of the late stage vortex state using the energy spectra. A summary and final remarks are presented in Chapter 6.

Chapter 2

Theory

A Bose-Einstein condensate is a state of matter in which virtually all the particles of the Bose gas occupy the same lowest energy single-particle state [15]. This was first predicted by Albert Einstein in 1924 [16], but not experimentally realised until 1995 [1] due to the challenges of obtaining the extremely low temperatures necessary for the gas to condense [17]. Today experiments can be conducted with a great degree of control, enabling us to study vortex dynamics in specific configurations [11] as well as in chaotic regimes [18]. The fact that BECs can flow without viscosity results in a quantisation of the vorticity. This is a key feature which sets the dynamics of superfluids apart from classical fluids. Chaotic fluid flows are often termed *turbulent*, and in the case of the chaotic motion of vortices in superfluids we call it quantum turbulence.

In what follows we will establish the foundation for a description of quantum turbulence in weakly interacting BECs. We start off in Section 2.1 by showing that a non-interacting Bose gas has a capacity to condense into the lowest lying single-particle state, and discuss how this phenomenon translates to interacting systems in $2D$. Next, in Section 2.2, we introduce the order parameter Ψ , which is the main quantity for describing the macroscopic dynamics of a BEC. Then we derive the Hamiltonian for a weakly interacting Bose gas (Section 2.3) and calculate the dispersion relation of its fundamental excitations. We also find the evolution of the field operators for the Bose gas in Section 2.5, giving us an evolution equation for the order parameter called the Gross-Pitaevskii equation (GPE). We will see how this equation can be mapped to a set of hydrodynamic equations (Section 2.6) which describe an inviscid, i.e. superfluid, flow. In Section 2.7 we present Landau's criterion for superfluidity and see from the dispersion relation we derived for the fundamental excitations in Bose gas that the BEC is a superfluid when the velocity is lower than the speed of sound in the condensate. Next we discuss vortices (Section 2.8), which are topologically protected in the BEC, as well as a minimal model of their motion. Finally we will briefly discuss classical turbulence in Section 2.9 and quantum turbulence in Section 2.10.

2.1 Bose-Einstein Condensation

2.1.1 Condensation in 3D

In this section, we will show that the ground state of a Bose gas is macroscopically occupied below the critical temperature in the non-interacting case. Since real gases are interacting this serves first and foremost as a proof of concept. We start out by writing the particle number, N , as a sum of the occupation number of the ground state, N_0 ,

and the combined occupation number of all other states, N_T ,

$$N = N_0 + N_T. \quad (2.1)$$

We will refer to the excited states as the thermal cloud. To find an expression for the occupation number of the Bose gas we start off with the grand canonical partition function, which is the sum of the Boltzmann factors over all possible configurations of all possible particle numbers

$$Z = \sum_{\{n_i\}} e^{-\beta \sum_i n_i (\varepsilon_i - \mu)} = \sum_{\{n_i\}} \prod_i e^{-\beta n_i (\varepsilon_i - \mu)},$$

where $\{n_i\}$ indicates a configuration of particles, $\beta = 1/k_B T$. ε_i and n_i are respectively the energy and occupation number of a single-particle state i , μ is the chemical potential. Since the particles are independent we can factor out each single-particle state and write the grand canonical partition function as

$$Z = \sum_{n_0} \left(e^{\beta(\mu - \varepsilon_0)} \right)^{n_0} \sum_{n_1} \left(e^{\beta(\mu - \varepsilon_1)} \right)^{n_1} \dots$$

The grand canonical potential is given by

$$\Omega = -k_B T \ln Z. \quad (2.2)$$

For the Bose gas then, we find that the grand canonical potential can be expressed as

$$\Omega = k_B T \sum_i \ln \left(1 - e^{\beta(\mu - \varepsilon_i)} \right),$$

which can again be used to find the particle number

$$N = - \left(\frac{\partial \Omega}{\partial \mu} \right)_{T,V} = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} = \sum_i \bar{n}_i.$$

The occupation number of the ground state is then given by

$$N_0 = \bar{n}_0 = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}. \quad (2.3)$$

We see from Equation 2.3 that the chemical potential has to satisfy $\mu < \varepsilon_0$ to prevent negative occupation numbers, which would be unphysical. It is also clear that N_0 would diverge if $\mu \rightarrow \varepsilon_0$. If the chemical potential of the gas approaches the energy of the lowest lying single-particle state the occupation number of this state will drastically increase.

We consider a non-interacting Bose gas in a 3D box. The number of particles in the thermal cloud can be calculated from the density of states and the Bose-Einstein distribution $n(\varepsilon)$, which gives the average occupation number for a given energy,

$$N_T = \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) n(\varepsilon). \quad (2.4)$$

In this case the density of states is given by [15]

$$\mathcal{D}(\varepsilon) = \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \delta(\varepsilon - \varepsilon_n) = X \sqrt{\varepsilon},$$

where $X = \frac{V}{\sqrt{2}} \frac{m^{\frac{3}{2}}}{\pi^2 \hbar^3}$, and n_x, n_y and n_z are the quantum numbers associated with a single-particle state. Inserting the expression for the density of states and the Bose-Einstein distribution into 2.4 and doing the substitution $x = \beta\epsilon$ we get the result

$$N_T = X (k_B T)^{\frac{3}{2}} \int_0^\infty dx \frac{\sqrt{x}}{e^{-\beta\mu} e^x - 1}. \quad (2.5)$$

Say that we want to cool the gas keeping N_T and V constant as we do so. As the temperature decreases the chemical potential has to rise, increasing the value of the integral in Equation 2.5. At a critical temperature μ will approach its upper limit $\mu = 0$, which is the energy of the lowest lying single-particle state. Now the number of particles in the thermal cloud is given by

$$N_T = X (k_B T_c)^{\frac{3}{2}} \int_0^\infty dx \frac{\sqrt{x}}{e^x - 1}. \quad (2.6)$$

Above the critical temperature T_c , we can assume that most of the particles are in the thermal cloud. Approaching the critical temperature from above we can then insert $N \approx N_T$ into Equation 2.6 and find an expression for the critical temperature

$$k_B T_c = \left(\frac{N}{V} \frac{X}{2.315} \right)^{\frac{2}{3}}, \quad (2.7)$$

where the dimensionless integral in Equation 2.6 has been evaluated to 2.315 [19]. Below T_c , the chemical potential is equal to zero, we can rewrite Equation 2.6 in terms of T_c

$$N_T = \left(\frac{T}{T_c} \right)^{\frac{3}{2}} N. \quad (2.8)$$

Using this result along with 2.1 we can find the occupation number in the ground state

$$N_0 = N - N_T = N \left[1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}} \right].$$

As we keep cooling our gas a large proportion of the particles will de-excite from the thermal cloud into the ground state. Moreover, if we add particles to the system at a constant temperature below T_c the number of particles in N_T will not grow as the apparent dependence on N in Equation 2.8 is cancelled by the particle number hidden in T_c . It is clear that for a sufficiently dense gas at temperatures below T_c the ground state of the gas can be macroscopically occupied, $N_0 \gg N_T$. In other words, the gas condenses. If we were to do the same exercise as above for an ideal uniform Bose gas in 2D we would find a critical temperature $T_c = 0$, indicating that Bose-Einstein condensation in 2D is not possible. True 2D is not realisable in physical experiments, but it is possible to confine the Bose gas to be quasi 2D.

2.1.2 Condensation in 2D

We have shown that a uniform ideal Bose gas in 3D will condense into a single energy state below a critical temperature. This phase transition is a result of Bose statistics, not interactions and is in fact the only known case of a phase transition in a non-interacting system [20]. If we were to do the same exercise as above for an ideal uniform Bose gas in 2D we would find a critical temperature $T_c = 0$, indicating that Bose-Einstein condensation in 2D is not possible. Furthermore, regardless of whether or not the gas

is interacting, long wavelength thermal fluctuations would destroy the condensate in a uniform system [21, 20]. If we confine an ideal Bose gas in a trap however, the density of states is altered raising the critical temperature to a finite value [19]. The trap also quenches the long-range thermal fluctuations, which might act to stabilise the condensate [20]. In a realistic system we also need to account for interactions. Interactions give rise to phase fluctuations in the Bose gas and cause the condensate to deplete. As we cool an interacting Bose gas it will first undergo a phase transition called the Berezinskii-Kosterlitz-Thouless transition at T_{BKT} . Below T_{BKT} the gas becomes superfluid, and the coherence length increases. The coherence length does not span the entire system however. The coherence length is smaller than the system size and the Bose gas is in a quasicondensate regime [22]. Above the critical temperature T_{BKT} the phase coherence decays exponentially [19], while below this temperature the phase coherence decays as a power law [20]. In the Berezinskii-Kosterlitz-Thouless transition free vortices, which form spontaneously above T_{BKT} , become bound in vortex anti-vortex pairs. The bound vortices are not as disruptive to the quasicondensate phase and allow for a quasi-long range order, making it possible to represent the system with one condensate wavefunction Ψ , within an adequately small region [23]. As we keep cooling the condensate the coherence length will grow until we eventually reach a truly condensed phase [20, 19]. A transition from a BKT-dominated regime to a BEC-dominated regime can also be associated with decreasing the interaction strength [24]. One approach to effectively decreasing the interaction strength would be to make the gas more dilute. A smooth transition between a BKT-dominated phase to a BEC-dominated phase due to varying interaction strength has been observed experimentally [25].

2.2 Order parameter

To model the Bose gas we use the field operators as our starting point. The field operators $\hat{\Psi}^\dagger$, and $\hat{\Psi}$ are given by the bosonic creation and annihilation operators, $a_{\mathbf{p}}^\dagger$, $a_{\mathbf{p}}$ and the wave function associated with the particle state $\psi_{\mathbf{p}}$,

$$\begin{aligned}\hat{\Psi}^\dagger &= \sum_{\mathbf{p}} \psi_{\mathbf{p}}^* \hat{a}_{\mathbf{p}}^\dagger \\ \hat{\Psi} &= \sum_{\mathbf{p}} \psi_{\mathbf{p}} \hat{a}_{\mathbf{p}}.\end{aligned}$$

We can interpret the field operators as creating and annihilating wave functions. Their operation is not directly analogous to a physical process, but we can naively think of annihilating one state and creating another as the system transitioning from the first state to the other. The particle creation and annihilation operators obey the bosonic commutation relations

$$\begin{aligned}[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] &= [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}'}^\dagger] = 0 \\ [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] &= \delta_{\mathbf{p}, \mathbf{p}'},\end{aligned}$$

as do the field operators. When the Bose gas is condensed it makes sense to split the field operators into the condensate part, $\mathbf{p} = 0$, and the thermal part, $\mathbf{p} \neq 0$

$$\hat{\Psi} = \psi_0 \hat{a}_0 + \sum_{\mathbf{p} \neq 0} \psi_{\mathbf{p}} \hat{a}_{\mathbf{p}}.$$

We can represent condensed part of the gas with a classical field with occupation number N_0 , i.e. replacing $\hat{a}_0$ with the number $\sqrt{N_0}$. This is known as the Bogoliubov approximation [19]. We can now express the field operator as

$$\hat{\Psi} = \Psi_0 + \delta\hat{\Psi},$$

where $\Psi_0 = \sqrt{N_0}\psi_0$ and $\delta\hat{\Psi} = \sum_{\mathbf{p} \neq 0} \psi_{\mathbf{p}} \hat{a}_{\mathbf{p}}$. We call Ψ_0 the order parameter of the condensate. Because of the macroscopic occupation of the ground state the condensate acts as a reservoir and its properties do not change if we add or remove a particle. $|N_0\rangle$ is physically equivalent to $|N_0 - 1\rangle$. This has the unusual consequence that the expectation value of the field operator is different from zero,

$$\langle \hat{\Psi} \rangle = \langle N_0 | \psi_0 \hat{a}_0 | N_0 \rangle = \Psi_0 \langle N_0 | N_0 - 1 \rangle = \Psi_0,$$

where we have used the approximation for the annihilation operator $\hat{a}_0 = \sqrt{N_0}$. If the time evolution of the state $|N_0\rangle$ is given as $|N_0\rangle e^{-iE(N_0)t/\hbar}$, then the time evolution of the classical field is given by

$$\Psi_0(t) = \Psi_0 e^{i[E(N_0) - E(N_0 - 1)]t/\hbar} = \Psi_0 e^{-i\mu t/\hbar}, \quad (2.9)$$

since $\mu = E(N_0) - E(N_0 - 1) \approx \frac{\partial E}{\partial N}$. Therefore Ψ_0 oscillates with the chemical potential of the condensate.

We mentioned that Ψ_0 is an order parameter of the condensate [19]. To explain generally what an order parameter is we can consider an example that is easy to visualise. Say we open a box of matches and drop them on a table. Some of them will lie parallel to their neighbours while others are perpendicular to the neighbours, most will have some angle in between these extremes. In such a case we could replace the specific angles of each matchstick with a field that quantifies their degree of alignment. Such a field would function as a an order parameter. The order parameter characterises the degree of underlying order in systems with broken continuous symmetries [26]. In the case of the BEC the broken symmetry is a rotational symmetry in the phase of Ψ_0 [19]. The Bose gas is ordered if the phase of its wave function is coherent and disordered if the phase is decoherent. A BEC is characterised by quasi long-range order, the decoherence of the phase would cause the condensate to melt. This is what happens above T_c , as the order parameter vanishes. It is also possible for the order parameter to vanish below T_c . Generally points where the magnitude of the order parameter goes to zero are called topological defects. In a BEC the topological defects take the form of vortices. Apart from the topological defects there is no vorticity in the BEC. Since Ψ_0 is the wave function of the condensate the occupation number when Ψ_0 vanishes is zero, which is equivalent to saying that the condensate density is zero at the vortices. In the rest of this thesis we will simply use Ψ as the symbol of the order parameter.

2.3 Hamiltonian of a Bose gas

The Bose gas is considered dilute when the mean inter-particle separation is significantly larger than the range of the interaction between the atoms, that is

$$r_{\text{Int}} \ll l,$$

where r_{Int} is the range of the interaction and l is the mean distance between the particles. When this condition is fulfilled we can assume that the event of three particles coming

close enough to all interact with each other is far less likely than two particles interacting, and we can therefore assume that two-body scattering processes dominate the dynamics of the gas [19]. At low temperatures, and consequently low momenta, the scattering amplitude is independent of energy and angle of the incoming particles. The only parameter needed to characterise the interactions in the gas is then the s -wave scattering length a [19], and the condition of the gas being weakly interacting can be expressed as

$$|a| \ll l.$$

The Hamiltonian of the gas can now be written in terms of field operators as

$$\hat{H} = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} \nabla \hat{\Psi}^\dagger(\mathbf{r}) \nabla \hat{\Psi}(\mathbf{r}) + \hat{\Psi}^\dagger(\mathbf{r}) V_{\text{Ext}}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right] + \frac{1}{2} \int d\mathbf{r}' d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}') \hat{\Psi}^\dagger(\mathbf{r}) U(\mathbf{r}' - \mathbf{r}) \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}), \quad (2.10)$$

where $\hbar$ is Planck's constant, m is the mass of the bosons, V_{Ext} is the external potential and U is the interaction potential. For a uniform gas in a volume V we can write the field operators on the form

$$\hat{\Psi}(\mathbf{r}) = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} \frac{1}{\sqrt{V}} e^{i\mathbf{p} \cdot \mathbf{r} / \hbar}. \quad (2.11)$$

Inserting the field annihilation operator and equivalent creation operator from Equation 2.11 into the Hamiltonian (Equation 2.10) we get the one-body operator

$$\hat{H}_1 = \sum_{\mathbf{p}} \left(\frac{p^2}{2m} + V_{\text{Ext}} \right) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}},$$

and the two-body operator

$$\hat{H}_2 = \frac{1}{2V} \sum_{\mathbf{q}, \mathbf{p}_1, \mathbf{p}_2} U_{\mathbf{q}} \hat{a}_{\mathbf{p}_1 + \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_2 - \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_2},$$

where the interaction potential has been Fourier transformed into momentum space. The total Hamiltonian is then

$$\hat{H} = \sum_{\mathbf{p}} \left(\frac{p^2}{2m} + V_{\text{Ext}} \right) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{q}, \mathbf{p}_1, \mathbf{p}_2} U_{\mathbf{q}} \hat{a}_{\mathbf{p}_1 + \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_2 - \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_2}.$$

The interaction part of the Hamiltonian can intuitively be interpreted to represent a process where two incoming particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ exchange momentum $\mathbf{q}$. A realistic potential will include an attractive and a repulsive part. However, since the interaction is characterised by the single parameter a , it turns out that the microscopic details of the potential are not important for the weakly interacting gas as long as the scattering length is the same. We can therefore replace the momentum-dependent potential with a coupling constant g given by the Born approximation [19]

$$g = \frac{4\pi\hbar^2 a}{m}.$$

This is equivalent to using the constant potential for the mode $\mathbf{q} = 0$ in all interactions. The Hamiltonian for a weakly interacting Bose gas simplifies to

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{g}{2V} \sum \hat{a}_{\mathbf{p}_1 + \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_2 - \mathbf{q}}^\dagger \hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_2}. \quad (2.12)$$

We can find a first estimate for the ground state energy of the condensate by using the Bogoliubov approximation $\hat{a}_0 = \sqrt{N_0}$, and neglecting states with higher momenta, so that

$$E_0 = \frac{1}{2} \frac{N_0^2 g}{V}.$$

The chemical potential for an interacting Bose gas can then be estimated as

$$\mu = \frac{\partial E_0}{\partial N} = g\rho,$$

where $\rho = N/V$. Unlike an ideal Bose gas an interacting Bose gas has non-zero chemical potential which is positive also at $T = 0$ [19].

2.4 Low energy excitations of a Bose gas

The most straight forward approximation of the ground state of the Bose gas is given by

$$\prod \hat{a}_0^\dagger |0\rangle,$$

i.e. a state where all particles have zero momentum. The natural next step to improve this approximation is a Hartree-Fock calculation, accounting for contributions from single particle excitations to the system. Expanding the Hamiltonian (Equation 2.12) in terms of low momenta $\mathbf{p}$, we get to leading order

$$\begin{aligned} \hat{H} = & \frac{U_0}{2V} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \\ & + \frac{U_0}{2V} \sum_{\mathbf{p} \neq 0} \left[4\hat{a}_0^\dagger \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \hat{a}_0 + \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger \hat{a}_0 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}} \right]. \end{aligned}$$

The normalisation condition can be expressed as

$$\hat{a}_0^\dagger \hat{a}_0 + \sum_{\mathbf{p} \neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} = N. \quad (2.13)$$

This can be used to rewrite the first term of the Hamiltonian as

$$\begin{aligned} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 &= \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0^\dagger \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_0 \\ &\approx \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0^\dagger \hat{a}_0 = N^2 - 2N \sum_{\mathbf{p} \neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \left(\sum_{\mathbf{p} \neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \right)^2. \end{aligned}$$

For a gas in the condensed phase the ground state is macroscopically occupied and we can neglect the last term. Expanding the interaction potential to include non-zero momenta we have [19]

$$U_0 = g \left(1 + \frac{g}{V} \sum_{\mathbf{p} \neq 0} \frac{m}{p^2} \right). \quad (2.14)$$

Using Equation 2.13 and Equation 2.14, the Hamiltonian becomes

$$\begin{aligned}\hat{H} = & \frac{gN^2}{2V} + \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \\ & + \frac{g\rho}{2} \sum_{\mathbf{p} \neq 0} \left[2\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}} + \frac{mg\rho}{p^2} \right],\end{aligned}\quad (2.15)$$

in the thermodynamic limit. Here ρ is the particle density N/V . We can now find the excitation spectrum of the gas by diagonalising the Hamiltonian. To do this we start by introducing a new set of creation and annihilation operators for the excitations, $\hat{b}_p^\dagger$, and $\hat{b}_p$

$$\begin{aligned}\hat{a}_{\mathbf{p}} &= u_{\mathbf{p}} \hat{b}_{\mathbf{p}} - v_{-\mathbf{p}}^* \hat{b}_{-\mathbf{p}}^\dagger \\ \hat{a}_{\mathbf{p}}^\dagger &= u_{\mathbf{p}}^* \hat{b}_{\mathbf{p}}^\dagger - v_{-\mathbf{p}} \hat{b}_{-\mathbf{p}}.\end{aligned}$$

This is called the Bogoliubov transformation [19]. Inserting for $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^\dagger$ in Equation 2.15 and requiring that the off diagonal elements $\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}}^\dagger$ and $\hat{b}_{\mathbf{p}} \hat{b}_{-\mathbf{p}}$ are zero we get that

$$\left(\frac{p^2}{2m} + g\rho \right) v_{-\mathbf{p}} u_{\mathbf{p}} + \frac{g\rho}{2} (|v_{-\mathbf{p}}|^2 + |u_{\mathbf{p}}|^2) = 0. \quad (2.16)$$

Enforcing that the excitations should obey the bosonic commutation relation $[\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = \delta_{\mathbf{p}\mathbf{p}'}$, we find that the coefficients $u_{\mathbf{p}}$ and $v_{-\mathbf{p}}$ must satisfy

$$|u_{\mathbf{p}}|^2 - |v_{-\mathbf{p}}|^2 = 1.$$

Using this relation, we can reformulate Equation 2.16 into a quadratic equation for $u_{\mathbf{p}}$ and $v_{-\mathbf{p}}$, which has the solution

$$u_{\mathbf{p}}, v_{-\mathbf{p}} = \pm \left(\frac{p^2/2m + g\rho}{2\varepsilon(p)} \pm \frac{1}{2} \right)^{\frac{1}{2}},$$

where $\varepsilon(p)$ is the dispersion relation of the excitations given by

$$\varepsilon(p) = \left[\frac{g\rho}{m} p^2 + \left(\frac{p^2}{2m} \right)^2 \right]^{\frac{1}{2}}. \quad (2.17)$$

The Hamiltonian can now be written as

$$\hat{H} = E_0 + \sum \varepsilon(p) \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}, \quad (2.18)$$

where the zero state energy is given by

$$E_0 = \frac{gN^2}{2V} + \frac{1}{2} \sum_{\mathbf{p} \neq 0} \left[\varepsilon(p) - g\rho - \frac{p^2}{2m} + \frac{mg^2\rho^2}{p^2} \right].$$

The Hamiltonian (Equation 2.18) describes the excitations as a non-interacting gas of quasiparticles with energies given by the dispersion relation. We have expanded the Hamiltonian to the first order in terms of single particle operators. This approximation can account for interactions between the excitations and the mean field, but not for

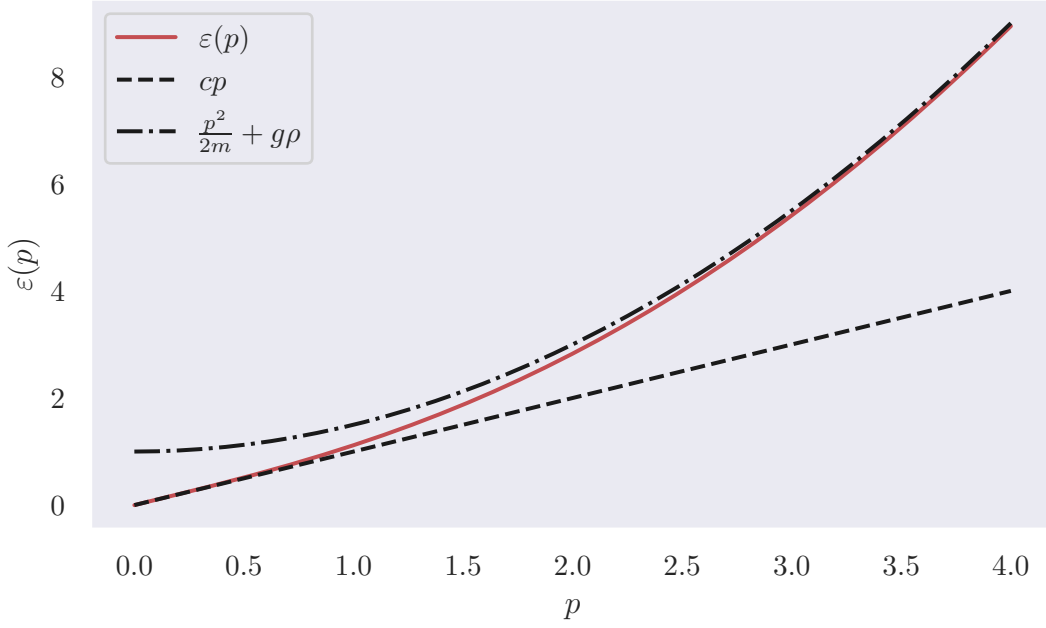


Figure 2.1: The dispersion relation of the lowest order excitations in a BEC (red) compared to the phonon like dispersion, cp , and a particle like dispersion, $p^2/2m + g\rho$.

interactions between the excited states. The dispersion relation (Equation 2.17) of the excitations is shown in Equation 2.1. At low momenta the dispersion relation is linear, like phonons in a solid. At higher momenta the dispersion becomes more particle like. We can find the length scale ξ at which the transition occurs through de Broglie's hypothesis $p = \hbar/\xi$

$$g\rho = \frac{p^2}{2m} \implies \xi = \sqrt{\frac{\hbar^2}{2mg\rho}}.$$

This is called the healing length, and describes the typical length scale of distortions in the wave function.

The occupation number of the excitation gas is given by [19]

$$N_{\mathbf{p}} = \langle \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \rangle = \frac{1}{e^{\beta\epsilon(p)} - 1}. \quad (2.19)$$

Note that the chemical potential of the excitations is zero, which means excitations can occur without any energy cost. We can now use the Bogoliubov transformation to rewrite the momentum distribution for particles with $p \neq 0$

$$n_{\mathbf{p}} = \langle \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \rangle = |v_{-\mathbf{p}}|^2 + |u_{\mathbf{p}}|^2 \langle \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \rangle + |v_{-\mathbf{p}}|^2 \langle \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} \rangle.$$

The last two terms vanish at zero temperature as can be seen from Equation 2.19. However, the first term does not vanish, indicating that there are particles with non-zero momenta at $T = 0$. The momentum distribution at zero temperature is then

$$n_{\mathbf{p}} = \frac{p^2/2m + mc^2}{2\epsilon(p)} - \frac{1}{2}.$$

At finite temperature it is given by

$$n_{\mathbf{p}} = \frac{mk_B T mc^2}{\epsilon(p)^2} = \frac{mk_B T}{p^2}.$$

The momentum distribution diverges as $p \rightarrow 0$, which is consistent with the ground state being macroscopically occupied.

2.5 The Gross-Pitaevskii equation

One possible approach to formulating an equation of motion for a quantum system is by taking the operators to be constant and letting the eigenstates evolve in time. This is the so called Schrödinger picture. A different but equivalent approach is letting the states be time independent while the operators evolve in time. This is called the Heisenberg picture, which we will now use to derive an equation of motion for the weakly interacting Bose gas. The time evolution of the field operator is given by

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi} = [\hat{\Psi}, \hat{H}].$$

We can calculate the commutator of the field operator with the Hamiltonian (Equation 2.10) term by term. The potential term of the Hamiltonian is given by

$$\hat{H}_V = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) V_{\text{Ext}}(\mathbf{r}) \hat{\Psi}(\mathbf{r}).$$

This gives the commutator

$$\begin{aligned} [\hat{\Psi}(\mathbf{r}'), \hat{H}_V] &= \int d\mathbf{r} [\hat{\Psi}(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r}) V_{\text{Ext}}(\mathbf{r})] \hat{\Psi}(\mathbf{r}) \\ &= \int d\mathbf{r} [\hat{\Psi}(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r})] V_{\text{Ext}}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \\ &= \int d\mathbf{r} \delta(\mathbf{r}' - \mathbf{r}) V_{\text{Ext}}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) = V_{\text{Ext}}(\mathbf{r}') \hat{\Psi}(\mathbf{r}'), \end{aligned}$$

Where we have used the commutation rule $[A, BC] = B[A, C] + [A, B]C$. Next we can find the commutator with the kinetic energy term. First we do integration by parts

$$\begin{aligned} \hat{H}_K &= \int d\mathbf{r} \nabla_{\mathbf{r}} \hat{\Psi}^\dagger(\mathbf{r}) \nabla_{\mathbf{r}} \hat{\Psi}(\mathbf{r}) \\ &= - \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r}), \end{aligned}$$

where we have assumed the fields to approach zero at the boundary. After this manipulation we can apply the same commutation rule as above to find the result

$$\begin{aligned} [\hat{\Psi}(\mathbf{r}'), \hat{H}_K] &= [\hat{\Psi}(\mathbf{r}'), - \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r})] \\ &= - \int d\mathbf{r} [\hat{\Psi}(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r}) \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r})] \\ &= - \int d\mathbf{r} \left\{ \hat{\Psi}^\dagger(\mathbf{r}) [\hat{\Psi}(\mathbf{r}'), \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r})] + [\hat{\Psi}(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r})] \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r}) \right\}. \end{aligned}$$

The Laplacian operator in the first term on the last line can be taken outside the commutator, since it only depends on $\mathbf{r}$, and not on $\mathbf{r}'$. We are then left with the commutator of the field annihilation operators, which is zero. The evolution due to the kinetic term is then

$$[\hat{\Psi}(\mathbf{r}'), \hat{H}_K] = - \int d\mathbf{r} \delta(\mathbf{r} - \mathbf{r}') \nabla_{\mathbf{r}}^2 \hat{\Psi}(\mathbf{r}) = - \nabla_{\mathbf{r}'}^2 \hat{\Psi}(\mathbf{r}').$$

Lastly, we find the commutator with the interaction term.

$$\hat{H}_I = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}') \hat{\Psi}^\dagger(\mathbf{r}) U(\mathbf{r}' - \mathbf{r}) \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}).$$

The interaction potential is a scalar function independent of $\mathbf{r}''$ and can be taken outside the commutator. We can manipulate the commutator inside the integral using the same commutator rule once again, neglecting the terms that are zero, and arrive at

$$[\hat{\Psi}(\mathbf{r}''), \hat{H}_I] = \int d\mathbf{r} U(\mathbf{r}'' - \mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}'').$$

If we approximate the potential with a constant coupling in momentum space, as above, we get $U(\mathbf{r}' - \mathbf{r}) \approx \delta(\mathbf{r}'' - \mathbf{r})g$

$$[\hat{\Psi}(\mathbf{r}''), \hat{H}_I] \approx g \hat{\Psi}^\dagger(\mathbf{r}'') \hat{\Psi}(\mathbf{r}'') \hat{\Psi}(\mathbf{r}'')$$

Well below the critical temperature for condensation the ground state occupation number is very large. In this case we can replace the field operators with classical fields $\hat{\Psi} \rightarrow \Psi$. This is valid for length scales much larger than the scattering length a . Effectively, this approximation neglects the contribution from the commutators of the field operators. The commutators are of order $1/V$, while the field operators themselves, are of the order $\sqrt{N_0/V}$, according to the Bogoliubov prescription $\hat{a}_0 \equiv \sqrt{N_0}$. We arrive at the mean field equation for evolution of the condensate

$$i\hbar \partial_t \Psi(t, \mathbf{r}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{Ext}}(\mathbf{r}) + g|\Psi(\mathbf{r})|^2 \right) \Psi(\mathbf{r}), \quad (2.20)$$

which is called the Gross-Pitaevskii equation. This equation accurately describes the dynamics of atomic Bose-Einstein condensates at low temperatures [2, 27]. Since the condensate oscillates with the chemical potential μ (Equation 2.9) it is natural to transform the GPE to the comoving frame $\Psi \rightarrow \Psi e^{-i\frac{\mu t}{\hbar}}$. We get the GPE on the form

$$i\hbar \partial_t \Psi(t, \mathbf{r}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{Ext}}(\mathbf{r}) - \mu + g|\Psi(\mathbf{r})|^2 \right) \Psi(\mathbf{r}).$$

The GPE as stated above applies to $T = 0$, at finite temperature however the particles in the condensate can interact with the states in the thermal cloud. The thermal cloud has a dissipative effect on the condensate, to account for this effect we can modify the GPE by adding a phenomenological damping coefficient γ [2] $i\partial_t \rightarrow (i - \gamma)\partial_t$, to obtain the damped Gross-Pitaevskii equation. The effective damping γ is typically very small. Experiments with ^{87}Rb can typically have an effective damping of $\gamma \sim 5 \times 10^{-4}$ [28].

It is easier to work with the GPE on dimensionless form. The characteristic velocity of the GPE is the speed of sound given by $c^2 = \mu/m$ and the length scale is set by the healing length $\xi = \hbar/\sqrt{m\mu}$. The characteristic time scale of the GPE is then $\tau = c/\xi = \mu/\hbar$. The energy scale is set by the chemical potential. Rescaling the GPE with these units, using primes to mark dimensionless quantities, we find

$$i\partial'_t \Psi(t, \mathbf{r}) = \left(-\frac{1}{2} \nabla'^2 + V'_{\text{Ext}}(\mathbf{r}) - 1 + \frac{g}{\mu} |\Psi(\mathbf{r})|^2 \right) \Psi(\mathbf{r}).$$

The chemical potential is estimated as $\mu = g\rho$, rescaling the order parameter with the number density in the condensate $\Psi \rightarrow \sqrt{\rho}\Psi$, we then find

$$i\partial_t \Psi(t, \mathbf{r}) = \left(-\frac{1}{2} \nabla^2 + V_{\text{Ext}}(\mathbf{r}) - 1 + |\Psi(\mathbf{r})|^2 \right) \Psi(\mathbf{r}), \quad (2.21)$$

where we have dropped the prime on the dimensionless operators. We can find an approximation to a static solution for the condensate from the time independent GPE

$$\Psi(\mathbf{r}) = -\frac{1}{2}\nabla^2\Psi(\mathbf{r}) + V_{\text{Ext}}(\mathbf{r})\Psi(\mathbf{r}) + |\Psi(\mathbf{r})|^2\Psi(\mathbf{r}).$$

If the interaction is repulsive $g > 0$ then the gas will expand in comparison to the non-interacting case. This smooths out the density profile of the condensate [19]. Far from the wall the density of the condensate is slowly varying and we can neglect the kinetic energy term. Neglecting this term is known as the Thomas-Fermi approximation. The density profile of the condensate in the Thomas-Fermi limit is given by

$$|\Psi_{TF}(\mathbf{r})|^2 = 1 - V_{\text{Ext}}(\mathbf{r}), \quad (2.22)$$

this will be useful as an initial guess for Ψ when we are solving the GPE numerically. We generally start off with the Thomas-Fermi solution and add perturbations, either in form of vortices or waves to the stationary solution.

2.6 Hydrodynamics of a Bose gas

The dynamics of the weakly interacting Bose gas can be described by the evolution of its order parameter Ψ , which evolves according to the GPE (Equation 2.20). Alternatively we can describe the condensate as a quantum fluid with a density ρ and a velocity $\mathbf{v}$. To find the set of hydrodynamic equations governing the evolution of these quantities we start by multiplying the GPE (Equation 2.20) by Ψ^* and subtract its complex conjugate. This gives us the continuity equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0, \quad (2.23)$$

where $\rho = |\Psi|^2$ and where $\mathbf{J}$ is the probability current of the particles given by

$$\mathbf{J} = -\frac{i\hbar}{2m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*). \quad (2.24)$$

We can rewrite Ψ through the Madelung transformation as $\Psi = \sqrt{\rho} \exp(i\theta)$ [29]. Inserting this into Equation 2.24 we find an alternative expression for the current

$$\mathbf{J} = \rho \frac{\hbar}{m} \nabla \theta. \quad (2.25)$$

This gives us the expression for the velocity of the condensate

$$\mathbf{v} = \frac{\hbar}{m} \nabla \theta. \quad (2.26)$$

To find a hydrodynamic equation for $\mathbf{v}$ we can insert Ψ in polar form into the GPE (Equation 2.20), this gives one equation in the real part and one in the imaginary part. The equation from the imaginary part is the continuity equation, which we already found above (Equation 2.23). The real part gives the equation

$$\hbar \partial_t \theta = \frac{\hbar^2}{2m\sqrt{\rho}} \nabla^2 \sqrt{\rho} - \frac{\hbar^2}{2m} (\nabla \theta)^2 - V_{\text{Ext}} - g\rho. \quad (2.27)$$

We can take the gradient on both sides of the equation to find the equation for the velocity

$$\partial_t \mathbf{v} = \frac{1}{m} \nabla \left(\frac{\hbar^2}{2m\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) - \frac{1}{2} \nabla (v^2) - \frac{1}{m} \nabla V_{\text{Ext}} - \frac{1}{m\rho} \nabla p, \quad (2.28)$$

where we have defined $p = \rho^2 g/2$ in line with Madeira et al. [27]. The equation we are left with then, is equivalent to the Euler equation for an irrotational fluid with an extra pressure term that depends on the density gradients in the fluid. This extra pressure term is called the quantum pressure. The prediction that the quantum fluid should be irrotational is not correct. There can be vortices in the fluid, in which case $\partial_t \omega \neq 0$. When vortices are present the phase is multivalued, and the straightforward application of the chain rule we used to arrive at Equation 2.27 does not hold [30]. We will discuss vortices further in Section 2.8. Another thing to note in Equation 2.28 is the absence of a dissipative term. The flow is inviscid. This is because of the form of the dispersion relation of the elementary excitations in the BEC which prohibit it from interacting with its surroundings below a critical velocity. Below this velocity, the flow of the condensate is superfluid.

2.7 Superfluidity

Superfluidity is a quantum effect allowing a fluid to flow without dissipation. It was first discovered in liquid ^4He in 1938, and explained by Landau in 1941 [19]. Superfluidity occurs when the available excitations of the fluid are not energetically favourable. In this situation the fluid cannot be thermally heated and internal friction does not impact the flow. The fluid becomes inviscid. Let us consider a fluid moving past a wall at velocity $\mathbf{v}$ with respect to the wall. The Galilean transformations of energy and momentum from reference frame R to R' moving at a velocity $\mathbf{v}$ with respect to each other are given by

$$E' = E - \mathbf{p} \cdot \mathbf{v} + \frac{1}{2} M v^2$$

$$\mathbf{p}' = \mathbf{p} + M \mathbf{v}$$

Friction between the wall and the fluid dissipates some of the kinetic energy into thermal energy, heating the fluid. We can look at this process in terms of single excitations of the fluid $\varepsilon(\mathbf{p})$. The energy of the fluid in its rest frame is then

$$E = E_0 + \varepsilon(\mathbf{p}).$$

In this frame the momentum of the excitation $\mathbf{p}$ is the only contribution to the momentum of the fluid. Transforming the energy into the frame of the wall, which moves with a velocity $-\mathbf{v}$ with respect to the fluid, we get

$$E' = E_0 + \varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v} + \frac{1}{2} M v^2.$$

We can identify $\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}$ as the energy associated with the excitation in the wall frame. For this to be energetically favourable it has to be negative

$$\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v} < 0.$$

When this condition is fulfilled the fluid interacts with the wall through friction and heat is generated. However when it is not fulfilled, i.e. when

$$v < \min_p \frac{\varepsilon(\mathbf{p})}{p},$$

then none of the possible excitations in the fluid will be energetically favourable and the fluid is essentially un-excitable. This is known as Landau's criterion for superfluidity.

Below this critical velocity the fluid will persistently flow past the wall without friction, and without heating. It becomes superfluid. For a BEC the dispersion is given by Equation 2.17. The critical velocity is then given by the speed of sound $c = \sqrt{g\rho/m}$.

2.8 Quantum vortices

The hydrodynamic equation (Equation 2.28) describes the dynamics of a non-rotational fluid. Nonetheless vortices can appear in the condensate. The circulation around a region in the fluid enclosed by the curve C is generally given by [31]

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l},$$

where $\mathbf{v}$ is the velocity of the fluid. The superfluid velocity is given by the gradient of the phase. Therefore the vorticity is zero everywhere except where the phase is undetermined [29], and the circulation is zero unless there is a singularity inside the curve C . For the phase to be single valued along the path around the singularity it will have to make a jump of an integer multiple of 2π at the start or end of the loop [29]

$$\Gamma = \oint_C \frac{\hbar}{m} \nabla \theta \cdot d\mathbf{l} = \frac{\hbar}{m} \oint_C d\theta = 2\pi \frac{\hbar}{m} q,$$

where q is an integer. We refer to this as the topological charge. In a superfluid then, vorticity is quantised. This is not in and of itself a quantum effect, but arises from the fluid being inviscid. It is possible for a vortex to have a charge that exceeds ± 1 , however this will be energetically unfavourable and the vortex will normally split up into multiple vortices with a charge of ± 1 [2]. For an inviscid fluid the circulation along a path $C(t)$ which encloses the same fluid particles over time is constant according to Kelvin's circulation theorem [31]. This applies also in the presence of a singularity and compressible effects as long as the forces acting on the fluid are conservative. This has the consequence that vorticity localised at the singularities has no way to disappear unless an oppositely charged vortex enters the loop $C(t)$. Vortices in the superfluid are therefore persistent structures and are a form of topological defect, characterised by the singularity in the phase as well as a stable core [32].

For the condensate to have finite energy, the density of the condensate has to go to zero at the singularities [32, 29]. The density profile of a single vortex at $\mathbf{r} = 0$ can be approximated by the ansatz [28]

$$\sqrt{\rho} = \rho_0 \frac{r}{\sqrt{(\Lambda^{-1}\xi)^2 + r^2}}, \quad (2.29)$$

where $\Lambda \approx 0.8249$ [28]. Far from the vortex the condensate density is constant, equal to the unperturbed background density ρ_0 . Near the core the magnitude of the order parameter increases linearly. The velocity field of a vortex is given by $\mathbf{v} = \frac{\hbar}{m} \nabla \theta$, where the phase is given as the polar angle with the vortex position as the origin [33]. In Cartesian coordinates we can write the phase as [34]

$$\theta = q \arctan\left(\frac{x - x_0}{y - y_0}\right). \quad (2.30)$$

This gives the velocity profile around a vortex

$$\mathbf{v} = q \frac{(y - y_0)\mathbf{i}_x - (x - x_0)\mathbf{i}_y}{|\mathbf{r} - \mathbf{r}_0|^2}. \quad (2.31)$$

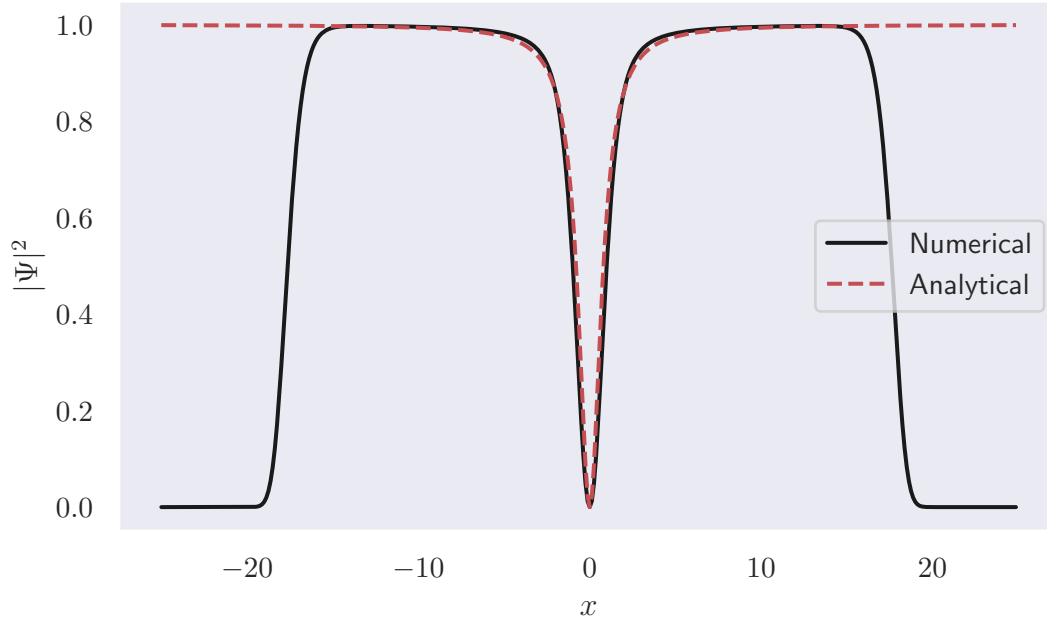


Figure 2.2: Density profile across an isolated quantum vortex.

The analytical density profile of an isolated vortex (Equation 2.29) is compared to the numerical solution in Figure 2.2. Near the core and in the far field the analytical profile and the numerical profile are near identical. The motion of vortices will be discussed further in Chapter 3. In the next section we will introduce classical turbulence before addressing quantum turbulence in the subsequent section.

2.9 Classical Turbulence

The motion of isotropic classical fluids is described by the Navier-Stokes equations

$$\begin{aligned}\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} - \mathbf{f}_{\text{Ext}} \\ \nabla \cdot \mathbf{v} &= 0,\end{aligned}$$

where $\mathbf{v}$ is the velocity of the fluid, ν is the kinematic viscosity, ρ is the mass density and p is the pressure. The first of these equations represent momentum balance, while the second equation represents incompressibility. These equations are applicable to a wide range of regimes from very small length scales, bordering on the domain of molecular dynamics [35], to extremely large length scales on the order of the size of planets [36]. Two of the central physical effects dictating the dynamics of fluid flow are inertia and viscosity. The balance between them is captured in the dimensionless Reynolds number [31]

$$Re = \frac{UL}{\nu},$$

where U and L are the characteristic velocity and length scale of the flow, and ν is the kinematic viscosity. When viscosity dominates, and Re is small, the flow is described as laminar, and is generally neat and tidy. At the opposite end of the spectrum, at high Re , the flow is chaotic and turbulent. When the Reynolds number of a flow increases above

some critical value, which is specific to that particular geometry of the flow [37], the flow becomes unstable and perturbations grow into large disturbances. Turbulence kicks in, and the flow becomes chaotic. It is characterised by an energy cascade, a flux of energy across different length scales. Energy that is fed into the system at one length scale cascades either up towards the size of the system, or down to where viscosity dissipates the energy into heat. Typically in 3D energy is fed into the system at large length scales and dissipated at small length scales. This type of cascade, from large to small length scale, is called a direct cascade. In 2D there is also an inverse cascade, transporting energy towards length scales comparable with the system size. This creates large scale coherent structures, such as typhoons for example [38].

3D turbulence is associated with a scaling law, given by $E(k) \sim k^{-5/3}$, called the Kolmogorov scaling. This scaling law is observed across a large range of systems at high Reynolds numbers and where there is a sufficiently large separation between the driving length scale and dissipative length scale [39]. The Kolmogorov scaling can be derived using dimensional arguments. We assume the turbulence to be in steady state, that the input of energy occurs at large length scales at the same rate as which it is dissipated at small length scales. The relevant parameters determining the lower length scale are the viscosity, ν , and the dissipation rate per mass, ε . We can combine ν and ε into a quantity of dimension $[L]$ as

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}},$$

this sets the lower length scale of the cascade. To characterise the flow at large scales we can use the auto-correlation function. We can think of turbulence as a patchwork of regions of rotational flow called eddies [39]. The auto-correlation function then quantifies the size of these eddies. We can assume that an eddy primarily interact with other eddies on roughly the same length scale. For sufficiently large length scales, $r \gg \eta$, we can then neglect viscous effects on the dynamics of the flow. This is called the *inertial subrange* [40]. Now, since the energy is transferred gradually in small steps from one length scale to the next, and the energy input at the large end of the cascade balances the dissipation at the lower end, the rate of energy dissipation ε has to be equal to the rate of energy transfer across the entire range. Since the dissipation rate set by the viscosity is the same across all scales, we can express the kinetic energy of an eddy in terms of the viscosity and the dissipation rate

$$V^2 \sim (\nu\varepsilon)^{\frac{1}{2}},$$

where V is the typical velocity of the eddy. The dissipation rate has units $[L^2T^{-3}]$, so if we want to express the kinetic energy in terms of the dissipation rate and a length we find that the exponent $2/3$ gives a quantity with units velocity squared

$$V^2 \sim (\varepsilon r)^{\frac{2}{3}} \sim \varepsilon^{\frac{2}{3}} k^{-\frac{2}{3}}.$$

The amount of energy in the window $[k, k + dk]$ is then

$$E = E(k)dk \sim \varepsilon^{\frac{2}{3}} k^{-\frac{2}{3}},$$

and the resulting energy spectrum is then $E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$. In 3D the main mechanism of energy transfer from one vortex to another is through smaller eddies being stretched by the flow field of a larger eddy. The stretching causes the eddy to 'spin up' [31], increasing its energy. In 2D this mechanism is not available, 2D turbulence will therefore not exhibit the same cascade as in 3D. At low k , where viscosity can be neglected, there

are two conserved quantities in 2D, enstrophy and kinetic energy. This gives rise to two cascades. The cascade related to the conservation of kinetic energy drives the energy towards lower k [40], and has the same scaling as in 3D, $E(k) \sim k^{-5/3}$. The cascade related to enstrophy on the other hand scales as $E(k) \sim k^{-3}$, and the energy in this cascade flows towards high k , where it is dissipated by viscous forces.

In contrast to classical turbulence quantum turbulence is constrained by the quantisation of vorticity. In classical turbulence vorticity can increase or decrease continuously, and eddies can be of any size between the viscous length scale and the system size. In quantum turbulence however the vorticity is constrained to local vortices on the order of the healing length, ξ , with vorticity $2\pi q$, where q is an integer. Vorticity cannot then dissipate gradually, vortices can only disappear through collisions with an anti-vortex, or by leaving the system. This difference drastically changes the dynamics of quantum turbulence in comparison to classical turbulence.

2.10 Quantum Turbulence

Quantum turbulence most resembles classical turbulence in the quasiclassical regime, which is characterised by the formation of clusters of like signed vortices [2]. Whereas enstrophy in a classical eddy can increase or diminish gradually, the enstrophy of a cluster of vortices must increase in discrete steps. If the vortices arrange into sufficiently large clusters the discrete quanta of vorticity become smaller relative to the total vorticity of the cluster. The clusters then effectively behave as classical eddies [12]. The presence of clusters in the BEC is also associated to the Kolmogorov $k^{-5/3}$ scaling [41]. The length scales for the inertial range in this case has to be well above the mean distance between the vortices $k \gg k_l$, [12] otherwise the discretisation effects are too strong for clusters to resemble classical eddies. The Kolmogorov scaling is associated with a concentration of energy at large length scales. Typically length scales in BEC experiments can be up to two orders of magnitudes above the healing length ξ [12], which in turn sets the lower length scale. In steady state quantum turbulence examples of the driving force can be a stirring laser [2], or shaking of the potential [42]. Experimentally there is always some dissipation due to interactions with the thermal cloud. As an example, with ^{87}Rb at 100 nK the damping coefficient is $\gamma \sim 6 \times 10^{-4}$ [28]. Even at $T = 0$ however, vortex reconnections (3D) or annihilation events (2D) can function as an effective dissipative mechanism [12], transforming the incompressible kinetic energy bound up to the vortices into sound waves. In addition to this accelerating vortices emit sound waves, in analogy to electric charges [10], reducing the vortex energy.

Without a driving force quantum turbulence at finite temperature will decay. In 2D decaying quantum turbulence vortices have been observed to assemble into clusters of like-signed vortices [8, 9, 4]. This phenomenon has been explained by a process called evaporative heating [8]. The low energy vortices disappear from the system effectively increasing the energy per vortex in the system, forcing them into a high energy configuration, i.e. a cluster. In a chaotic configuration of vortices some of them will collide and annihilate. The vortices cannot disappear on their own, but only through annihilation with an anti-vortex. At finite temperature vortices and anti-vortices drift towards each other and annihilate, vortices can annihilate in a two-vortex process. In the absence of boundaries or dissipation vortex annihilation is a four-vortex process [3]. An isolated dipole will have a constant separation between the vortex and the anti-vortex in the absence of dissipation [4]. With a third vortex the energy of the dipole can be transferred to the third vortex as the dipole collapses. This collapse creates a solitary

wave, in which the positive and negative vortices are tightly bound to each other. This wave preserves the signature of the vortices until it scatters off a fourth vortex, or splits up again into defects [13]. If we assume a n -body process to follow the rate equation

$$\frac{d}{dt}N_v = \Gamma_n N_v^n, \quad (2.32)$$

where Γ is a rate coefficient and N_v the total vortex number, we expect the vortex number to decay as $N_v \sim t^{-1/(n-1)}$. The four-vortex annihilation process then, has the decay signature $N_v \sim t^{-1/3}$. In this process the incompressible kinetic energy is converted into compressible kinetic energy as the vortices annihilate and the wave scatters. The essential quantity in the evaporative heating mechanism is the *incompressible* kinetic energy per vortex. Whether or not clusters emerge depends, among other things, on the effectiveness of the energy transfer from incompressible kinetic energy to other forms of energy during the annihilation process. Kanai and Guo [9] found that the energy transfer to sound waves during the annihilation process suppresses the formation of clusters in a periodic system. For a trapped condensate however, dipoles can annihilate with their ghosts at the boundary. In this process all of the kinetic energy of the dipole is not converted into compressible kinetic energy. This process has been shown to be the main mechanism of clustering [9].

Energy decomposition

A common way of characterising turbulent flow is through the energy spectrum of the kinetic energy. The distribution of energy across k indicates at which length scale the energy is concentrated. The energy of the BEC is given by the Hamiltonian,

$$H = \int d\mathbf{r} \frac{\hbar^2}{2m} |\nabla \Psi|^2 + V |\Psi|^2 + \frac{g}{2} |\Psi|^4, \quad (2.33)$$

where the first term represents the kinetic energy of the superfluid, the second term represents the energy from an external potential, and the last term the interaction energy. Applying the Madelung transformation $\Psi = \sqrt{\rho} e^{i\theta}$, the kinetic energy is split into two terms

$$K = E_q + E_k,$$

The first term of the kinetic energy comes from the quantum pressure [28] and has been called the *quantum energy* [43]. It is given by

$$E_q = \frac{\hbar^2}{2m} \int d\mathbf{r} |\nabla \sqrt{\rho}|^2. \quad (2.34)$$

The second term is the kinetic energy associated with the superfluid velocity and is simply referred to as kinetic energy [28, 14]. Meanwhile

$$E_k = \frac{m}{2} \int d\mathbf{r} \rho |\mathbf{v}|^2,$$

where $\mathbf{v}$ is the superfluid velocity (Equation 2.26). In the few cases where we need to distinguish the full kinetic energy $E_k + E_q$ from the kinetic energy from the superfluid velocity E_k we will refer to the latter as the hydrodynamic kinetic energy. As the superfluid is compressible the kinetic energy carries a dependence on the particle density. Since the superfluid velocity $\mathbf{v}$ is ill defined at the vortex core it will not lend itself useful

for analysing superfluid flows with vortices present. However we can use an alternative velocity field

$$\mathbf{u} = \sqrt{\rho} \mathbf{v} \quad (2.35)$$

to study the kinetic energy in turbulent flows [28, 14]. Furthermore we want to do a Helmholtz decomposition, splitting the velocity field $\mathbf{u}$ into a compressible and incompressible part, $\mathbf{u} = \mathbf{u}^c + \mathbf{u}^i$. The compressible part of the velocity field is irrotational $\nabla \times \mathbf{u}^c = 0$, the incompressible velocity field is divergence free $\nabla \cdot \mathbf{u}^i = 0$. As a result they are also orthogonal

$$\int d\mathbf{r} \mathbf{u}^i \cdot \mathbf{u}^c = \int d\mathbf{r} \mathbf{u}^i \cdot \nabla \phi = \int_{\Gamma} d\Gamma \phi \mathbf{u}^i \cdot \mathbf{n} - \int d\mathbf{r} \phi \nabla \cdot \mathbf{u}^i = 0,$$

where we have expressed $\mathbf{u}^c$ in terms of its potential ϕ and assume the flow at the boundary to be negligible. The Helmholtz decomposition is found in Fourier space using the equations [44]

$$\tilde{\mathbf{u}}_i^c(\mathbf{k}) = \sum_j \frac{k_i k_j}{k^2} \tilde{\mathbf{u}}_j(\mathbf{k}) \quad (2.36)$$

$$\tilde{\mathbf{u}}_i^i(\mathbf{k}) = \sum_j \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \tilde{\mathbf{u}}_j(\mathbf{k}), \quad (2.37)$$

where $\tilde{\mathbf{u}}(\mathbf{k}) = \mathcal{F}[\mathbf{u}(\mathbf{r})]$. The kinetic energy of the compressible and incompressible velocity fields are given by

$$\begin{aligned} E_c &= \int d\mathbf{r} |\mathbf{u}^c|^2 \\ E_i &= \int d\mathbf{r} |\mathbf{u}^i|^2. \end{aligned} \quad (2.38)$$

The incompressible kinetic energy, E_i , is associated to the rotational flow around vortices, the incompressible energy spectrum can be used to characterise vortex configurations. The compressible kinetic energy, E_c , is primarily associated with perturbations in the superfluid density that evolve with time, such as waves or moving vortices. The quantum energy, E_q , comes from density gradients in the condensate, static or moving.

Chapter 3

Theoretical results

3.1 Locating vortices from the pseudo-vorticity

We saw in Section 2.8 that describing the flow field of the superfluid with vortices present *only* through the phase of the condensate does not fully account for the vortex dynamics. In addition to describing the vortices as singularities of the phase, we also use the fact that the condensate density vanishes locally to construct a pseudo-vorticity field, which is finite at the vortices, and zero otherwise [45]. We first construct a vector field

$$\underline{\Psi} = [\Psi_1, \Psi_2], \quad (3.1)$$

where Ψ_i are the real and imaginary part of the wave function respectively, and then take its curl [32]. This allows us to track the position of the phase singularities with a non-singular field [45, 32]. In 2D, this pseudo-vorticity is a real scalar field given by

$$D = \frac{1}{2i} \varepsilon_{ij} \partial_i \Psi_1 \partial_j \Psi_2 = \frac{1}{2i} \varepsilon_{ij} \partial_i \Psi^* \partial_j \Psi, \quad (3.2)$$

where ε_{ij} is the Levi-Civita symbol. The D -field captures high-energy excitations related to flows induced by both density and phase disturbances [45]. For a condensate with a uniform background condensate density it is zero everywhere in the condensate except at the vortices, where its local sign determines the vortex charge. Accounting for the finite core size of the vortices the D -field has a magnitude of $|D| \approx 0.7$ at the vortices [32]. Other excitations are typically lower than $|D| \approx 0.7$. At $T > 0$ the non-topological excitations are generally short lived due to the damping in the condensate, so that only topological defects are picked up by the D -field. We can express the defect density field, using the D -field, as $\rho_v = D \delta^2(\underline{\Psi})$ [32]. The vortex positions are given by the zeros of the wave functions. In the limit where vortices are far apart this position corresponds to where D is non-zero, so the D -field in and of itself can be used to locate the vortices. Generally however the D -field captures high-energy excitations related to flows induced by both density and phase disturbances [45].

In analogy to an electrical dipole, we can also have a dipole of vortices. The dipole moment is generally given by

$$\mathbf{p} = \int d\mathbf{r}' \rho_v(\mathbf{r}')(\mathbf{r}' - \mathbf{r}),$$

where ρ_v is the density field for topological charge. Using the D -field as the charge density we can then define the quantity

$$\mathbf{d}(\mathbf{r}) = \int d\mathbf{r}' D(\mathbf{r}')(\mathbf{r}' - \mathbf{r}). \quad (3.3)$$

This is related to the dipole moment, with the difference that we have neglected the Dirac delta in ρ_v . This has the consequence that $\mathbf{d}(\mathbf{r})$ is defined regardless of whether or not there are any defects in the condensate. For our purposes this is an advantage since it enables us to use the same quantity, $\mathbf{d}(\mathbf{r})$, to characterise dipoles as well as solitary waves. When we refer to $\mathbf{d}(\mathbf{r})$ later in this work we will simply call it a dipole moment.

3.2 Tracking vortex motion from the pseudo-vorticity evolution

The vortex density field in a superfluid can be written as

$$\rho_v = \sum_{\alpha} q_{\alpha} \delta^2(\mathbf{r} - \mathbf{r}_{\alpha}) = D \delta^2(\underline{\Psi}),$$

where q_{α} is the charge of a vortex, $\mathbf{r}_{\alpha}$ is the position of a vortex in space and D is given by Equation 3.2. To find the motion of the vortices we start by calculating the time derivative of the D -field

$$\partial_t D = \frac{1}{2i} \varepsilon_{ij} \partial_t (\partial_i \Psi^* \partial_j \Psi) = \frac{1}{2i} \varepsilon_{ij} \partial_i (\partial_t \Psi^*) \partial_j \Psi + \frac{1}{2i} \varepsilon_{ij} \partial_i \Psi^* \partial_j (\partial_t \Psi).$$

We can rewrite using the relation $\varepsilon_{ij} \partial_i \Psi^* \partial_j \Psi = -\varepsilon_{ij} \partial_j \Psi^* \partial_i \Psi$

$$\partial_t D = \frac{1}{2i} \varepsilon_{ij} \partial_i (\partial_t \Psi^*) \partial_j \Psi - \frac{1}{2i} \varepsilon_{ij} \partial_j \Psi^* \partial_i (\partial_t \Psi).$$

Now we can pull the ∂_i outside, since $\varepsilon_{ij} \partial_i \partial_j = 0$ the cross terms do not contribute. Thus,

$$\partial_t D = \frac{1}{2i} \partial_i \varepsilon_{ij} (\partial_t \Psi^* \partial_j \Psi - \partial_t \Psi \partial_j \Psi^*) = \partial_i \varepsilon_{ij} \text{Im}(\partial_t \Psi^* \partial_j \Psi) = -\partial_i \varepsilon_{ij} \text{Im}(\partial_t \Psi \partial_j \Psi^*) = -\partial_i J_i^D,$$

we have a continuity equation for the D -field where the current of pseudo-vorticity is given by $J_i^D = \varepsilon_{ij} \text{Im}(\partial_t \Psi \partial_j \Psi^*)$. The time derivative of the vortex density field yields

$$\partial_t \rho_v = \partial_t D \delta^2(\underline{\Psi}) + D \partial_t \delta^2(\underline{\Psi}) = -\partial_i (J_i^D \delta^2(\underline{\Psi})) + J_i^D \partial_i \delta^2(\underline{\Psi}) + D \partial_t \delta^2(\underline{\Psi})$$

The second term can be rewritten

$$J_i^D \partial_i \delta^2(\underline{\Psi}) = J_i^D \partial_i \Psi \delta^{2'}(\underline{\Psi}) = \frac{1}{2i} \varepsilon_{ij} [\partial_t \Psi \partial_i \Psi \partial_j \Psi^* - \partial_t \Psi^* \partial_i \Psi \partial_j \Psi] \delta^{2'}(\underline{\Psi})$$

the last term is zero since it is symmetrical

$$J_i^D \partial_i \delta^2(\underline{\Psi}) = \frac{1}{2i} \varepsilon_{ij} \partial_i \Psi \partial_j \Psi^* \partial_t \Psi \delta^{2'}(\underline{\Psi}) = -D \partial_t \Psi \delta^{2'}(\underline{\Psi}) = -D \partial_t \delta^2(\underline{\Psi})$$

where we have flipped the indices to get the D -field. Here $\delta^{2'}(\underline{\Psi}) = \frac{\partial}{\partial \Psi} \delta^2(\underline{\Psi})$. Inserting this result into the equation above we get the conservation law for topological charges

$$\partial_t \rho_v = -\partial_i (J_i^D \delta^2(\underline{\Psi})) - D \partial_t \delta^2(\underline{\Psi}) + D \partial_t \delta^2(\underline{\Psi}) = -\partial_i [J_i^D \delta^2(\underline{\Psi})] = -\partial_i J_i^{\rho_v}$$

The vortex current is given by $J^{\rho_v} = \sum_{\alpha} q_{\alpha} v_{\alpha} \delta^2(\mathbf{r} - \mathbf{r}_{\alpha})$, which is essentially ρ_v . Now we have that

$$\mathbf{J}^{(\rho_v)} = \mathbf{J}^D \delta^2(\underline{\Psi}) = \sum_{\alpha} q_{\alpha} \frac{\mathbf{J}^D}{D} \delta^2(\mathbf{r} - \mathbf{r}_{\alpha})$$

3.2. Tracking vortex motion from the pseudo-vorticity evolution

and we recognise the vortex velocity as determined by the current of pseudo-vorticity evaluated at the vortex position

$$\mathbf{v}_\alpha = \frac{\mathbf{J}^D(\mathbf{r}_\alpha)}{D(\mathbf{r}_\alpha)}. \quad (3.4)$$

It is possible to find a stationary vortex solution to the GPE. If we assume that Ψ_0 is such a stationary vortex solution we can find response to perturbations in the phase, θ' , and density, ρ' . Note that prime signifies that the fields are perturbations in this context, not a derivative. The full wave function can be written as

$$\Psi = \Psi_0 e^{\ln \rho' + i\theta'} = \Psi_0 e^\phi.$$

To leading order in $\nabla\phi$, the time evolution of the perturbed wave function is given by

$$\partial_t \Psi \approx i \nabla \Psi_0 \cdot \nabla \phi e^\phi.$$

The gradient of the wave function is given by $\nabla \Psi_0 = e^\phi \nabla \Psi_0 + \Psi_0 e^\phi \nabla \phi$, the second term will make the expression second order in $\nabla\phi$, so we only insert the first term. This gives the vortical current

$$\begin{aligned} J_i^{(D)} &= 2 \operatorname{Im}(i \partial_i \Psi_0^* \partial_j \Psi_0 \partial_j \phi e^{\phi^* + \phi}) = 2 \operatorname{Re}(\partial_i \Psi_0^* \partial_j \Psi_0 \partial_j \phi) e^{\phi^* + \phi} \\ &= \rho'^2 (\partial_i \Psi_0^* \partial_j \Psi_0 \partial_j \phi + \partial_i \Psi_0 \partial_j \Psi_0^* \partial_j \phi^*) \end{aligned}$$

If we write out the two terms on the last line in 2D and consider the imaginary part only, $\phi = i\theta'$, the terms where the derivative is taken in the x - or y -direction only cancel and we are left with the D -field times the rotated gradient of θ' . Note that the D -field in this case contains a factor ρ'^2 . We get the pseudo-vorticity current

$$J_i^{(D, \operatorname{Im})} = -2D \varepsilon_{ij} \partial_j \theta'$$

this gives the rotated current of pseudo-vorticity

$$\varepsilon_{ij} J_j^{D, \operatorname{Im}} = -D \varepsilon_{ij} \varepsilon_{jk} \partial_k \theta' = D \partial_i \theta'.$$

Very close to the vortex core we can approximate the vortex wave function as $\Psi_0 \approx r e^{iq\theta}$, with the phase given by Equation 2.30. In that case we have the relation $\partial_j \Psi_0 = -iq \varepsilon_{jk} \partial_k \Psi_0$, and can use this to find the real part

$$\begin{aligned} J_i^{(D, \operatorname{Re})} &= \rho' 2 \operatorname{Re}(\partial_i \Psi_0^* \partial_j \Psi_0 \partial_j \rho') \\ &= 2\rho' q \operatorname{Re}(-i \partial_i \Psi_0^* \varepsilon_{jk} \partial_k \Psi_0 \partial_j \rho') \\ &= -2i\rho' q \varepsilon_{jk} (\partial_i \Psi_0^* \partial_k \Psi_0 - \partial_i \Psi_0 \partial_k \Psi_0^*) \partial_j \rho' \\ &= 2i\rho' q \varepsilon_{kj} (\partial_i \Psi_0^* \partial_k \Psi_0 - \partial_i \Psi_0 \partial_k \Psi_0^*) \partial_j \rho' \\ &= 2q \frac{D}{\rho'} \partial_i \rho'. \end{aligned}$$

The vortical current from the density perturbation becomes

$$J_i^{D, \operatorname{Re}} = q \frac{D}{\rho'} \varepsilon_{ij} \partial_j \rho',$$

and the total velocity is given by

$$v_i = \frac{J_i^D}{D} |_{\mathbf{r}=\mathbf{r}_\alpha} = (\partial_i \theta' + q \varepsilon_{ij} \partial_j \ln \rho') |_{\mathbf{r}=\mathbf{r}_\alpha}, \quad (3.5)$$

which shows that vortices move due to disturbances in the condensate phase and density. The phase term is simply the superfluid velocity induced by the phase perturbation. Notice that the drift induced by density disturbances is perpendicular to the density gradient. The velocity of a vortex induced by a passing planar density wave will be perpendicular to the direction of propagation for the wave.

3.3 Energy spectrum of a single vortex

To study turbulence, we can look for signatures in the spectral energy to characterise the dynamical regime. We can derive analytically the incompressible contribution to the energy spectrum from a single vortex. The phase of a vortex located at $[0, 0]$ is given by $\theta = \arctan(x/y)$. In radial components the velocity of the superfluid can be written as

$$\mathbf{v} = \nabla\theta = \frac{\sin\theta\mathbf{i}_x - \cos\theta\mathbf{i}_y}{r}.$$

The kinetic energy of the flow is given by

$$E = \frac{m}{2} \int d\mathbf{r} \rho |\mathbf{v}|^2 = \frac{m}{2} \int d\mathbf{k} |\sqrt{\rho} v_x|^2 + |\sqrt{\rho} v_y|^2,$$

due to Parseval's theorem. We can transform the integral to polar coordinates in k -space. The kinetic energy in the region $[k, k + dk]$ is then proportional to

$$E(k) \propto k \left(|\sqrt{\rho} v_x|^2 + |\sqrt{\rho} v_y|^2 \right).$$

For the velocity field specified above we have

$$\sqrt{\rho} v_x = \frac{1}{2\pi} \int d\mathbf{r} \frac{\sin\theta}{r} \sqrt{\rho(r)} e^{-ikr \cos\theta} = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dr \sin\theta \sqrt{\rho(r)} e^{-ikr \cos\theta} = 0,$$

so the spectrum of the kinetic energy only depends on the y -component

$$\begin{aligned} \sqrt{\rho} v_y &\propto \frac{1}{2\pi} \int d\mathbf{r} e^{-ik\cdot\mathbf{r}} \frac{\sqrt{\rho(r)}}{r} \cos\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dr e^{-ikr \cos\theta} \sqrt{\rho(r)} \cos\theta \\ &= -\frac{1}{2\pi} \frac{d}{dk} \int_0^{\infty} dr r \frac{\sqrt{\rho(r)}}{r} \int_{-\pi}^{\pi} \frac{d\theta}{ir \cos\theta} e^{-ikr \cos\theta} \cos\theta. \end{aligned}$$

The cosines cancel and the integral over θ gives a zeroth order Bessel function of the first kind,

$$\sqrt{\rho} v_y \propto \frac{d}{dk} \int_0^{\infty} dr \frac{\sqrt{\rho(r)}}{r} J_0(kr).$$

Now, doing the substitution $u = kr$, we can rewrite this expression as

$$\sqrt{\rho} v_y \propto \frac{d}{dk} \left(\frac{1}{k} \int_0^{\infty} du \frac{\sqrt{\rho(u/k)}}{u/k} J_0(u) \right) = - \int_0^{\infty} \frac{du}{u} J_0(u) \frac{d}{dk} \sqrt{\rho(u/k)}$$

In the near core region, $k\xi \gg 1$, we can approximate the density profile $\sqrt{\rho} \sim r$ [19]

$$\sqrt{\rho} v_y \propto - \int_0^{\infty} \frac{du}{u} J_0(u) \frac{d}{dk} (u/k) \sim \frac{1}{k^2} \int_0^{\infty} du J_0(u)$$

and kinetic energy scales as

$$E(k)_{k \gg 1} \propto k |\widetilde{\sqrt{\rho}v_y}|^2 \propto k^{-3}.$$

We now consider the opposite limit i.e. in the far-field where $k\xi \ll 1$. In this case, the density profile is constant

$$\widetilde{\sqrt{\rho}v_y} \propto \int_0^\infty \frac{dr}{r} \frac{d}{dk} J_0(kr) \propto - \int_0^\infty \frac{dr}{r} r J_1(kr) \propto - \int_0^\infty dr J_1(kr).$$

By using the same substitution, $u = kr$, we then get

$$\widetilde{\sqrt{\rho}v_y} \propto -\frac{1}{k} \int_0^\infty du J_0(u).$$

From which it follows that

$$E(k)_{k \ll 1} \propto k^{-1},$$

which is the energy spectrum induced by an isolated vortex in the far-field. For many interacting vortices, deviations from this -1 can be used to characterise the regime of the vortex configuration. If the incompressible spectrum at low k has a gentler slope than k^{-1} this implies that the vortices are screening each other. In the opposite case if the slope at low k is steeper than k^{-1} this implies that the vortices have some degree of clustering.

Chapter 4

Numerical methods

In this chapter, we describe the numerical implementation of the GPE from Equation 2.21 and the PVM (Equation 3.5). For the GPE we use the spectral method in space and the operator splitting method in time, this is presented in Section 4.1. The method has been benchmarked and tested in earlier work by Rønning and Angheluta [32] and Rønning et al. [46]. For the PVM (Section 4.3) we use the forward Euler scheme to update the vortex positions. We also discuss how we set the initial conditions for the GPE in Section 4.2, and the PVM in Section 4.3, as well as presenting two approaches for tracking vortices in our solution for Ψ in Section 4.4. Finally, in Section 4.5 we will present the implementation of the energy spectra.

4.1 Spectral method for solving the GPE

In general, a non-linear PDE can be written as the sum of the non-linear part and the linear part

$$\partial_t \Psi(\mathbf{r}, t) = \omega(\nabla) \Psi(\mathbf{r}, t) + N(\Psi, t),$$

where $\omega(\nabla)$ is linear differential operator and $N(\Psi, t)$ contains the non-linear terms. For brevity we drop the dependence of Ψ on $\mathbf{r}$ and t in the non-linear term. Transforming into Fourier space gives us a set of algebraic equations in $\mathbf{k}$

$$\partial_t \tilde{\Psi}(\mathbf{k}, t) = \tilde{\omega}(\mathbf{k}) \tilde{\Psi}(\mathbf{k}, t) + \tilde{N}(\Psi, t),$$

multiplying the equation with $e^{-\tilde{\omega}(\mathbf{k})t}$ allows us to rewrite the equation as

$$\partial_t \left(\tilde{\Psi}(\mathbf{k}, t) e^{-\tilde{\omega}(\mathbf{k})t} \right) = \tilde{N}(\Psi, t) e^{-\tilde{\omega}(\mathbf{k})t}.$$

Integrating from t to $t + \Delta t$ we get

$$\tilde{\Psi}(\mathbf{k}, t + \Delta t) = \tilde{\Psi}(\mathbf{k}, t) e^{\tilde{\omega}(\mathbf{k})\Delta t} + \int_t^{t+\Delta t} dt' \tilde{N}(\Psi, t') e^{-\tilde{\omega}(\mathbf{k})t'}.$$

We can make the substitution $\tau = t' - t$, this gives

$$\tilde{\Psi}(\mathbf{k}, t + \Delta t) = \tilde{\Psi}(\mathbf{k}, t) e^{\tilde{\omega}(\mathbf{k})\Delta t} + e^{\tilde{\omega}(\mathbf{k})\Delta t} \int_0^{\Delta t} d\tau \tilde{N}(\Psi, \tau) e^{-\tilde{\omega}(\mathbf{k})\tau}.$$

Assuming the non-linear part is constant, $\tilde{N}_0$, over the range $\tau \in [0, \Delta t]$ we find the solution

$$\tilde{\Psi}(\mathbf{k}, t + \Delta t) = \tilde{\Psi}(\mathbf{k}, t) e^{\tilde{\omega}(\mathbf{k})\Delta t} + \frac{\tilde{N}_0}{\tilde{\omega}(\mathbf{k})} \left(e^{\tilde{\omega}(\mathbf{k})\Delta t} - 1 \right).$$

We can improve our initial estimate for $\tilde{\Psi}_1 \approx \tilde{\Psi}(\mathbf{k}, t + \Delta t)$ by calculating $\tilde{N}_1 = \tilde{N}(\Psi_1, t + \Delta t)$ and letting $\tilde{N}$ vary linearly from $\tilde{N}_0$ to $\tilde{N}_1$

$$\tilde{N}(\Psi(t + \tau), t + \tau) = \tilde{N}_0 + \frac{\tau}{\Delta t} (\tilde{N}_1 - \tilde{N}_0).$$

We now get the additional term

$$(\tilde{N}_1 - \tilde{N}_0) \frac{1}{\tilde{\omega}(\mathbf{k})} \left[\left(e^{\tilde{\omega}(\mathbf{k})\Delta t} - 1 \right) \frac{1}{\Delta t \tilde{\omega}(\mathbf{k})} - 1 \right]$$

The scheme can then be written as

$$\begin{aligned} \tilde{\Psi}_1 &= \tilde{\Psi}(\mathbf{k}, t) e^{\tilde{\omega}(\mathbf{k})\Delta t} + \frac{\tilde{N}_0}{\tilde{\omega}(\mathbf{k})} \left(e^{\tilde{\omega}(\mathbf{k})\Delta t} - 1 \right) \\ \tilde{\Psi}(\mathbf{k}, t + \Delta t) &= \tilde{\Psi}_1 + (\tilde{N}_1 - \tilde{N}_0) \frac{1}{\tilde{\omega}(\mathbf{k})} \left[\left(e^{\tilde{\omega}(\mathbf{k})\Delta t} - 1 \right) \frac{1}{\Delta t \tilde{\omega}(\mathbf{k})} - 1 \right]. \end{aligned}$$

To solve the damped GPE we use the operators [29]

$$\tilde{\omega}(\mathbf{k}) = (i + \gamma) \left(1 - \frac{1}{2} k^2 \right) \quad (4.1)$$

$$N(\mathbf{r}, t) = - (i + \gamma) \left[V\Psi + |\Psi|^2\Psi \right]. \quad (4.2)$$

When solving the GPE we use the Thomas-Fermi solution (Equation 2.22) as the starting point for the initial condition for Ψ .

4.2 Confining potential and generating vortices

Unless otherwise specified we use a disk shaped potential to trap the condensate. This gives a uniform condensate density in the bulk, while the density vanishes at the trap wall. The trap is radially symmetric and has the form

$$V_{\text{Ext}} = \frac{V_0}{2} \left[1 + \tanh\left(\frac{r - R}{\chi}\right) \right], \quad (4.3)$$

where V_0 gives the height of the potential wall, R the radius of the disk, χ dictates the steepness of the wall, and r is the radial coordinate. Typical values of the trap parameters are given in Tabel 4.1. With a condensate at rest the potential does not

Table 4.1: Typical trap parameters for a quadratic domain of size $L \times L$.

V_0	$10\mu_0$
R	$0.55 \times \sqrt{2}/2L$
χ	1.0ξ

need to be higher than $V_0 = 1\mu_0$ for the condensate density to approach zero at the boundaries. In simulations with high vortex densities however, the condensate will slosh up the walls and the trap needs to be significantly deeper to prevent the condensate from reflecting off the walls of the numerical domain. For low energy set-ups with few vortices, we can increase the trap radius $R = 0.65 \times \sqrt{2}/2L$. This effectively reduces the height of the potential wall, which is okay as long as the condensate does not have the energy to reflect off the numerical boundary. Generally it is beneficial to use a

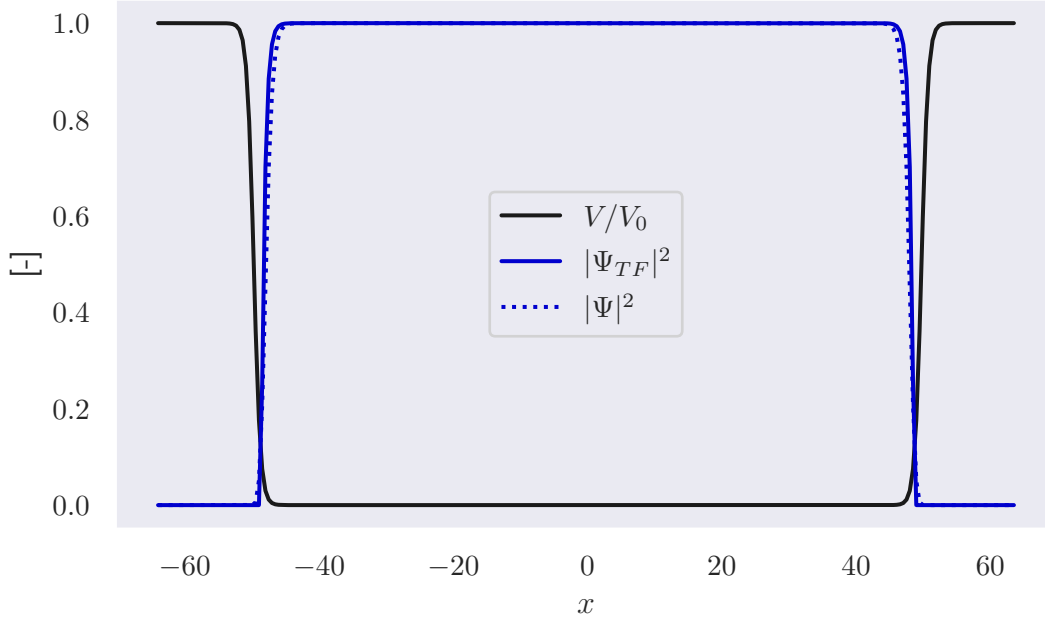


Figure 4.1: The ground state density profile of the condensate (blue), and the normalised potential (black). The solid line is the Thomas-Fermi density profile, while the dotted line is the density profile of the condensate after letting the system evolve until $5i\tau$.

disk with larger radius if we can, since the areas outside of the disk are essentially a waste of computational resources. As an initial guess to the solution of the GPE we use the Thomas-Fermi approximation, as given by Equation 2.22. Close to the walls this solution is inaccurate. To get the proper density profile we can propagate the system in imaginary time, that is, we replace $i\partial_t$ with ∂_t in the GPE and let the system evolve towards its ground state [29]. In this case we replace $(i + \gamma)$ with 1 in the operators above (Equations 4.1 and 4.2). We refer to propagating the system in imaginary time as relaxing the condensate. The Thomas-Fermi approximation is compared to the numerical solution after imaginary time propagation in Figure 4.1. We see that the Thomas-Fermi approximation works well in the bulk, while it deviates slightly near the wall.

To study dynamics we need to introduce excitations to the condensate. We have used two different strategies for exciting the condensate and creating vortices. The first is to imprint the condensate with a singular phase, θ , given by Equation 2.30. The condensate is multiplied with the vortex phase

$$\Psi = \Psi_0 e^{-i\theta}.$$

The negative sign is there for the vortex charge to be consistent with the sign of the D -field. Next, the condensate is relaxed. Letting the system evolve until $it = i\tau$ is often sufficient for the vortex profile to develop. This approach allows for imprinting vortices in specific configurations. The other approach is to perturb the Thomas-Fermi solution with a random phase and relax the system until we have an appropriate amount of vortices. This results in a random vortex configuration.

As a general rule we use a numerical domain twice the size of the physical system, giving a resolution of 0.5ξ . When evolving the system we use the time step $\Delta t = 0.01\tau$. In the cases where we use higher resolution the time step is varied accordingly to keep the Courant number, $v\Delta\tau/\Delta x$, approximately the same across all simulations. Of course

there will be some variation in the velocity, v , according to the specific set-up. The solver used for the GPE has been used in previous work [47, 32, 46] with the same temporal and spatial resolution as just stated.

4.3 Numerical discretisation of the PVM

The PVM is stated in Equation 3.5. The phase of the vortices are given by Equation 2.30 with the corresponding velocity field in Equation 2.31. To calculate the motion of the vortices from the PVM we define a set of initial positions, $\mathbf{r}_i(t_0)$, where i is the vortex index. The positions are updated according to

$$\mathbf{r}_i(t_1) = \mathbf{r}_i(t_0) + \mathbf{v}_i(t_0)\Delta t. \quad (4.4)$$

A vortex flows with the superfluid velocity induced by its neighbours. To find the velocity, $\mathbf{v}_i(t_0)$, for vortex i , we sum all the velocity fields from the other vortices, $j \neq i$, evaluated at the vortex position, $\mathbf{r}_i(t_0)$. It is also possible to calculate a velocity induced by the change in the density profile due to the neighbouring vortices, this is only relevant when the vortices are close together. If we assume that the density profile of the vortices is frozen, and given by the ansatz in Equation 2.29, then the second term of Equation 3.5 gives

$$\nabla^\perp \ln \rho = 2q_j \Lambda^{-2} \xi^2 \frac{(y_i - y_j)\mathbf{i}_x - (x_i - x_j)\mathbf{i}_y}{|\mathbf{r}_i - \mathbf{r}_j|^2(|\mathbf{r}_i - \mathbf{r}_j|^2 + \Lambda^{-2}\xi^2)},$$

where Λ^{-2} sets the area of the vortex core. The density correction is aligned in the same direction as the velocity induced by the vortex, so it will primarily effect the velocity of the vortices and not their trajectories.

To implement periodic boundary conditions we need to keep the vortices within a box of size $L \times L$ given by $x \in [0, L]$ and $y \in [0, L]$. For every time step we shift the vortex positions using the modulus operator

$$\mathbf{r}_i(t_1) \leftarrow \mathbf{r}_i(t_1) \bmod L. \quad (4.5)$$

Since there are ghost vortices beyond the boundaries of the domain which impact the motion of the vortices we also make eight copies of the vortex configurations, shifting the position of the vortices by $\pm L$ in the x -direction and y -direction. The update of the position of a vortex is then calculated summing over all the other vortices within the domain as well as all the ghosts in the eight neighbouring domains. For each time step the positions of the ghosts are updated by copying the positions of the vortices within the domain. To annihilate the vortices we simply set their charge to zero whenever two vortices are closer together than 2ξ , this way they stop moving and stop influencing the motion of the other vortices. We also set the charge of their ghosts to zero. The start positions of the vortices are set to replicate GPE simulations. Since the GPE simulations are done on the domain $x \in [-L/2, L/2]$ and $y \in [-L/2, L/2]$ we have to shift the initial vortex positions by $[L/2, L/2]$ before starting the simulations, and shift them back by $[-L/2, -L/2]$ afterwards for comparison.

4.4 Tracking

We have two different kinds of numerical set-ups, which require two different tracking methods suited to their separate needs, we call these the D -tracker and the ρ -tracker.

In the turbulence simulations we are interested in determining the vortex number as accurately as possible. In the dipole-dipole collisions and wave-vortex collisions we already know how many defects we are expecting to find, but need a method that determines the location of the phase slip to sufficient accuracy. The D -tracker uses the pseudo-vorticity field to locate the vortices. The D -field is non-zero at vortices, as well as for some other features in the condensate with steep density gradients such as near a wall or in a solitary wave. Generally the magnitude of the D -field due to vortices are higher, and can be discerned from other features. The ρ -tracker is based on finding the minimum of the condensate density.

4.4.1 Pseudo-vorticity based approach

The implementation of the D -tracker is modified from the previous work of Rønning [47]. The first step is to filter out the regions which constitute vortices. We identify all pixels where the pseudo-vorticity exceeds a threshold, $\pm\varepsilon_D$, and the density is below the threshold $\rho < \varepsilon_\rho$. This is done separately for positive and negative values of D . Then the pixels are sorted into clusters. The clusters with an area that is either too large or too small to constitute a stable vortex are removed. At this stage one can simply count the number of clusters to find the vortex number. If it is necessary to also find the vortex position this is done by averaging the positions of the pixels within each cluster.

In theory $\varepsilon_D \approx 0.7$ [32], in practice however not all defects reach that amplitude. After visually inspecting a number of different vortex configurations, and how they evolve, we have found that most features that show up using $\varepsilon_D = 0.3$ are persistent, and can be interpreted as vortices. In addition to the magnitude of the D -field we know that the vortices have low density and that they generally span a consistent area. In addition to the threshold on the D -field we have implemented an upper threshold of the density on $\rho = 0.3\rho_0$, and an upper and lower limit on the area of the vortices as $A_{\text{upper}} = 12.5\xi^2$ and $A_{\text{lower}} = 0.75\xi^2$. In damped systems this is often unimportant, but at $T = 0$, in systems with a high vortex density there may be other features than vortices that surpass the threshold on $|D|$. Some of these are physically realistic high energy excitations, others arise from numerical noise. The additional requirements on density prevent the classification of non-topological high energy excitations as vortices. The lower threshold on the area of the vortex mostly acts as a noise filter. If the system has a global topological charge, the tracking algorithm sometimes mistakes the region outside the trap wall as a vortex. Setting an upper limit on the area of a vortex prohibits this. $A_{\text{upper}} = 12.5\xi^2$ is set a lot higher than the area of single vortices to allow the tracker to still locate defects that are colliding or merging.

4.4.2 Condensate density based approach

The D -tracker is designed to reliably find the number and position of topological defects. In the dipole-dipole collisions and the wave-vortex collisions we also need to track the location of phase slips of defects that are either very close together, or phase slips which lack a stable core altogether. The phase slips in these cases are not located at the centre of the non-zero region D -field. The D -field cannot be used to determine their positions with high accuracy. We have therefore implemented a different tracker, the ρ -tracker, which locates the minima of the order parameter.

Again, the first step is to filter out the regions that satisfy $|D| < \varepsilon_D$ and $\rho < \varepsilon_\rho$. The threshold for the condensate density is set to $\varepsilon_\rho = 0.3$, but the threshold for the pseudo-vorticity is now set to $\varepsilon_D = 0.1$ to include more features. Since the collision event

happens within a small region, everything outside the collision region can be excluded. The pixels that satisfy the requirements on $|D|$ and ρ are clustered, and counted. Since there is an upper limit on the number of phase slips one will find during the collision these two steps are performed recursively, adjusting ε_ρ down slightly, until the number of clusters that satisfy $\rho < \varepsilon_\rho$ falls within the expected range. To ensure that there are enough pixels in each cluster to find a sensible minima we include all pixels within a radius of $\sqrt{2}dx$ of the centre of the cluster. This gives a minimal cluster size of nine pixels. The next step is to interpolate the order parameter within each cluster using Clogh-Tocher interpolation and find the minima of the magnitude of the interpolated order parameter. This gives the estimate for position of the zero of the order parameter, or equivalently, for the phase slip.

4.5 Implementation of energy spectra

The kinetic energy density in real space is given by $e(\mathbf{r}) = |\mathbf{u}(\mathbf{r})|^2$, where $\mathbf{u}(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}\mathbf{v}(\mathbf{r})$. The superfluid velocity is given by

$$\mathbf{v}(\mathbf{r}) = \mathbf{J}(\mathbf{r})/\rho(\mathbf{r}) = \text{Im} \left\{ \frac{\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r})}{\Psi^*(\mathbf{r})\Psi(\mathbf{r})} \right\} = \text{Im} \left\{ \frac{\nabla\Psi(\mathbf{r})}{\Psi(\mathbf{r})} \right\}.$$

Taking the gradient in Fourier space and multiplying by $\sqrt{\rho(\mathbf{r})} = |\Psi(\mathbf{r})|$, we get an expression for $\mathbf{u}(\mathbf{r})$

$$u_i(\mathbf{r}) = |\Psi(\mathbf{r})\Psi'(\mathbf{r})| \text{Im} \left\{ \frac{\mathcal{F}^{-1}\{ik_i\mathcal{F}[\Psi'(\mathbf{r})]\}}{\Psi(\mathbf{r})} \right\}. \quad (4.6)$$

where the index i signifies the x or y component of the vector field and $\Psi'(\mathbf{r})$ is a filtered order parameter, to ensure that it is periodic. The compressible and incompressible velocity fields $\mathbf{u}^c(\mathbf{r})$ and $\mathbf{u}^i(\mathbf{r})$ are found by implementing the Helmholtz transformation give in Equation 2.36 and Equation 2.37. To find the energy spectrum of the velocity field in question the energies are sorted according to the magnitude of k in Fourier space. The kinetic energy associated with a velocity field $\mathbf{u}$ is given by $\int d\mathbf{r}|\mathbf{u}(\mathbf{r})|^2 = \int d\mathbf{k}|\tilde{\mathbf{u}}(\mathbf{k})|^2$ according to Parseval's theorem. The first step to find the energy per wavenumber, $e(k)$ is to calculate the norm of the velocity field at each point in k -space. Next, we make a set of bins spanning $[k_i, k_i + dk]$. For each bin, i , the energies $\epsilon(\mathbf{k})$ where $|\mathbf{k}| \in [k_i, k_i + dk]$ are summed up,

$$e(k_i) = \sum_{|\mathbf{k}|=k_i}^{|\mathbf{k}|=k_i+dk} \epsilon(\mathbf{k}).$$

This gives a distribution of energy in 1D k -space, $e(k)$.

Chapter 5

Results on collisions events

In this chapter we will first compare the velocity of a dipole to theoretical estimates using the PVM (Equation 3.5) and the pseudo-vorticity current (Equation 3.4), showing that core effects are significant when vortices are close together. In Section 5.2 we will look at head-on collisions between dipoles. This set-up enables us to study both vortex annihilation and vortex scattering, and to study the role of the compressible kinetic energy in these two scenarios. Next, in Section 5.3 we look at the collision between a solitary wave and a vortex. Previous studies have drawn attention to the formation of this wave during annihilation [5, 4, 13], and identified it as the last stage of the annihilation process. Here we study the interaction between the solitary wave and vortex in further detail. Finally, in Section 5.4 we address 2D decaying quantum turbulence directly, looking at vortex number decay and the structure of the turbulent state through its spectrum.

5.1 Dipole velocity

To compare the theoretical velocity to the actual vortex velocity in the full solution we imprint a vortex and an anti-vortex in a disk trap, and let the density profile evolve through imaginary time propagation while keeping the imprinted phase fixed. This way the vortices do not drift towards each other during the imaginary time propagation. As the vortices start moving there is a sudden burst of acceleration. For the first 5τ of the simulation we add a damping coefficient of $\gamma = 0.01$ throughout the system to dissipate some of the disturbances. Then we set $\gamma = 0$ and let the simulation run until $t = 20\tau$, collecting data between $t = 10\tau$ and $t = 20\tau$. We use three ways of finding the velocity of the vortices. The first is simply tracking the position of the vortices over time and doing a linear regression of the position as a function of time. The slope gives us the velocity of the vortex. In the second method we calculate the velocity using Equation 3.4, the implementation of which is discussed in Appendix D. The third method is to use the approximation $v = 1/\Delta r$, which comes from the phase term of the PVM (Equation 3.5).

Since the phase term of the PVM neglects core effects we expect $v = 1/\Delta r$ to break down when Δr is small. From Figure 5.1 we see that the PVM estimate is generally good, but undershoots at small vortex separations. Here the density correction to the PVM is necessary. The estimate from the current of pseudo-vorticity Equation 3.4 is less stable than the estimate from the PVM. Apart from one outlier between $\Delta r = 2\xi$ and $\Delta r = 3\xi$ however, it is more accurate at low Δr . This is because the D -field inherently

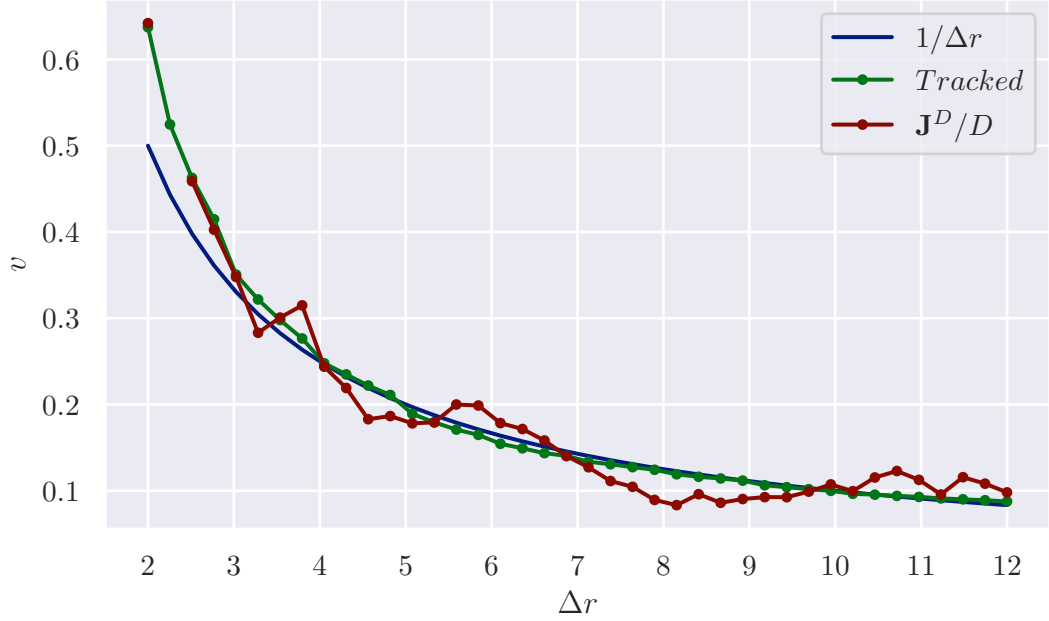


Figure 5.1: The velocity of the dipoles for different vortex separations Δr . The green line shows the velocity tracked from the full GPE solution. The blue line, $1/\Delta r$, is a theoretical prediction from the PVM. The red line shows the vortex velocity calculated from the current of pseudo-vorticity (Equation 3.4). Between $\Delta r = 2\xi$ and $\Delta r = 3\xi$ we have removed one outlier in the red curve.

accounts for density modulations in the core structure of the vortices, which is significant when the vortices are close together.

5.2 Dipole-dipole collisions

In decaying quantum turbulence at zero temperature there is no energy dissipation, but there can be an energy transfer between different forms of energy. There could be several mechanisms for the transfer of energy between the incompressible modes and the compressible modes. For example, during acceleration quantum vortices radiate compression waves [10]. Another mechanism of energy transfer is the annihilation event. As vortices annihilate their density profiles flatten out. This substantial change in the density profile of the condensate creates waves. We wish to study these mechanisms closer in a controlled setting. A set-up which enables us to look into energy transfer between the incompressible and compressible modes both in an annihilation event as well as from radiation is a four vortex collision. A similar set-up has been studied experimentally by Kwon et al. [11]. Below a critical initial vortex separation Δr_c the vortices annihilate. By varying Δr above and below Δr_c we can study the energy transfer during annihilation, as well as for interacting vortices that do not annihilate.

We imprint two dipoles in a disk trap of radius $R = 29.4\xi$ such that they propagate towards one another. Initially the dipole pairs are separated by 20ξ . After imprinting the vortices we propagate the system in imaginary time until $it = i\tau$ to let the density profiles of the vortices develop. The relaxation in imaginary time decreases the distance between the phase slips slightly, when plotting against Δr we are using the measured Δr during the approach. For the dipoles in the lower range we are imprinting the vortices

with separations in the range, $\Delta r \in [3\xi, 3.5\xi]$. We run the simulations until $t = 140\tau$. In the upper range we have $\Delta r \in [4\xi, 8\xi]$, and let the system evolve until $t = 100\tau$.

5.2.1 The collision process

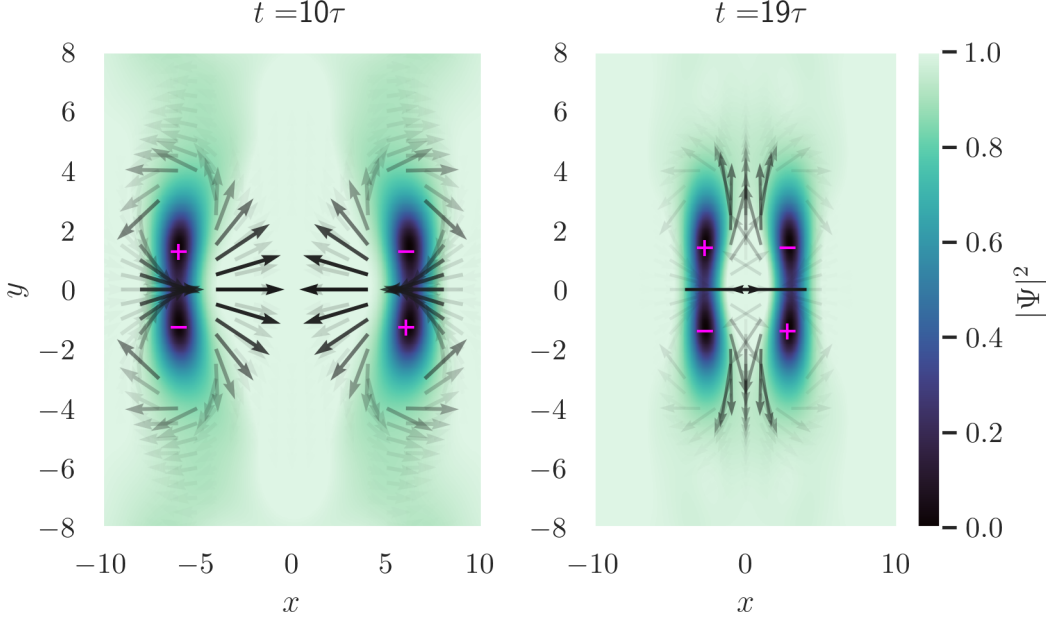


Figure 5.2: The colour map indicates the condensate density while the arrows represent the superfluid velocity. The opacity of the arrows as well as their length is scaled with v to highlight where the superfluid velocity is high. The + and – indicate the topological charge of the vortices.

Initially the dipoles propagate towards each other. As they come closer however the velocities induced by the topmost vortex-antivortex pair and the bottom most vortex-antivortex pair is directed away from the centre of the trap, and pull the dipoles apart. This process is seen in Figure 5.2. As the dipoles split up, they partner up with a vortex from the other dipole, forming two new dipoles propagating in the y -direction, perpendicular to the original direction of motion to the incoming dipoles. The collision event is a partner exchange.

As the dipoles switch partners the low density region connecting the dipoles disappears and a new one forms. We can see the transition from one set of dipole pairs to another in the top row of Figure 5.3. The modulation of the density profile acts as a source of radiation.

In Figure 5.3 we see the two possible outcomes of the dipole collision. In the top row there is a partner exchange and two dipoles are propagating away from each other in the y -direction. In the row below the dipoles are annihilating and a wave is spreading out in all directions from the collision site. There is a critical vortex separation, Δr_c , below which the dipoles annihilate during the collision. Note that there will be some variation in Δr_c depending on how we imprint the vortices, so the value we have found is specific to our set-up. The defects have annihilated once the phase singularities disappear. We compare the phase after the collision at $\Delta r = 2.45\xi$ to $\Delta r = 2.35\xi$ in Figure 5.4. At $\Delta r = 2.35\xi$ there is a steep phase gradient in the top and bottom of the outgoing wave, but no phase slips. In $\Delta r = 2.45\xi$ there are still phase slips present after the collision, so Δr_c must be somewhere between 2.35ξ and 2.45ξ . The wave that is emitted from

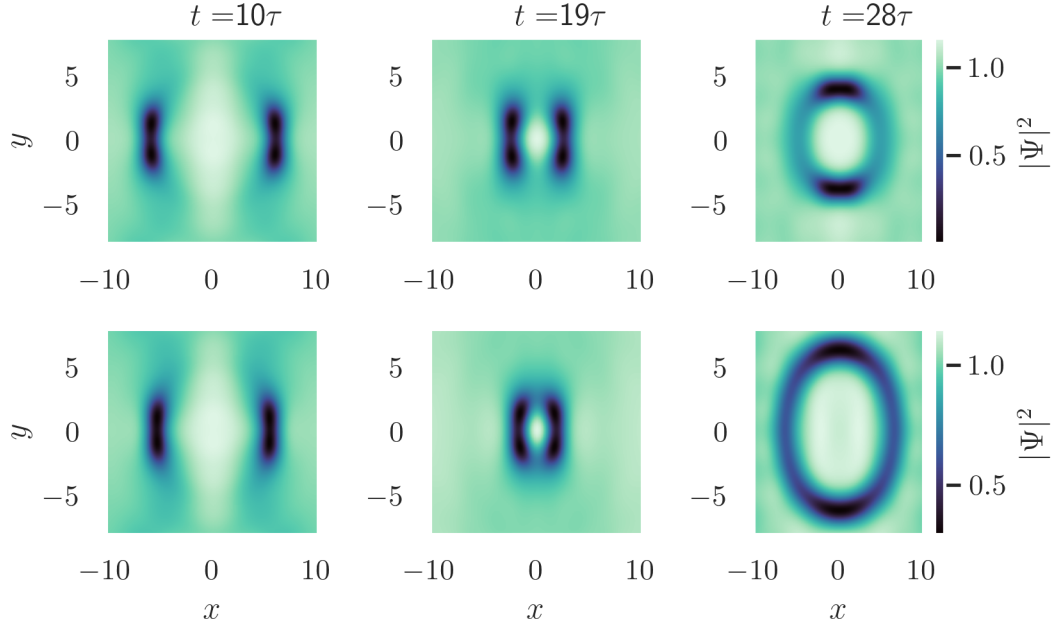


Figure 5.3: The density of the condensate at three different times during the collision process. The top three panels are for initial $\Delta r = 2.45\xi$ while the bottom panels are with an initial $\Delta r = 2.0\xi$. At $t = 10\tau$ we see the incoming dipoles, they are more or less identical. At $t = 19\tau$ we see the vortices as they are very close to each other, just before they are pulled apart. At $t = 28\tau$ we see the wave after the collision in the bottom panel, and the dipoles propagating away from the collision site in the top panel.

the condensate just below Δr_c , where the phase slips annihilate, resemble a dipole if we look at its pseudo-vorticity. There are no phase slips present, but there are clear regions of negative and positive pseudo-vorticity. When this wave hits the wall we observe that it does not simply reflect like a purely density driven wave, but splits up and traverses along the boundary of the trap like a dipole. The way a solitary wave just below the threshold of annihilation interacts with the wall closely resembles the dynamics of a dipole. This also applies to how it interacts with vortices.

5.2.2 Evolution of energy during the collision

In the cases where the dipoles approach each other and annihilate there is a flux of energy from the incompressible modes to the compressible modes. We can see this transfer in Figure 5.5. Between $t = 20\tau$ and $t = 25\tau$ the incompressible energy decreases rapidly as the compressible energy increases. A large proportion of the incompressible kinetic energy bound to the vortices is transferred to compressible kinetic energy during the annihilation event. As the vortices annihilate there is a spike in the quantum energy, E_q , which corresponds to a drop in $E_i + E_c$. After the collision E_q drops to a level below its initial value. The quantum energy is mostly localised around the vortices, as well as along the boundaries of the system if the condensate is trapped. Once the density profile of the vortices disappear only the contribution from the trap wall remains. The fact that the drop in $E_i + E_c$ corresponds to the peak in E_q is an indication that the energy transfer from the incompressible modes to the compressible modes is mediated by the quantum energy

$$E_i \rightarrow E_q \rightarrow E_c.$$

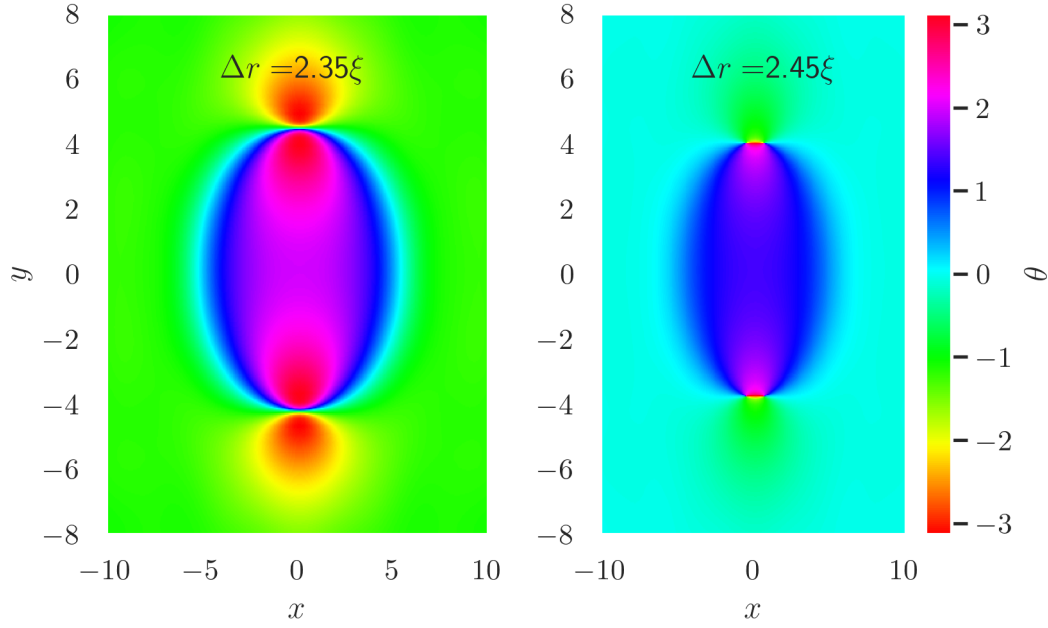


Figure 5.4: Both panels display the outgoing wave or dipole at $t = 28\tau$, as shown in the rightmost panels in Figure 5.3. In the left panel the dipoles have an initial vortex separation of $\Delta r = 2.35\xi$. We see the steep density gradients at the top and bottom of the wave, but no phase slips. In the right panel the initial vortex separation is $\Delta r = 2.45\xi$. We see the phase slips at the front of the wave. The phase gradient across $y = 0$ in the x -direction is higher in the left panel than in the right panel.

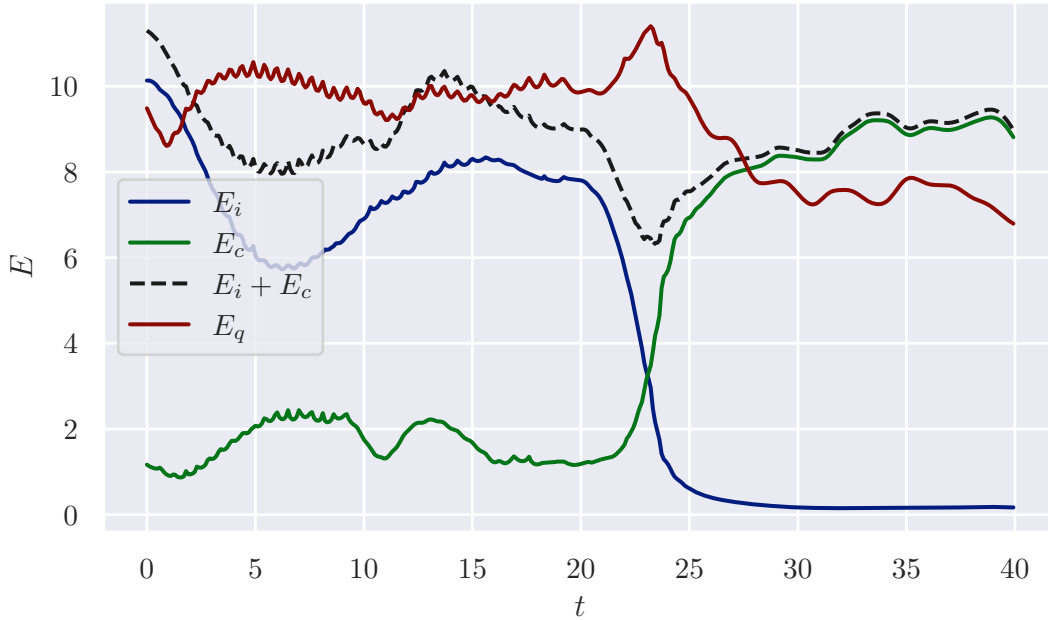


Figure 5.5: The kinetic energy over time during an annihilation event of dipoles with an initial vortex separation of $\Delta r = 2.1\xi$. As the vortices annihilate between $t = 20\tau$ and $t = 25\tau$ the incompressible kinetic energy, E_i , drops rapidly while the compressible kinetic energy, E_c , increases. The quantum energy increases during the annihilation event before it decreases again to a lower energy than its initial energy. The peak in E_q corresponds to a drop in the sum $E_i + E_c$. E_q has been shifted down by a constant $25\mu_0$ to make the figure more readable.

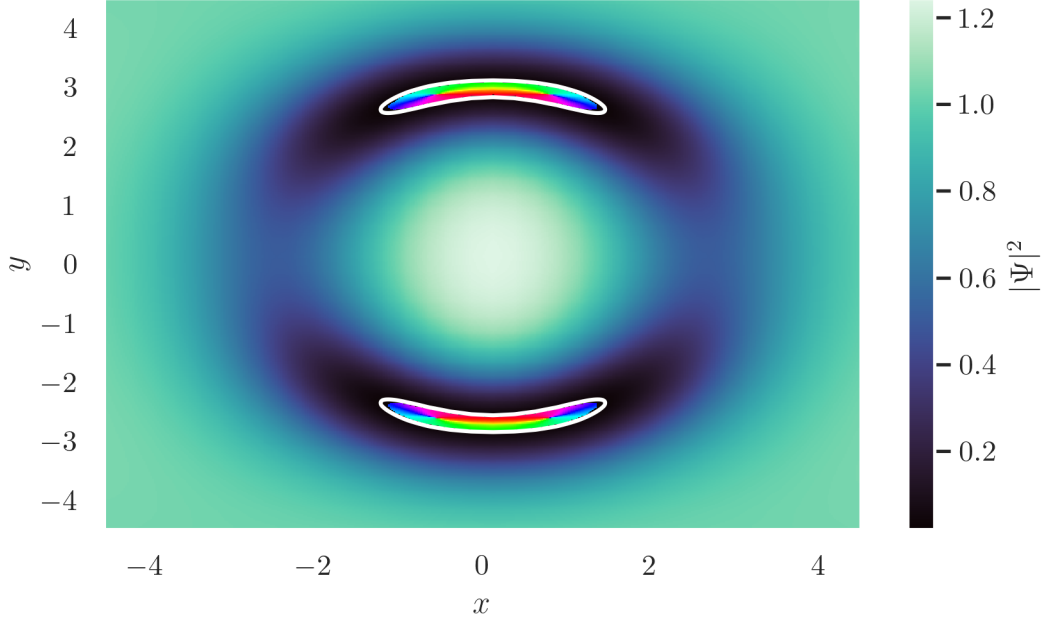


Figure 5.6: The density profile of the condensate at $t = 23.3\tau$ for $\Delta r = 2.1\xi$, which corresponds to the peak in E_q , as seen in Figure 5.5. The phase of the condensate is plotted within the region where $\rho < 0.03\rho_0$. In our simulations we have generally found phase slips below $\rho \approx 0.02\rho_0$ (Figure E.1), in our context this constitutes the core of the vortex. The core region where $\rho < 0.02\rho_0$ has a combined area of $1.15\xi^2$.

Annihilation occurs when the phase slips overlap. Right before the annihilation the cores of the defects merge, forming an extended core. This coincides with the peak in E_q . Figure 5.6 shows the condensate density, and phase in the core region, when E_q is at its highest. The combined area of the core region is $1.14\xi^2$. For reference, in our simulations the area of the core for a single stationary and isolated vortex is $0.11\xi^2$. When the vortices come close together there is first an energy flux from E_i to E_q . As the vortices screen each other more effectively E_i decreases. Meanwhile the extended core is surrounded by steep density gradients which increase E_q . Once the phase slips annihilates the core disappears and the condensate density flattens out, producing ripples and waves. This transfers energy from E_q to E_c .

Figure 5.7 shows the change in kinetic energy over time for a dipole collision above the threshold for annihilation. Again we see a simultaneous drop in incompressible kinetic energy and increase in compressible kinetic energy. In this case the increase in E_c does not directly correspond to the decrease in E_i – radiation can only partly account for the drop in E_i . Radiation is emitted as the density profile of the dipoles change from forming a low density region around the incoming dipole pairs to the outgoing dipole pairs. The reduction in E_i is due to screening, the outgoing dipoles generally have lower vortex separation, such that the velocity fields of the vortices in the dipoles cancel each other out more effectively. After the initial drop in E_i it starts increasing again. This is a detail we will come back to when looking at the trajectory of the vortices in Section 5.2.4.

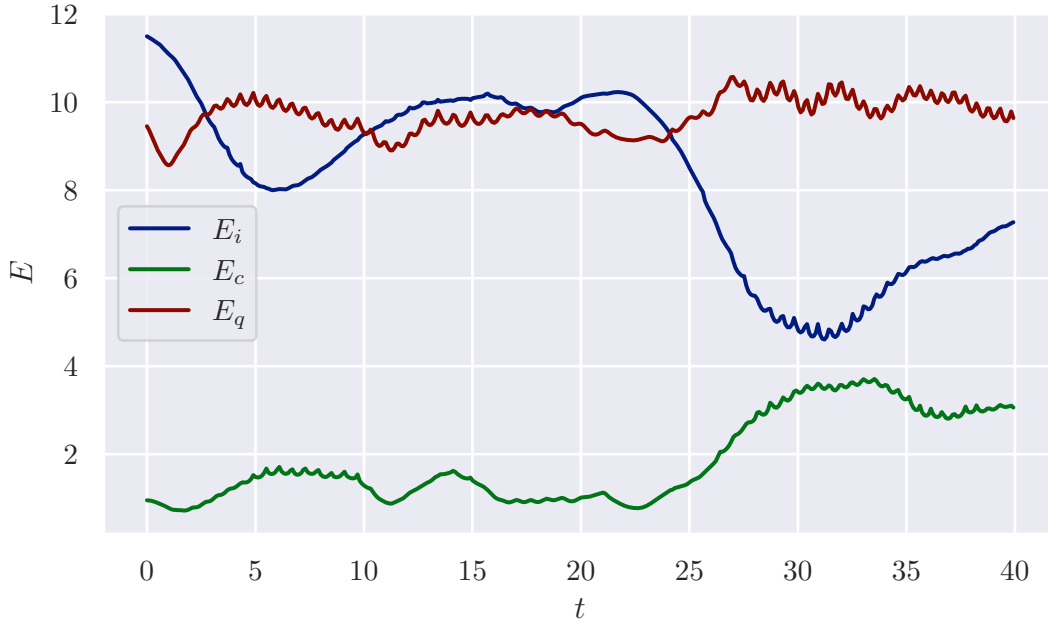


Figure 5.7: The kinetic energy over time for a pair of dipoles with an initial vortex separation of $\Delta r = 2.5\xi$. As the vortices accelerate there is an increase in E_c . There is also a decrease in E_i during the collision event itself. Afterwards there is a gradual increase in E_i . E_q has been shifted down by $25\mu_0$ to be on the same scale as E_i and E_c .

5.2.3 Change in kinetic energy

We calculate the change in kinetic energy for different Δr to see how the transfer of energy depends on the initial vortex separation. The variation in dipole velocities in the range $\Delta r \in [1.6\xi, 2.8\xi]$ is small enough that all start after $t = 15\tau$ and end before 40τ . We can then calculate the time averaged energy before 15τ and calculate the difference from time averaged energy after 40τ . The results are shown in Figure 5.8.

The vortices were found to annihilate somewhere between $\Delta r = 2.35$ and $\Delta r = 2.45$, in Figure 5.8 this is close to where ΔE_i starts increasing and ΔE_c decreasing. As expected the change in incompressible kinetic energy as well as in compressible kinetic energy is larger below the threshold for vortex annihilation. ΔE_i and ΔE_c have similar magnitudes, but with opposite signs, indicating that the incompressible kinetic energy bound up to the vortices has been transferred to compressible kinetic energy in the wave. There is also a negative change in the quantum energy. For vortices with a low initial separation, $\Delta r \approx 1.6\xi$, the profile vanishes completely and there is a significant change in E_q . Vortices have a quantum energy of $E_q = 1\mu_0$, as we will see in Section 5.3.5. Below $\Delta r_c = 2\xi$ the change in the quantum energy is close to $\Delta E_q = -4\mu_0$, which is the value expected if the density profile of four vortices disappeared completely. This also indicates that the density profile of vortices that are as close together as $\Delta r = 1.6\xi$ have the approximately same energy as well separated vortices, which is not given. As Δr increases, approaching Δr_c the wave leaving the collision gets steeper and steeper. Therefore the energy, E_q , after the collision increases with Δr , and $|\Delta E_q|$ decreases. Above Δr_c the energies transferred during the collision is significantly lower than below Δr_c . Note that the transition across Δr_c is smooth and spans an interval of approximately 0.3ξ . The phase slips are either present or not, so one could expect a sharp transition from where they show up to where they annihilate. This is not what we

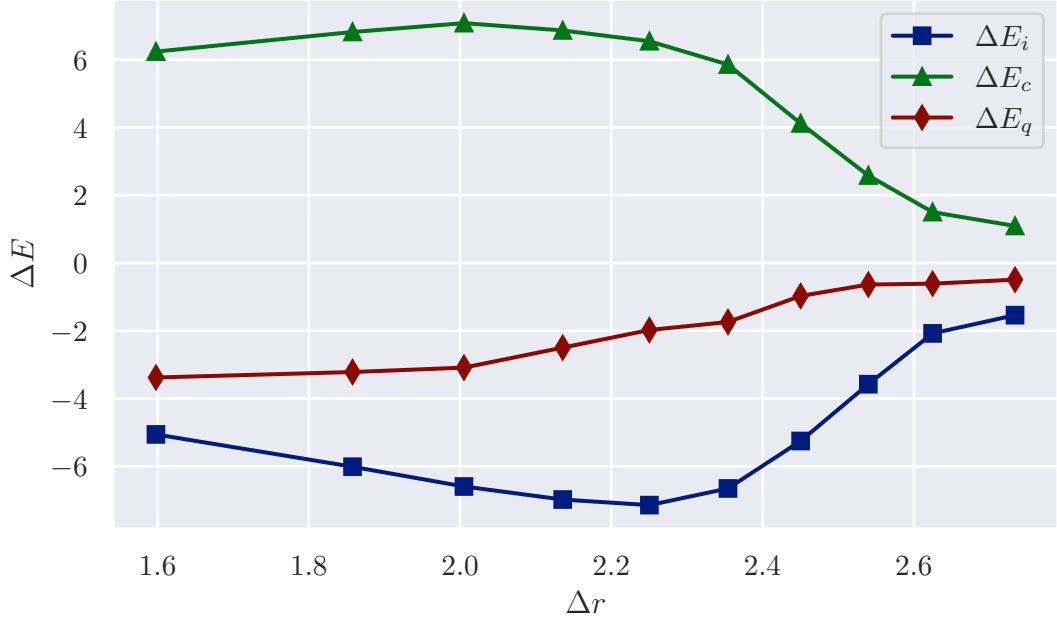


Figure 5.8: The change in kinetic energy during the collision. The phase slips annihilate at around $\Delta r = 2.4\xi$. The quantum energy, E_q changes by $\Delta E_q \approx -4\mu_0$ at $\Delta r = 1.6\xi$. The incompressible kinetic energy, E_i , and compressible kinetic energy, E_c , follow similar curves with opposite signs and slightly shifted magnitudes.

see. After the phase slips annihilate at around $\Delta r \approx 2.4\xi$ the change in energy for the dipoles scattering is similar to the change in energy for the formation of solitary waves.

5.2.4 Comparison between the PVM and the GPE

In the GPE the vortices have a coupling to the compressible modes that is not present in the PVM. To compare the difference between the two we have performed a set of simulations with vortex separations in the range $\Delta r \in [3\xi, 8\xi]$. The trajectories of the vortices for an initial vortex separation of $\Delta r = 4\xi$ is shown in Figure 5.9. The PVM trajectories in this figure are calculated from the phase term only. This is because in this set-up the density correction using the ansatz for the vortex profiles (Equation 2.29) gives velocities parallel with the phase throughout, so the trajectory looks the same in both cases. The only difference is that the density correction gives slightly higher velocities for the vortices. As long as the vortices propagate in a straight line the PVM and the GPE predict the same trajectories. During the acceleration however they deviate. The trajectories for the entire range of initial dipole moments, $[3\xi, 8\xi]$, are qualitatively similar to the trajectories shown in Figure 5.9, despite the fact that the distance at which the vortices are being pulled apart varies. This excludes overlapping density profiles as a source of radiation. It seems what is crucial to the deviation between the GPE and the PVM that we see in Figure 5.9 is the fact that the vortices are accelerating.

In Figure 5.10 the vortices are closer together than in Figure 5.9. When we look at the comparison between the trajectories from the PVM with the trajectories tracked from the GPE in this case, we get deviations which go beyond the effects from acceleration alone. As the vortices come close together their density profiles overlap and perturbations in the density profile influence the vortex motion. The vortices start deviating from a straight path when they are around 2ξ apart from each other. They seem to change

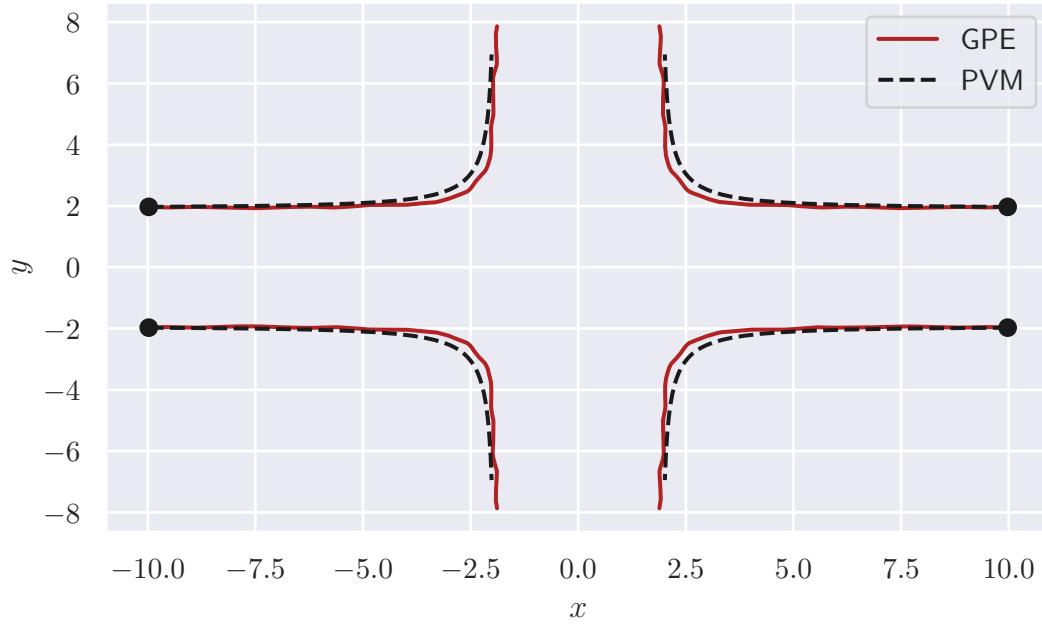


Figure 5.9: The trajectories of the vortices tracked from the full solution of the GPE (red) and the PVM (black dashed). We have an initial vortex separation of $\Delta r = 4\xi$. The vortices start at the positions indicated by the black dots.

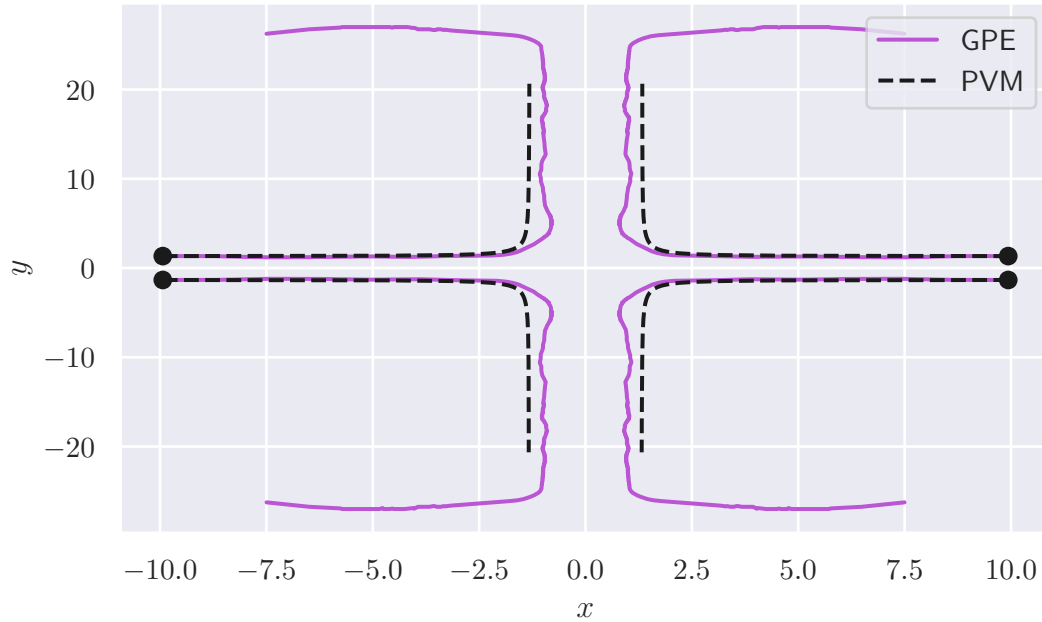


Figure 5.10: The initial vortex separation is $\Delta r = 2.55\xi$. The pink line indicates the tracked motion of the vortex from the full solution of the GPE, while the black dashed line gives the trajectory as calculated by the PVM.

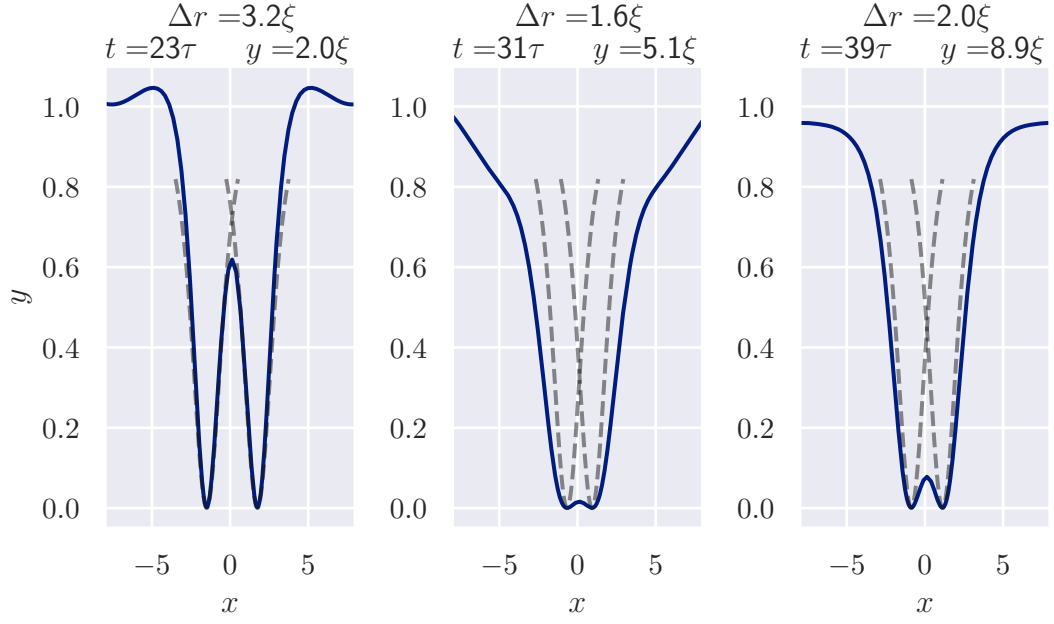


Figure 5.11: Cross sections of the condensate density for a dipole-dipole collision with initial separation of $\Delta r = 2.55\xi$, the same as in Figure 5.10 and Figure 5.7. The grey dotted lines indicate the density profile of an isolated vortex. In the centre panel the density profile of the vortices deviate significantly from the isolated vortex close to the core. The kink at around $0.8\rho_0$ comes from the wave surrounding the defects soon after the partner exchange.

direction in two stages, as the PVM takes a 90° turn the vortices in the GPE shift their direction by some smaller angle. When they are at their closest they repel each other before they propagate, with some oscillations, in the y -direction. At the bottom and top of Figure 5.10 the dipoles split up as they interact with the wall of the trap.

In the PVM there are no central forces unless we introduce a nonzero damping, $\gamma \neq 0$ [32], corresponding to $T > 0$. Yet the trajectories in Figure 5.10 show that there can be repulsion between the vortices in the GPE at $T = 0$. The cross section of the density profile along the x -direction through two of the phase slips is shown in Figure 5.11. The first panel shows the density of the vortices at $t = 23\tau$, as they start changing direction, which is also when the PVM starts deviating from the GPE (Figure 5.10). The second panel shows the vortices when they are at their closest, at $t = 31\tau$. This is also when E_c is around its maximum (Figure 5.7) and E_i at its minimum. The density between the phase slips is very low, and the density profile of the defects deviate significantly from the density profile of an isolated vortex. The kink at $0.8\rho_0$ is a signature of the wave surrounding the dipoles briefly after the exchange. After $t = 31\tau$ we see from Figure 5.7 that there is an increase in E_i and a decrease in E_c . At the same time we see from Figure 5.11 that the vortex separation is increasing from 1.6ξ to 2ξ . We take this to show that the vortices are reabsorbing radiation energy as they drift apart. It is possible to interpret radiation as a form of effective dissipation. Since radiation can both be absorbed and radiated the sign of γ_{eff} can be both positive or negative. From the second term in Equation 27 in Rønning and Angheluta [32],

$$v_i = -\gamma \partial_i \rho' + \dots, \quad (5.1)$$

where ρ' represents a perturbation of the density profile, we see how the dissipative attraction of vortices depends on the shape of the density profile of the vortices to the

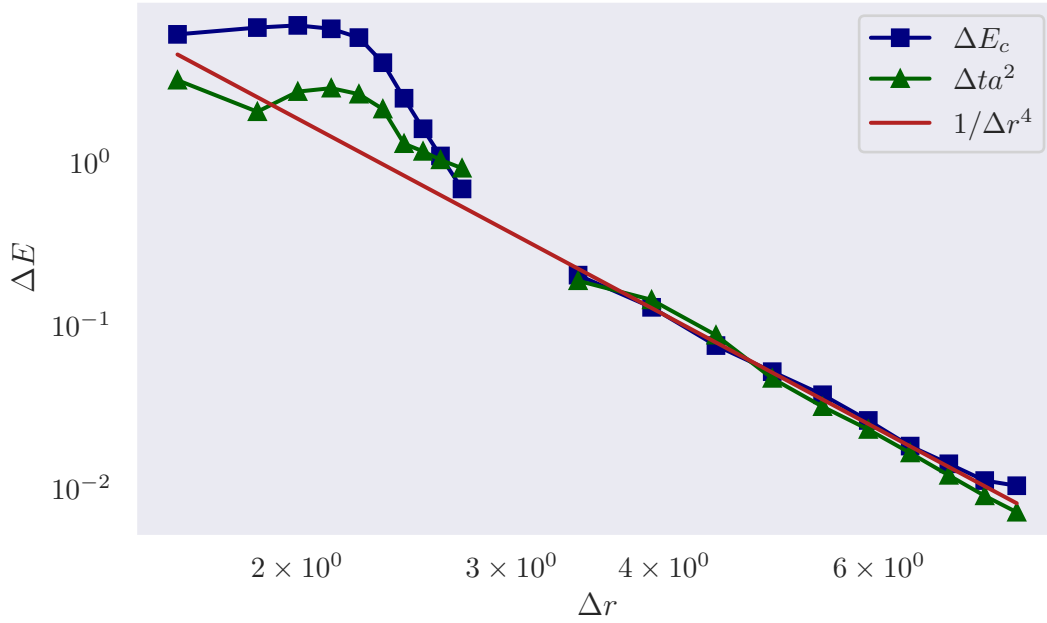


Figure 5.12: The blue squares shows the radiated compressible kinetic energy calculated directly from integrating over the compressible velocity field (Equation 2.38). The radiated energy according to Larmors formula is shown by the green triangles, and the energy radiated from an ideal head-on collision is shown in by the red line. The theoretical estimates include the contributions for all four vortices, and have both been scaled by a factor 30 to be on the same scale as ΔE_c .

leading order. At $t = 31\tau$ there is a significant deviation between the ideal vortex profile (grey dotted lines in Figure 5.11) and the actual profile. With a $\gamma_{eff} < 0$ due to absorption of radiation the vortices obtain a velocity away from each other according to Equation 5.1.

5.2.5 Energy radiation

When the vortices are far apart the primary mechanism of energy transfer between the incompressible modes and the compressible modes is radiation due to the acceleration of the vortices. The energy radiated from a vortex due to acceleration can be approximated by Larmors formula, $E \sim \int dt a^2$ [11], where a is the acceleration of the vortex. From this we can show that the radiated energy scales as $\Delta E_c \sim 1/\Delta r^4$, for ideal head-on collisions (Appendix A).

The radiated energy during the collisions are plotted in Figure 5.12. In the range $\Delta r \in [3\xi, 8\xi]$ the radiated energy calculated from the trajectories of the vortices (green) and directly from the compressible kinetic energy (blue) have the same scaling as the theoretical estimate $1/\Delta r^4$. However, the theoretical values (green and red) have been scaled up by a factor 30 to be on the same scale as ΔE_c . These approaches are based on point vortices, and underestimate the radiated energy since they do not account for the core structure.

The vortex density profiles deviate from the background density most significantly within the range of a couple of healing lengths. In the range $\Delta r \in [1.6\xi, 2.8\xi]$ there is a discontinuity in the energy calculated from the trajectories of the vortices (green) at around $\Delta r \approx 2.4$. As the vortices annihilate the description of vortices with negligible

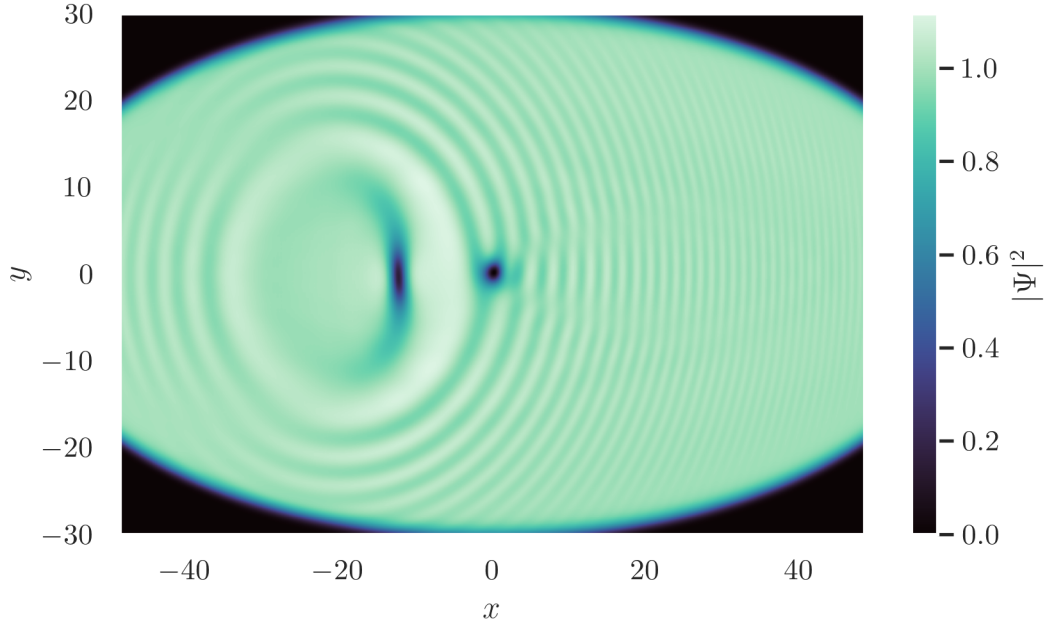


Figure 5.13: The condensate density showing a wave approaching the vortex at the centre of the trap from the left. The wave is formed by two annihilating vortices.

cores breaks down, as expected. The actual radiated energy ΔE_c plateaus somewhere between $\Delta r = 2\xi$ and $\Delta r = 3\xi$. In this range ΔE_c deviates significantly from the PVM prediction. The main effects in the radiation process for well separated vortices comes from the distortions in the density profiles around the vortices as they accelerate. Below Δr_c the cores of the vortices vanish, causing ripples and waves in the process. This is a fundamentally different process. In the plateau most of the incompressible kinetic energy, E_i bound to the vortices is transferred to compressible kinetic energy, E_c .

A head-on dipole-dipole collision is a highly idealised case of a four-vortex process. In chaotic flows with many vortices collisions might involve more vortices, and not all of them will collide simultaneously. From our simulations of 2D decaying quantum turbulence we observe that it is common for dipoles to go through a series of subsequent partner exchanges radiating energy until the dipole collapses into a solitary wave, which eventually disperses when it collides with a vortex. We will look at such a collision in the following section.

5.3 Wave-vortex collision

We want to look at the interaction between a wave created by an annihilation event and a vortex to see how the wave affects the vortex and give a detailed description of the collision event. This is the last stage of the type of annihilation process described in Groszek et al. [4]. The interaction between a solitary wave and a vortex also provides a direct way of studying the effect compressible modes might have on vortex dynamics.

The set-up is shown in Figure 5.13. We imprint a single, positive, vortex in the centre of the numerical domain using the vortex phase given by Equation 2.30. Since the domain has a net topological charge we do the simulations in a trapped condensate to prevent defects from forming at the boundaries. The numerical domain is 512×256 , with a system size of $128\xi \times 64\xi$. The trapping potential is set by a tanh-profile like in

Equation 4.3, but the radius R is modified to make the trap elliptical. After relaxing the system we imprint two vortices close together such that they immediately annihilate and form a wave propagating towards the vortex. In the first run of simulations we keep the dipole moment of the wave fixed and vary the initial distance between the wave and the vortex by shifting the wave gradually further from the centre of the domain. In the second run we keep the initial distance between the wave and the vortex fixed, and vary the dipole moment of the wave. Since the solitary waves are imprinted in an identical manner in each simulation, the dipole moment of the wave (Equation 3.3) relative to the centre of the wave is uniquely determined by the initial distance between the imprinted vortices, Δr . We store data from $t = t_0$ until $t = t_0 + 35\tau$. For the smallest r_0 we set $t_0 = 0$ and linearly increase the time from which we start storing data to $t_0 = 30\tau$ at $r_0 = 40\xi$. During this run the initial dipole moment in the wave is given by $\Delta r = 1.5\xi$. In the second run we vary the dipole moment by imprinting the vortices with separations in the range $\Delta r \in [0.8\xi, 1.8\xi]$, and imprint the wave at constant distance of $r_0 = 20\xi$.

5.3.1 Critical distance for vortex annihilation

To find the transition between a wave and a dipole we imprint two vortices with varying dipole moments in a periodic system and let them propagate until $t = 15\tau$. We then calculate the time averaged dipole moment of the wave or dipole from $t = 5\tau$ to 15τ using Equation 3.3, with the reference position at the centre of the wave or dipole. Through the dipole moment, $|\mathbf{d}(\Delta r)|$, we can find the critical distance Δr_c where the dipole collapses into a wave.

The dipole moment of the wave, or dipole, is plotted against the initial inter-vortex distance in Figure 5.14. There is a clear jump in $|\mathbf{d}(\Delta r)|$ for a critical initial Δr_c of about twice the coherence length $\Delta r_c \approx 2\xi$. If a vortex and an anti-vortex are imprinted closer together than Δr_c they annihilate and form a wave with significantly lower magnitude in the pseudo-vorticity field than the dipole. Double checking with the phase confirms that the critical distance for annihilation is $\Delta r_c \approx 2\xi$. There is also an equivalent drop in the total pseudo-entrophy of the wave, $\Omega = \int \mathbf{dr} |2D|^2$. Above Δr_c the vortices form a dipole, which left undisturbed will propagate the condensate indefinitely. As the wave annihilates the pseudo-vorticity does not completely vanish at once, but gradually diminishes over time as the wave disperses.

5.3.2 The collision process

During the collision between the wave and the vortex, a pair of phase slips form in the wave. The negative phase slip annihilates the vortex, leaving the positive phase slip behind to become a stable defect. The collision takes place as a partner exchange, and we can use the time at which the phase slips form and the time at which they annihilate to demarcate three distinct stages in the collision process. The first stage is the approach, before the new phase slips form. The second stage we call the exchange, and is when three phase slips are present in the collision region. The last stage is the relaxation, when there is again one positive phase slip present.

After the wave has formed there are still two regions where the pseudo-vorticity D is non-zero within the wave. In a uniform system, we found that the pseudo-vorticity diminishes over time as the wave disperses. In the presence of a vortex, however, we see that the pseudo-vorticity associated with the wave localises into two well-separated blobs as the wave approaches the vortex. This can be seen in Figure 5.15. At $t = 17\tau$, the magnitude of the pseudo-vorticity in the wave is very low and the contour lines

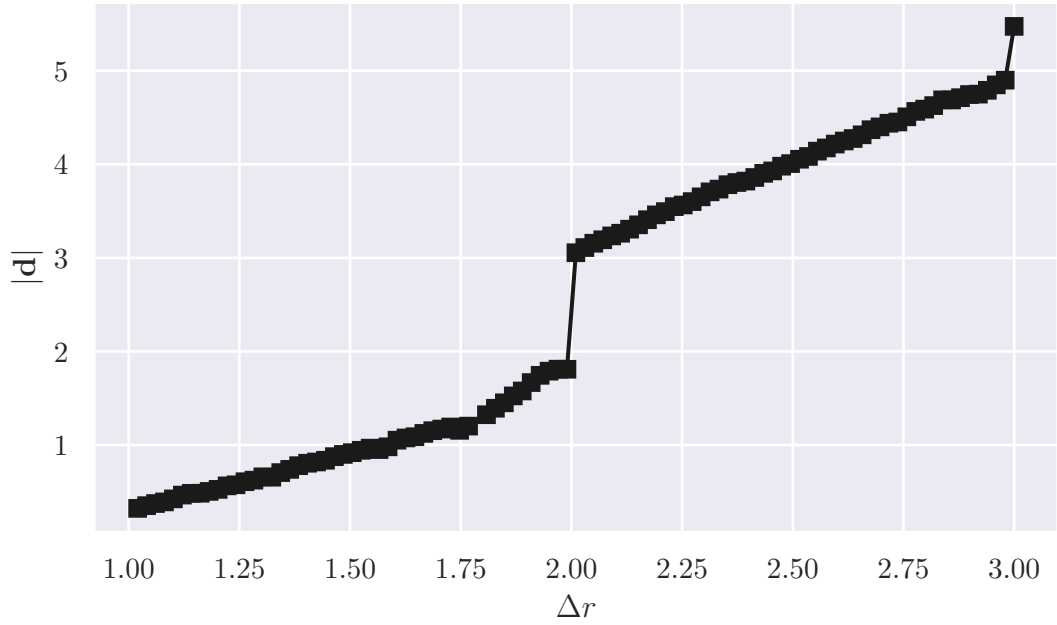


Figure 5.14: The dipole moment of the wave or dipole relative to its centre, averaged between 5τ and 15τ after the vortices were imprinted. The x -axis shows the distance at which they were imprinted.

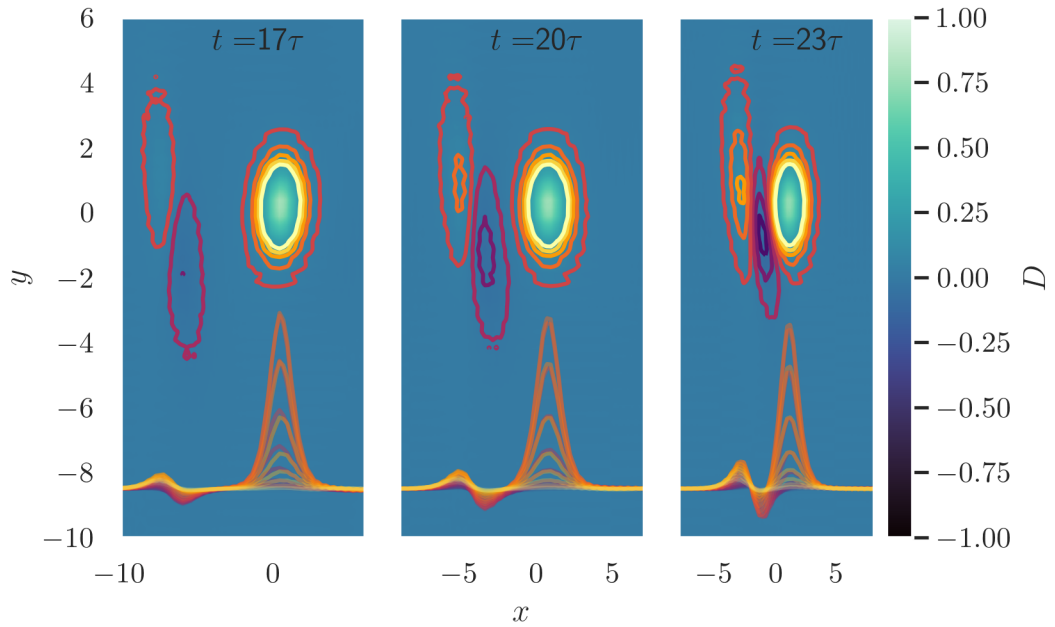


Figure 5.15: The pseudo-vorticity of the wave approaching the vortex. The contour lines have steps of approximately $0.15|D|$. The lines below the vortex are cross sections of D along x for y between $\pm 3.9\xi$, the dark lines are at negative values of y , while the yellow lines are at positive values of y . The peak magnitude of $|D|$ in the vortex is at about 0.8, while the positive region of the wave at $t = 23\tau$ peaks at roughly 0.1. The cross sections are scaled by a factor 7 for readability. The initial vortex separation is $\Delta r = 1.5\xi$.

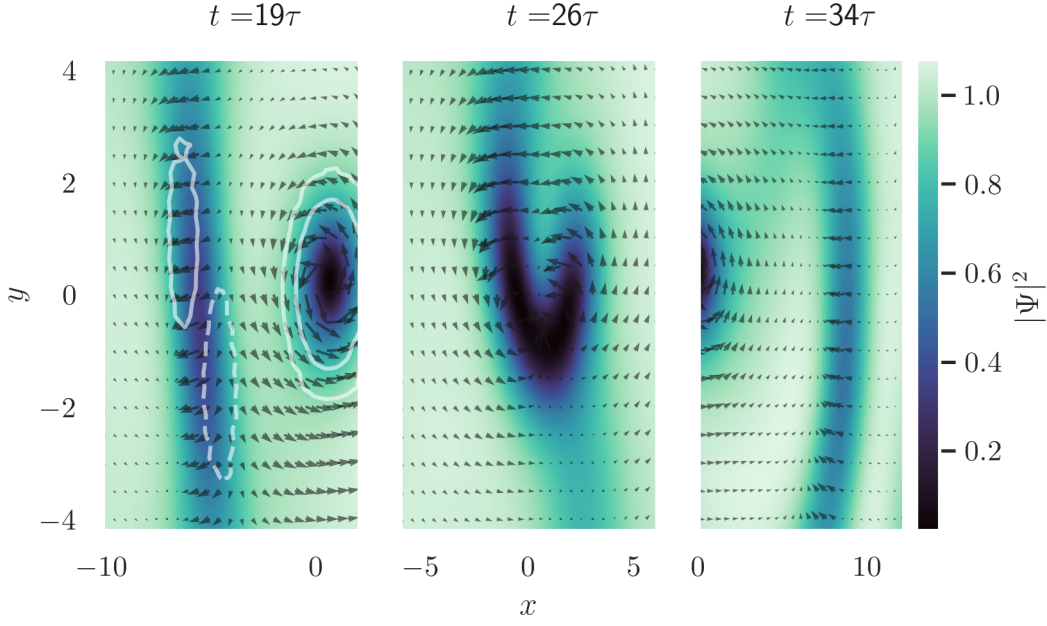


Figure 5.16: A time series of the density profile of a wave initialised by two vortices with imprinted initial vortex separation $\Delta r = 1.5\xi$. The panels show the approaching wave to the left, the merged density profiles of the wave and the vortex in the centre, and the dispersive wave after the collision to the right. In the left panel the circles indicate where D is non-zero. The solid lines show where the D -field is 0.04 and 0.12, while the dashed line shows where the D -field is -0.04 . The velocity field is $\mathbf{u} = \sqrt{\rho}\mathbf{v}$.

indicate that D is slowly-varying in space. At $t = 23$ the magnitude of D in both the positive and the negative region have increased, forming clear peaks. At the centre of the solitary wave the velocity of the superfluid is directed in the opposite direction of the wave propagation. At the ends of the wave, the condensate is flowing away from the centre of the wave in front of the wave, and in towards the centre behind the wave. This gives rise to pseudo-vorticity in the wave. We can see how the velocity field from the vortex influences the wave during the approach in the first panel of Figure 5.16. In front of the wave the velocity field of the vortex amplifies the region where D is negative. Behind the wave, the vortex amplifies the positive region of pseudo-vorticity. This drives the polarisation of the wave. It also contributes to rotating the wave such that the region where D is negative comes closer to the vortex than the positive region. We may intuitively understand the rotation of the wave as induced by the net torque when the negative and positive regions of pseudo-vorticity in the wave are slightly off-set from $y = 0$. From Figure 5.16 we see that the vortex field induces a tiny velocity in the negative x -direction for above $y = 0$, and vice versa below $y = 0$.

Towards the end of the approach a trench forms, connecting the vortex and the wave within one low density region. This is seen in the centre panel of Figure 5.16. In fact, the core of the vortex spreads out to encapsulate the pseudo-vorticity in the wave. The phase of the wave at this stage resembles a steep wall, as can be seen in the first frame of Figure 5.17. The wall steepens as it approaches the vortex, eventually forming a discontinuity and breaking apart into two new phase slips. The phase step from the turquoise, $\theta \approx 0$ to the red region, $\theta \approx \pi$, is continuous and connected by the blue line. This line is then "pinched off" when the gradient becomes too steep, forming two new phase slips. We can refer to the phase slips as quasi-defects. Persistent

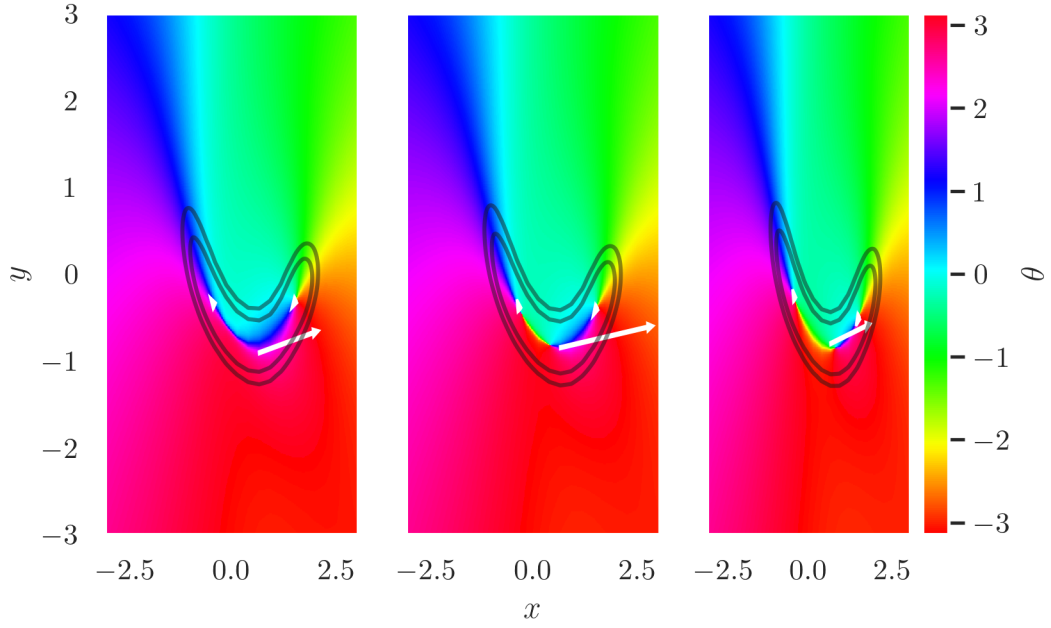


Figure 5.17: The formation of a phase slip during the wave-vortex collision event. The dark lines are iso-lines in the density at $\rho = 0.05$ and $\rho = 0.1$. The arrows indicate the velocity of the phase slips given by $\mathbf{J}^D/D$. The velocities are calculated as described in Appendix D. The images are taken within the interval $[26\tau, 27\tau]$. The initial vortex separation is $\Delta r = 1.5\xi$.

topological defects have a phase singularity as well as a stable core [32], the quasi-defects are constituted by the phase slip only.

During the exchange the positive quasi-defect and the negative quasi-defect split up and travel in opposite directions. The estimated velocity of the phase slips is indicated by the vortical current (Equation 3.4) in Figure 5.17. The negative quasi-defect follows the trench until it annihilates with the positive phase slip of the vortex. In the final stage the annihilation of the negative quasi-defect and the vortex form a new wave, seen in the last panel of Figure 5.16. For incoming solitary waves with relatively high dipole moments the wave formed after the exchange seems to preserve its shape. At lower dipole moments however, the outgoing wave quickly disperses. At the time of the annihilation event the quasi-defect that remains is still within the trench, and does not have a stable core yet. This forms during the last stage. The location of the new vortex is also shifted from the centre of the trap. As the new wave leaves, the remaining defect slowly back towards the position of the original defect, though it does not exactly return to this position. The trajectory of the vortex during the final stage is seen in Figure 5.18, in the blue segment right of the black square.

5.3.3 Trajectories of the phase slips during the collision

Tracking the trajectories of the phase slips throughout the collision enables us to give a more detailed picture of the collision process, and to find the time of formation and annihilation of the quasi-defects. The phase slips have been tracked using the ρ -tracker presented in Section 4.4.2. The ρ -tracker is constructed to find a set number of minima of the condensate density, regardless of whether or not the minima actually harbour phase slips. For that reason we also have data for the trajectory of the minima in the condensate density before the phase slips form, but they do not follow a continuous

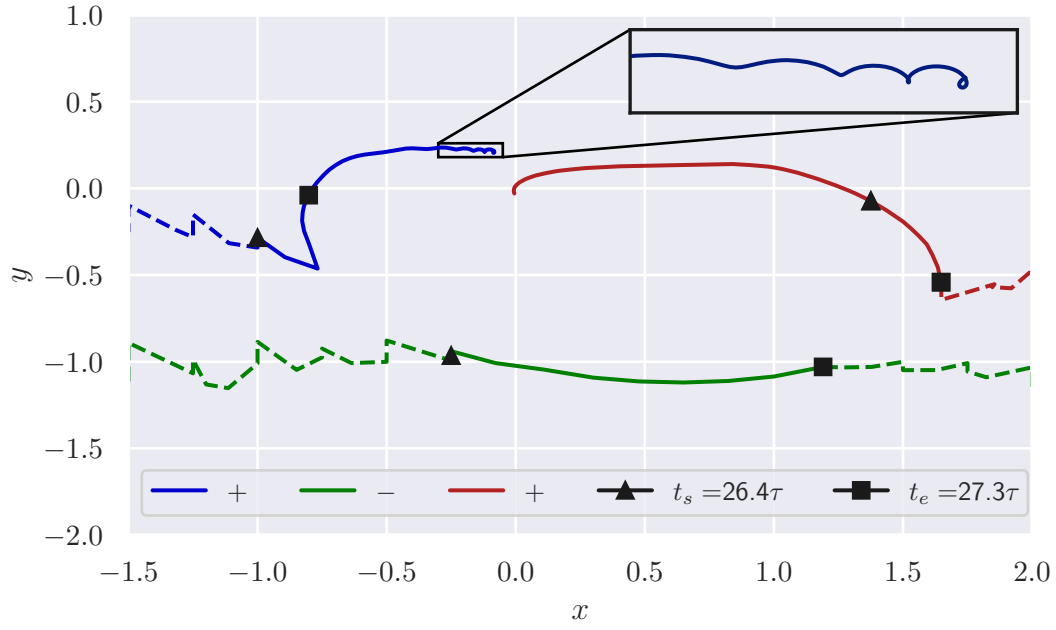


Figure 5.18: Trajectories of the three minima of the condensate density for a simulation where $\Delta r = 1.5\xi$ and the wave is imprinted $r_0 = 20\xi$ from the vortex. The squiggly part of the signal are minima in the waves before and after the exchange, while the smooth parts correspond to the position of the phase slips during the exchange. The red line is the position of the original vortex, while the green and blue lines show the negative and positive phase slips formed in the wave. The black triangle indicates a time estimate for the formation of the phase slips, while the black square marks an estimate for the time of annihilation after the exchange. The zoomed in rectangle shows the oscillations of the remaining vortex during the final stage of the event.

trajectory. Once the phase slips spawn however, their trajectories are smooth. This transition is clearly seen in Figure 5.18, the trajectories associated with the positive phase slips (blue and red) also have a sudden change of direction as the phase slips annihilate. We can use this to find the time of formation for the phase slips, t_s , as well as the time of annihilation, t_e , using change point detection (Appendix B).

The duration of the exchange depends both on the initial vortex separation Δr , and the initial distance between the wave and the vortex, r_0 . For vortices imprinted with $\Delta r = 1.5$ at a distance of $r_0 = 20\xi$ from the vortex, the duration of the exchange is roughly 1τ . The positions of the minima at the time of creation and annihilation of the phase slips are marked by black triangles (creation) and black squares (annihilation) in Figure 5.18. Judging from their locations the time of creation and annihilation of the negative phase slips is not fully in agreement with the time of creation and annihilation of the positive ones. This is not a physical feature, but a numerical artefact. The ρ -tracker masks out three areas, and locate the phase slips within these areas. When the positive and negative phase slips are very close together there are no pixels separating the positive from the negative region, so the algorithm can only find one of the phase slips.

When the wave is sufficiently far away from the vortex it causes a small perturbation in the phase and density at the vortex position, and we can describe the vortex motion through the PVM. In Figure 5.18 we see that from t_0 until the time of nucleation t_s , the vortex moves predominantly in the x -direction. This is a response to the phase of the wave. As the wave comes closer there is also a velocity in the y -direction, which comes from the density profile of the wave. During the second stage, between t_s and t_e , there are three phase slips present. The quasi-defects move in almost opposite directions after they are formed. If they were to follow a PVM-like dynamic they would have the same direction of travel, like a dipole. Granted, there would be some deviation due to the third vortex. We take this as an indication that the motion of the quasi-defects during the collision should be attributed to their density profile, more so than their phase. Lastly, during the third phase, the newly formed annihilation wave propagates away from the collision sight and rapidly disperses. The new vortex is left behind to drift in the positive x -direction towards $x = 0$. It does not return to the initial position of the original vortex. Towards the end of the trajectory of the new vortex we see oscillations in the path of the vortex, shown in the box in Figure 5.18. This oscillatory motion comes from background waves passing the vortex.

5.3.4 Comparison with the PVM

To compare the PVM to the full GPE solution we simulate a wave propagating through an empty system and calculate the velocity of a hypothetical vortex that starts out at the centre of the system from the only the phase term of the PVM, as well as the full PVM (Equation 3.5). This gives us the vortex trajectory due to interaction with the wave if the wave was unaffected by the vortex.

The results are shown in Figure 5.19. We see that the velocity calculated from the PVM is correct for the first part of the interaction. As the wave passes the vortex it changes direction and moves back towards its initial position. This is clearly seen in the left panel of Figure 5.19. The trajectory of the hypothetical vortex (green and blue) starts deviating from the actual vortex (black dashed) close to $t \approx 12\tau$ in the figure. This is shortly before the vortex changes direction indicating that the dynamics predicted by the PVM only breaks down for this set-up when the distance from the wave to the vortex is a few healing lengths. Tracking the position of the wave as well we can

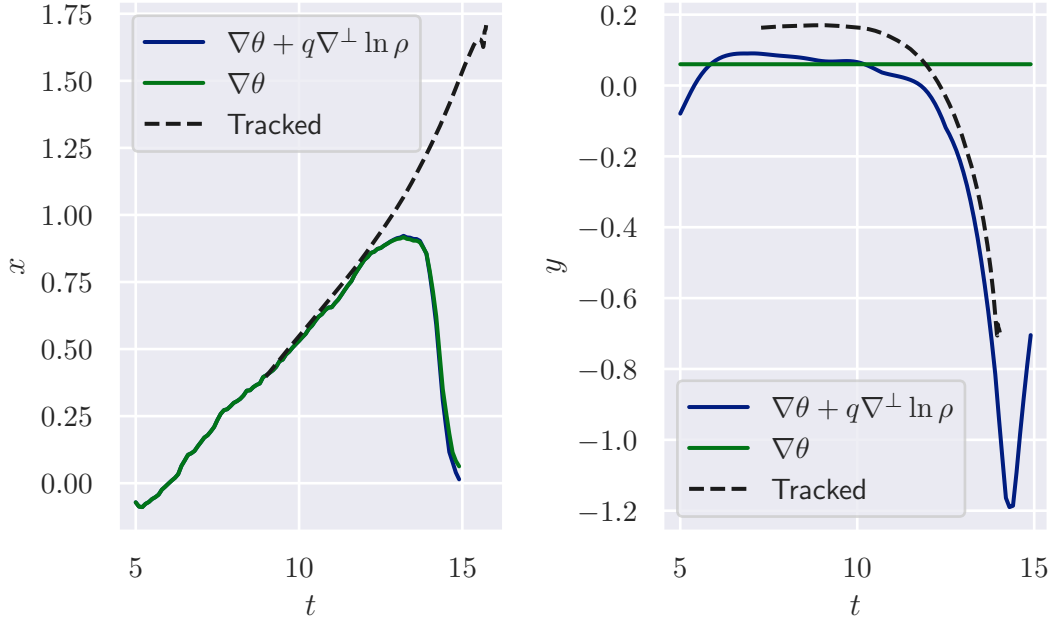


Figure 5.19: The position of the hypothetical vortex along the x -axis (left) and y -axis (right) over time. The green line is calculated using just the phase of the condensate near the hypothetical vortex position, while the blue line is calculated using both the phase and the density perturbations. The dashed black line is a tracking signal from a simulation of a wave interacting with a vortex. The dashed lines have been shifted for sake of comparison since the time of collision for the tracked vortex and the hypothetical vortex are not the same.

find the distance between the wave and vortex at the time when the PVM and tracking starts to deviate in the x -coordinate. This distance is approximately 3.2ξ measured from the centre of the wave. In the right panel of Figure 5.19 we see how the phase alone predicts that the vortex does not have any motion along the y -axis, perpendicular to the direction of travel of the wave. Including the density perturbation however gives rise to motion along the y -axis.

5.3.5 Isolating the energy of the wave

We wish to understand the interaction between the wave and the vortex in terms of transfer of kinetic energy. To study the effect of interaction on the energy of the wave and the vortex we perform three different simulations. The first one is of a wave in isolation, the second is of a wave and a vortex, and the last one is just a stationary vortex, where we pretend that a wave is present by defining an empty region where the wave would be if it were present. We use a numerical domain of size 150×150 with a system size of $75\xi \times 75\xi$, and set-up the condensate in a circular trap with radius is $R = 22.4\xi$. For the first simulation we imprint a wave with $\Delta r = 1.5\xi$ at $x = -20\xi$ and let it propagate towards the centre of the trap until $t = 25\tau$. We locate the wave masking out where $\rho < 0.9\rho_0$ and calculate the incompressible kinetic energy E_i , the compressible kinetic energy E_c , and the quantum energy E_q , within this region. In the next simulation with both the wave and vortex present we use the same approach for the vortex and wave separately. In the last simulation we look at a stationary velocity field with a vortex at the centre and calculate the energy within a region of the same area as the isolated wave at each point in time. This way we can compare the energy

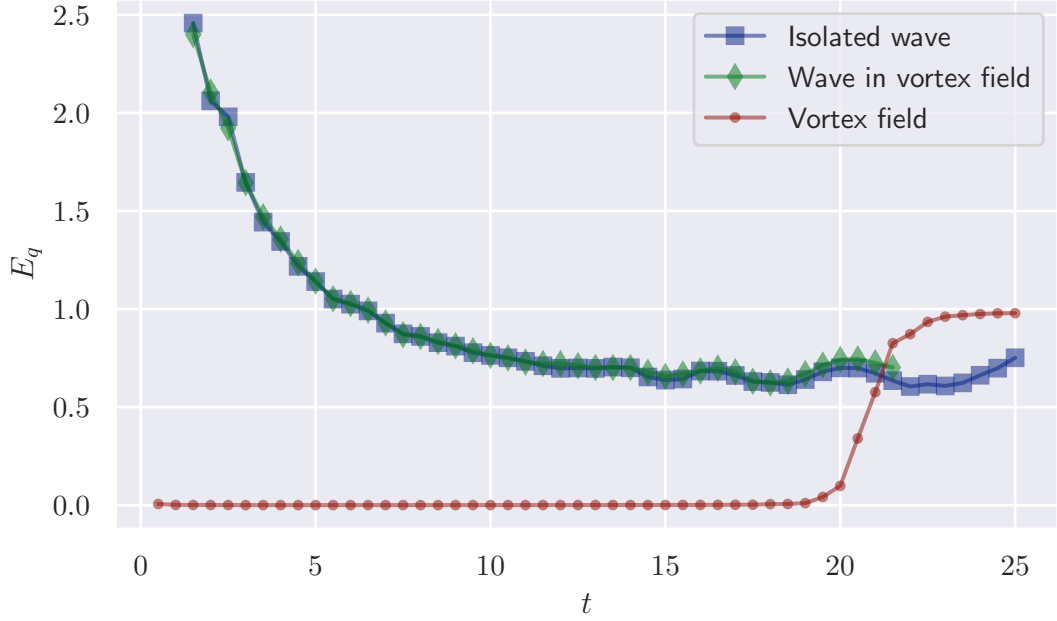


Figure 5.20: The quantum energy within a wave. The blue squares show the energy in a wave propagating freely in a disk trap. The green rhombuses show the energy for a wave propagating in a system with a vortex near the centre. The red circles show the energy within a region with the same area as the wave, moving at the same velocity as the wave, in a vortex field.

of the wave in the absence and presence of a vortex also accounting for possible effects from interaction between the two.

Comparing the quantum energy from the three simulations in Figure 5.20 we see that the energy associated with the quantum pressure is the same for an isolated wave as it is for a wave in the vortex field. Meanwhile E_q for the vortex field is close to zero. We conclude that the quantum energy in the wave is independent of the vortex. Similarly in Figure 5.21 we see that the incompressible kinetic energy in the wave in the vortex field is similar to E_i in the vortex field with the wave absent. We take this to indicate that the incompressible kinetic energy in the wave is not inherent to the wave but comes from the vortex field. In Figure 5.21 the energy from the wave in the vortex field starts deviating from the energy in the vortex field at around $t = 15\tau$, this is an effect of the interaction between the wave and the vortex. From Figure E.2 we see that this the same time as the wave starts splitting up. At around $t = 15\tau$ the dipole moment of the wave interacting with the vortex starts increasing, while the dipole moment of the isolated wave keeps decreasing. Lastly, looking at the compressible kinetic energy in Figure 5.22, we see that E_c in the wave is the same independently of the presence of the vortex while E_c in the vortex field alone is zero when the wave is far from the vortex. The compressible kinetic energy in the wave also does not change much as it approaches the vortex. The compressible kinetic energy of the wave is primarily inherent to the wave and is not influenced by the presence of the vortex.

The energy inside the vortex as the wave approaches is shown in Figure 5.23. There are some fluctuations in E_i . As the wave starts splitting up, at $t = 15\tau$, we see a decrease in E_i within the vortex. The quantum energy of the vortex by itself is $E_q = 1\mu_0$. As the wave merges with the vortex we see an increase in the quantum energy, as well as the compressible kinetic energy.

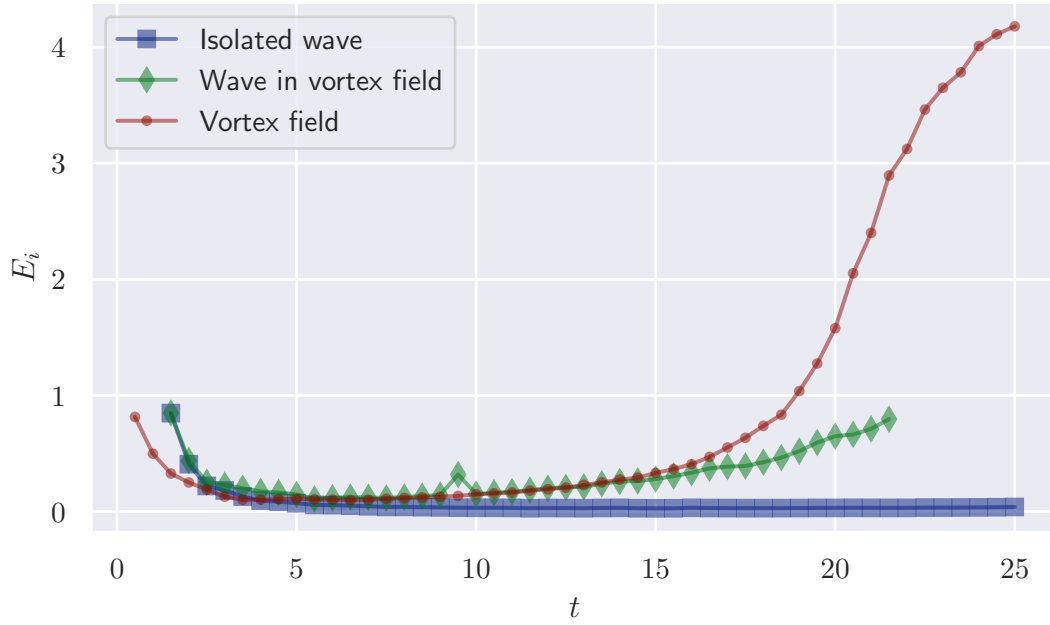


Figure 5.21: The incompressible kinetic energy within a wave, defined as a region of low density where $\rho < 0.9\rho_0$. The blue squares show the energy in a wave propagating freely in a disk trap. The green rhombuses show the energy for a wave propagating in a system with a vortex near the centre. The red circles show the energy within a region with the same area as the wave, moving at the same velocity as the wave, in a vortex field.

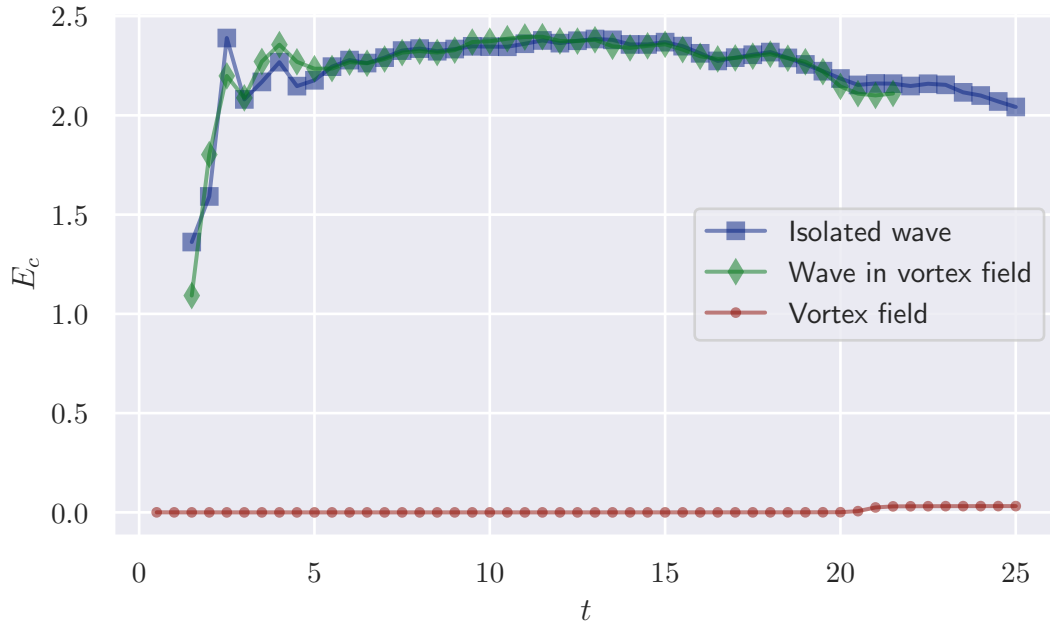


Figure 5.22: The compressible kinetic energy within a wave. The blue squares show the energy in a wave propagating freely in a disk trap. The green rhombuses show the energy for a wave propagating in a system with a vortex near the centre. The red circles show the energy within a region with the same area as the wave, moving at the same velocity as the wave, in a vortex field.

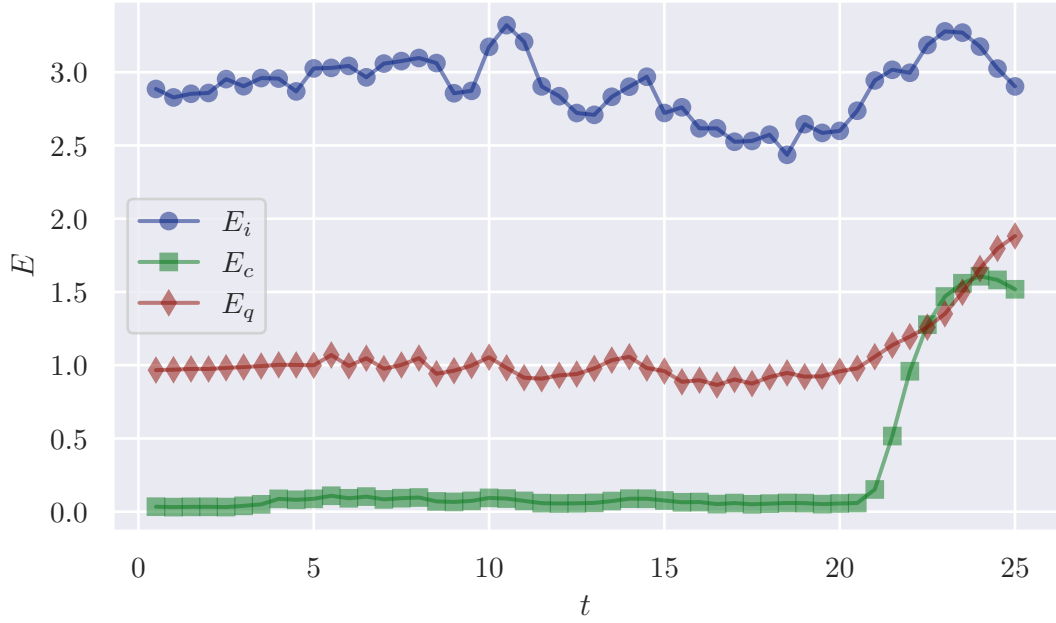


Figure 5.23: The incompressible, compressible and quantum energy within the low density region around the vortex as the wave approaches. We see that the quantum energy and the compressible kinetic energy are constant during the approach of the wave, and increase as the wave comes close to the vortex. The incompressible energy has some variation during the approach, and also increases as the wave and the vortex merge.

5.3.6 Collision process at different initial dipole moments

The collision process between the wave and vortex depends on the initial vortex separation Δr , as well as the distance from the vortex at which the wave is imprinted, r_0 . Until now we have presented the collision process looking at waves with an initial vortex separation of $\Delta r = 1.5\xi$. At higher Δr the phase slips form further from the vortex compared to waves where Δr is smaller. If we reduce the initial vortex separation below $\Delta r \approx 1.4\xi$ the quasi-defects do not form at all. The collision process for $\Delta r = 1.8\xi$ is shown in Figure 5.24. Since the phase slips are further separated the quasi-defects resemble independent vortices more closely. The peaks of the pseudo-vorticity reach higher values, closer to the magnitude of the vortex $D = 0.7$, during the process than for $\Delta r = 1.5$ (Figure 5.15). As a result of this we can discern three separate density profiles from the quasi-defects during the exchange, as can be seen in Figure 5.25. While the process at $\Delta r = 1.5\xi$ has one extended core region, there are three separate cores present during the exchange process at 1.8ξ . The outgoing wave at $\Delta r = 1.8\xi$ seems to preserve its shape, whereas the outgoing wave at $\Delta r = 1.5\xi$ disperses shortly after the exchange. The wave with higher dipole moment splits up more easily than a wave with low dipole moment. The outgoing wave at $\Delta r = 1.8\xi$ therefore resembles a dipole in its interaction with other vortices and the trap wall. In our turbulence simulations (Section 5.4) we frequently see partner exchanges between solitary waves and dipoles. When approaching the wall the wave tends to split up into two vortices rather than reflect.

The trajectories for simulations between $\Delta r = 1.24\xi$ and $\Delta r = 1.8\xi$ are shown in Figure 5.26. As mentioned, for larger Δr the phase slips form further from the vortex and travels a longer distance during the exchange than for smaller Δr . Somewhere close

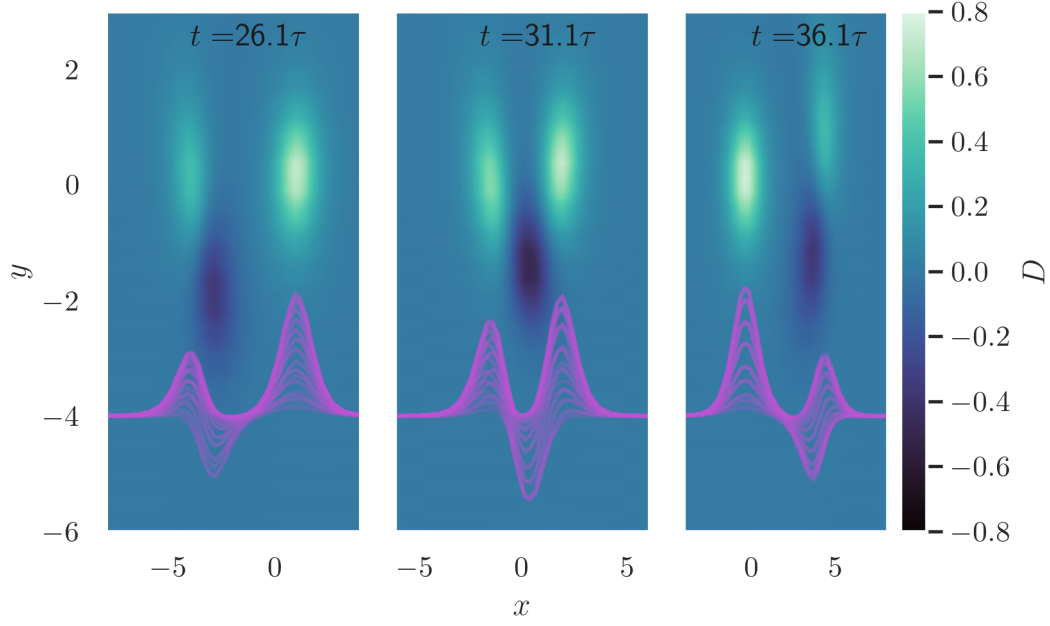


Figure 5.24: Snapshots from the collision process between a wave imprinted with a vortex separation of $\Delta r = 1.8\xi$. Apart from the initial vortex separation the process is the same as in Figure 5.15. The pink lines show slices in the x -direction at y -values between $\pm 3.8\xi$, the brightest lines are located at $y = 0$. The peaks of the vortex is around $D = 0.7$ in all images, while the negative trough is at its lowest at around $D = -0.35$ at $t = 26\tau$ and at $t = 36\tau$. At $t = 31.1\tau$ it is around $D = -0.5$. The lines are scaled for easier visibility.

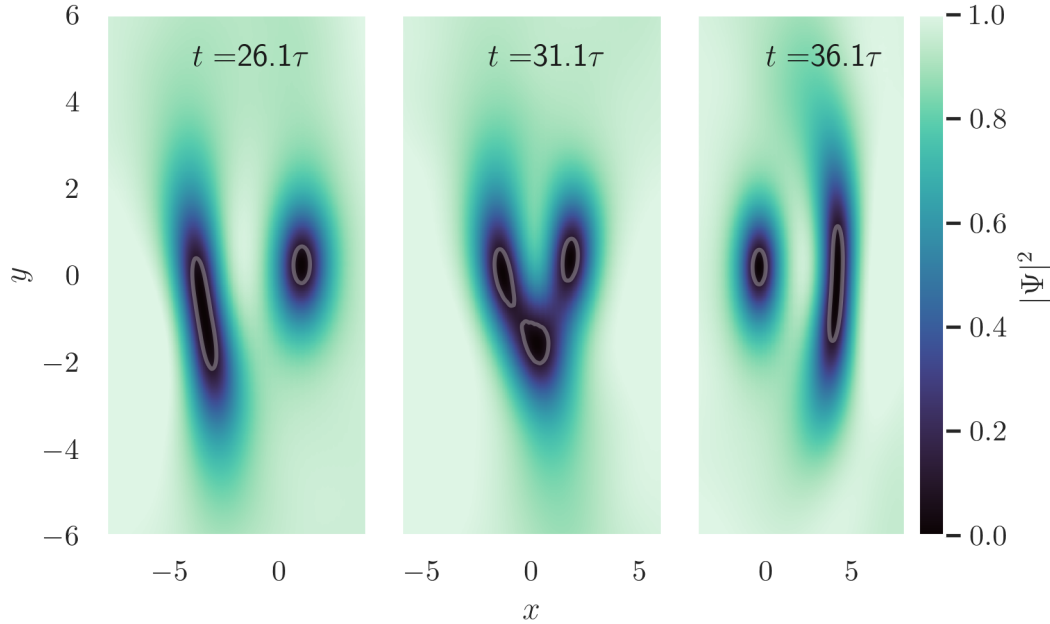


Figure 5.25: The collision process for $\Delta r = 1.8\xi$. Re see the wave approaching the vortex in the first panel. The second panel shows the density of the vortices during the exchange. In this image we can see three separate wells around each quasi-defect. In the last image we see the wave after the collision. The grey lines indicate where $\rho = 0.1\rho_0$.

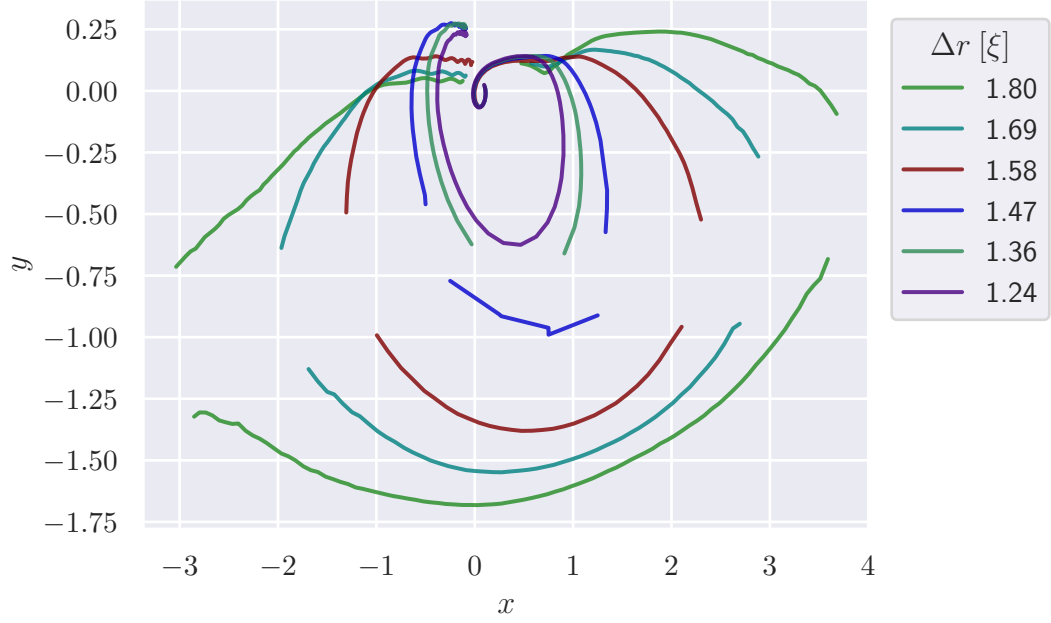


Figure 5.26: The trajectories of the phase slips for simulations with varying initial vortex separation used for imprinting the wave, between $\Delta r = 1.8\xi$, and $\Delta r = 1.24\xi$. The distance from the vortex to the phase slips as they form decrease with decreasing Δr . At $\Delta r = 1.36\xi$ (green) we only pick up two separate signals, from the vortex. In this case the quasi-defects do not form, and the vortex is shifted by the wave.

to $\Delta r = 1.4\xi$ there is a critical distance below which the phase slips do not form. We see this in Figure 5.26 at $\Delta r = 1.24\xi$, here the wave does not form quasi-defects and the vortex is simply shifted by the wave as it passes. The final position of the vortex in this case is similar to the final position of the new vortex formed during the exchange event for slightly larger Δr . As Δr decreases the dynamic of the solitary wave transitions from being dipole-like to resemble what we expect from a pure density perturbation.

The final stage of the collision is shown in Figure 5.27. Snapshots of the largest coherent area where $\rho < 0.9\rho_0$ is plotted every 2τ . The result is a time series of the wave leaving the collision. To the left in the figure, for $\Delta r = 1.8\xi$ (purple), the wave retains its shape and propagates in almost a straight line after the collision. In the next two collision events (green and blue), we see how the wave initially heads in the negative y -direction straight after annihilation, but is rotated to get a positive velocity component in the y -direction. This effect is similar to the rotation of the incoming wave.

The pseudo-vorticity within the wave is advected by the flow induced by the vortex itself. Thus, as the peaks in pseudo-vorticity are off-set from the y -axis, they will experience different advective velocity generating a net torque that rotates the wave. The magnitude of D decrease from left to right in the figure, where the pseudo-vorticity is higher this effect happens over a shorter length scale. When $\Delta r = 1.8\xi$ (purple) it is almost immediate. For $\Delta r = 1.69\xi$ (green) and $\Delta r = 1.58\xi$ it is more gradual. Further right in the figure the wave leaving the collision is more spread out. The outgoing wave is more closely resembles a pure density perturbation than a dipole-like solitary wave. Interestingly, the waves seem to have disclinations similar to the ripples in Figure 5.13.

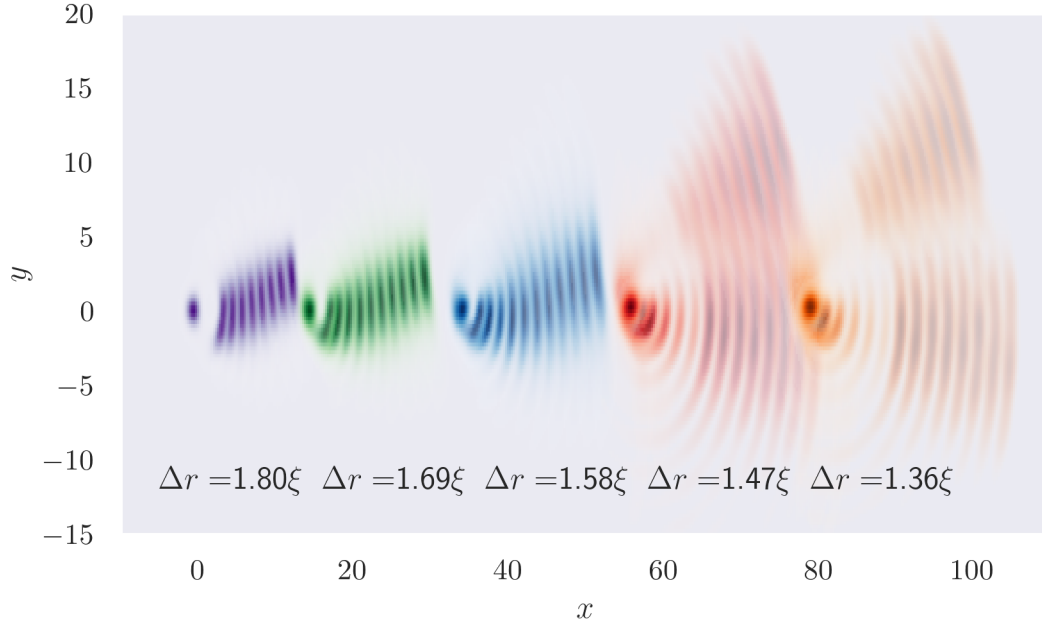


Figure 5.27: A time series of the outgoing wave after the exchange event. Every 2τ we plot the region with the largest area where $\rho < 0.9\rho_0$. The figure thus shows the motion of a single wave over time. The colours separate the collisions with different initial dipole moments, which is indicated below each of the collisions. The vividness of colour indicates the depth of the wave. From left to right we see how the outgoing wave becomes more spread out as its initial dipole moment decreases. In the purple figure the wave stays quite compact and retains its depth. On the other end of the scale in the red and orange figure we see a localised wave in the first snapshot only, close to the vortex. These waves quickly spread out into low amplitude ripples.

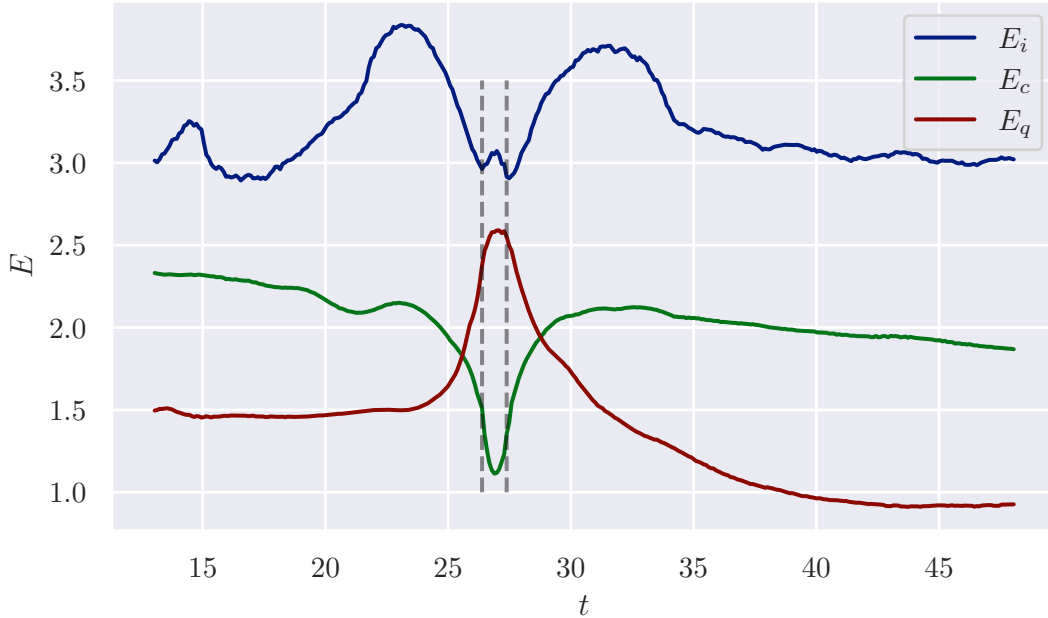


Figure 5.28: The incompressible kinetic energy (blue), compressible kinetic energy (green) and quantum energy (red) during the wave-vortex collision. The energies are calculated within a low density region around the wave and vortex, defined by $\rho < 0.9\rho_0$. The dotted vertical lines indicate the time of nucleation and annihilation of the phase slips.

5.3.7 Evolution of energy during the collision

The energy within the wave and vortex during the approach, exchange, and relaxation is shown in Figure 5.28. The quantum energy of the wave and vortex is approximately $E_q = 1.5\mu_0$, higher than it is in the vortex alone. As the wave and the vortex merge E_q increases, and after the wave disperses we are left with only the quantum energy of the vortex of $E_q = 1\mu_0$. The incompressible kinetic energy increases during the approach and drops as the density profile of the wave merges with the vortex. This effect is amplified by only including the energy within the wave and vortex, but is also seen in the global quantities. The drop in E_i happens on the same time scale as the drop in D within the vortex, as we will see in Section 5.3.11. So it is possible that the decrease in E_i is due to transfer of pseudo-vorticity between the vortex and the wave. As the phase slips form we see an increase in E_i . We can assume that the tiny peak in E_i occurs when the negative quasi-defect exerts the minimal combined screening on the positive phase slips, that is, when it is in the middle of its path between them. During the collision event there is also a drop in E_c corresponding to the peak in the quantum energy. This indicates a transfer of compressible kinetic energy associated with the motion of the wave to the quantum energy associated with the trench.

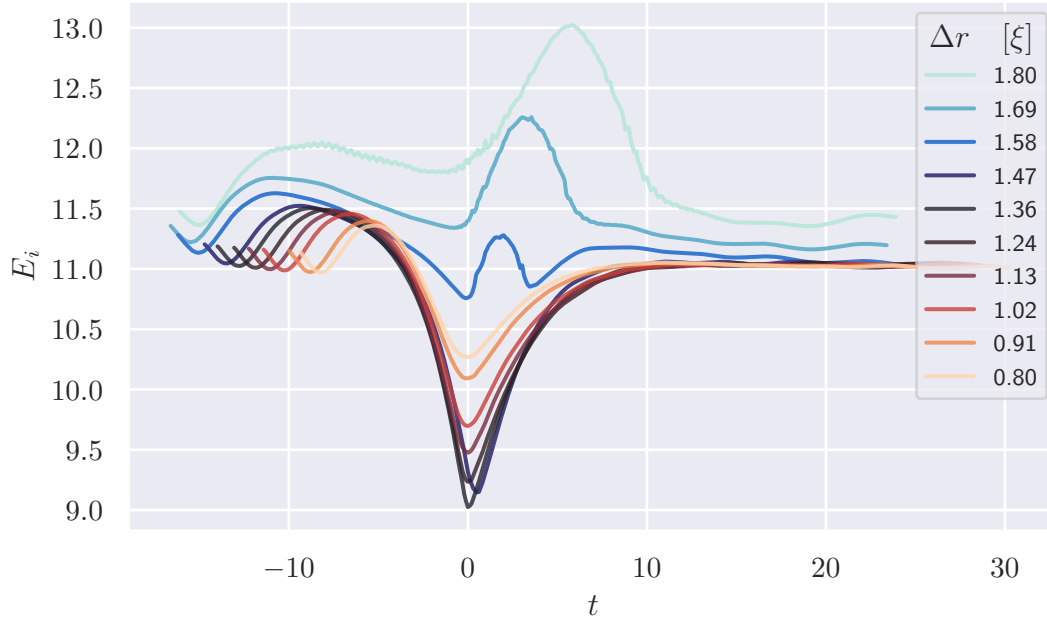


Figure 5.29: Incompressible kinetic energy over time for different initial dipole moments, scaled with its initial value. The time is shifted to set the start of the exchange at $t = 0$. From $\Delta r = 1.24\xi$ and below we do not observe any phase slips forming. For large Δr there is clearly a peak in E_i associated to the formation of the quasi-defects. As Δr decreases the peak associated with the phase slips decrease, and the overall incompressible energy dips more as the wave passes the vortex. There is a sudden transition in the evolution of the energy between $\Delta r = 1.58\xi$ and $\Delta r = 1.47\xi$. At $\Delta r = 1.47\xi$ there is no peak associated with the quasi-defects, but they are still clearly discernible in the phase.

5.3.8 Evolution of energy varying Δr

We look at the evolution of incompressible kinetic energy scaled with its initial value for waves with different dipole moments in Figure 5.29. From $\Delta r = 1.58\xi$ to $\Delta r = 1.47\xi$ there is an abrupt change in how E_i evolves. The peak associated with the quasi-defects disappear and instead E_i dips as the wave passes the vortex. For $\Delta r < 1.4\xi$, the wave passes the vortex without forming new phase slips, whereas for $\Delta r = 1.8\xi$ the quasi-defects are quite similar to vortices. From looking at the phase there is still a clearly discernible exchange event at $\Delta r = 1.47\xi$. Despite this, the evolution of energy in this case more closely resembles that of the wave like dynamic ($\Delta r < 1.4\xi$) than the vortex-like dynamic ($\Delta r = 1.8\xi$). The incompressible kinetic energy returns to approximately the same energy as the initial state of the system, while the compressible energy, seen in Figure 5.30, decreases. The decrease of kinetic energy in the system occurs primarily through the damping of the compressible modes. The amount of compressible kinetic energy that is radiated to the boundaries, where $\gamma \neq 0$, depend on the details of the collision. For different Δr the rate of decrease of E_c varies. The smallest perturbation, $\Delta r = 0.8\xi$, has the highest rate of decrease, following a power law $E_c/E_c(t_0) \propto t^{-0.4}$. For larger Δr the compressible kinetic energy decreases at a lower rate.

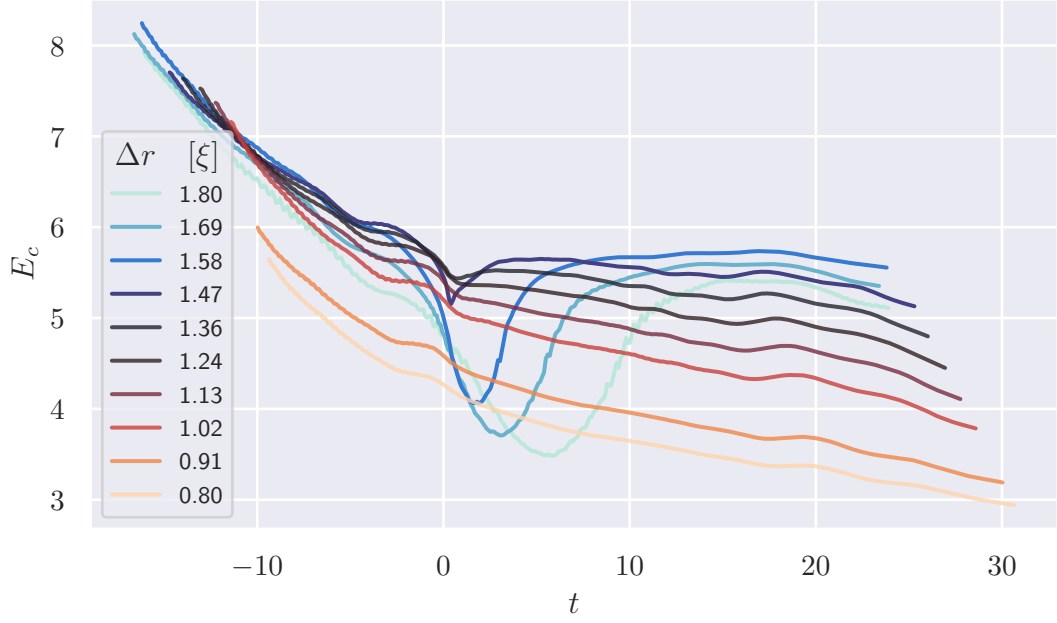


Figure 5.30: The compressible kinetic energy as it changes over time, scaled with its initial value. The curves are shifted to that the exchange starts at $t = 0$.

5.3.9 Evolution of energy varying r_0

The evolution of the incompressible kinetic energy, E_i , for different initial wave vortex distances, r_0 , is shown in Figure 5.31. Overall, E_i decreases as the wave approaches the vortex, and increases again after the collision event. Except for when the wave is imprinted closer than 10ξ from the vortex, E_i returns to approximately its initial value. It is interesting to notice that for the different collision events with $r_0 > 10\xi$, E_i increases during the formation of the two quasi-defects within the waves and its peak value seems to depend on r_0 . The fact that the quasi-defects cause a more prevalent increase in E_i at larger r_0 is probably due to the waves becoming more polarised when they travel further through the vortex field. The value of E_i drops again during the annihilation of the vortex with the negative quasi-defect, and then increases monotonically toward its initial value as the solitary wave reforms and disperses away.

While the incompressible kinetic energy, E_i , returns more or less to its initial value after the collision the compressible kinetic energy, E_c , diminishes during the collision. In Figure 5.32 we see that during the approach there is a drop in E_c with a subsequent increase after the collision. The closer the wave is imprinted to the vortex, the smaller the increase after the collision.

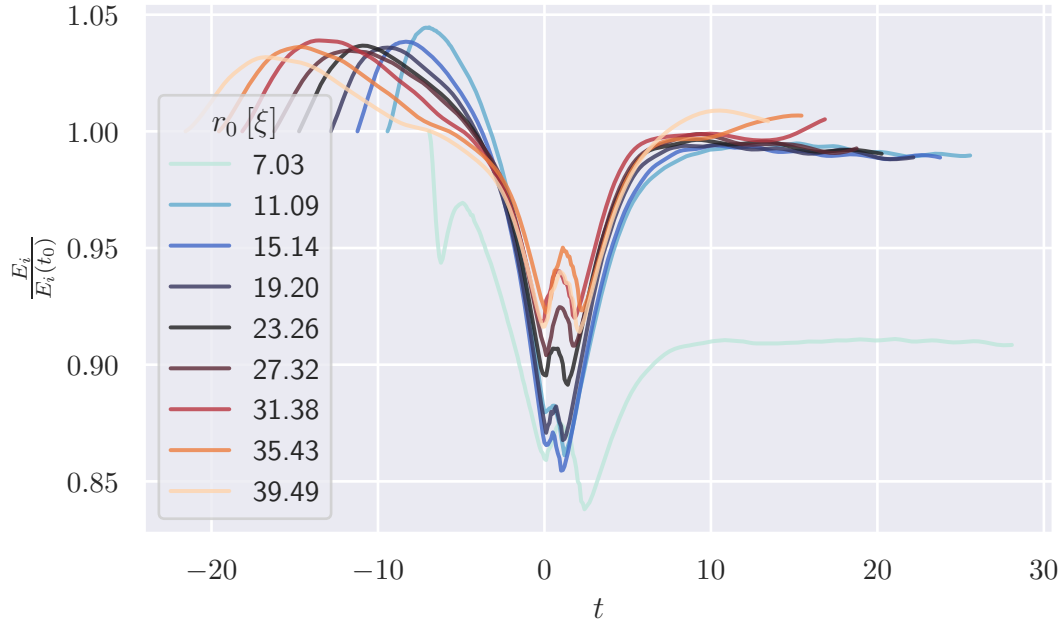


Figure 5.31: Incompressible kinetic energy in the system for different initial r_0 scaled with its initial value. Apart from when the wave is imprinted close to the vortex, $E_i/E_i(t_0)$ returns to its initial value after the collision. The curves are shifted so that the start of the exchange happens at $t = 0$.

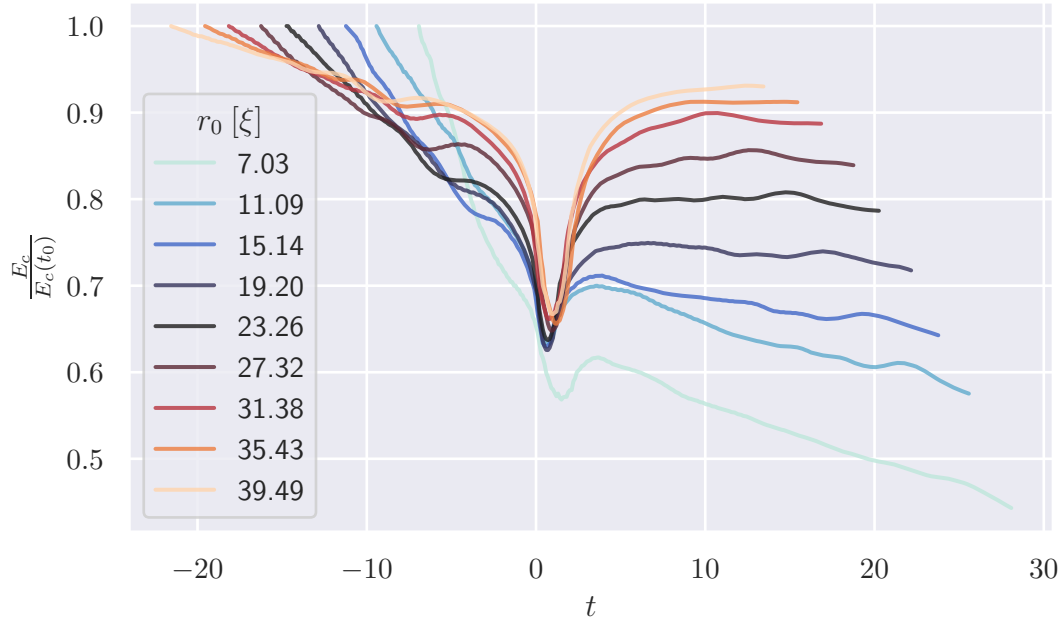


Figure 5.32: The compressible kinetic energy for different initial r_0 scaled with its initial value, $E_c(t_0)$. Overall there is a drop in E_c . The curves have been shifted for the exchange to start at $t = 0$.

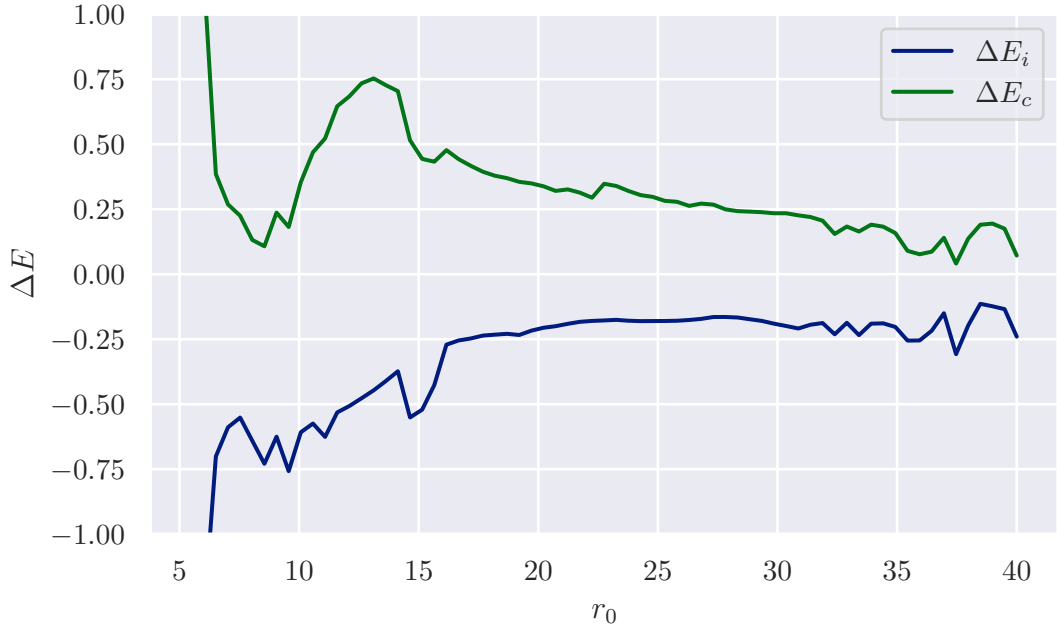


Figure 5.33: The difference in incompressible kinetic energy and compressible kinetic energy in the system before and after the wave-vortex interaction. As the wave propagates further to reach the vortex the difference in incompressible energy stabilises somewhere below $|\Delta E_i| \approx 0.25\mu$.

5.3.10 Change in the incompressible energy at different r_0

To find an estimate for the change in incompressible kinetic energy during the collision event we have run a set of simulations with $\Delta r = 1.5\xi$ and different r_0 where there is no damping present in the system. At the start of the simulation there is a burst of waves coming from the formation of the wave profile and the sudden acceleration of the solitary wave. Without dissipation these waves reflect back and forth in the trap causing oscillations in the trajectories of the phase slips during the exchange. This makes it challenging to find good estimates for t_s and t_e from the trajectories. However, the propagation of the solitary wave is similar in both cases. The estimates for t_s and t_e found in the simulation with damping also applies to the non-dissipative simulations. This can be seen from the trajectories of the phase slips (Figure E.3). Since E_i varies continuously during the approach, it is not entirely straightforward to identify a clear beginning and end of the interaction between the wave and the vortex in terms of energy exchange. However, we can still locate the trough in $E_i(t)$, as seen in Figure 5.31, and compare the time averaged incompressible energy before and after. We find the point in time before and after the trough where E_i is sufficiently flat, and take these to be the start and end of the energy exchange. The results plotted in Figure 5.33 are found with the threshold on the slope $|\Delta \hat{E}_i / \Delta t|$ of $0.025\mu_0/\tau$, where $\hat{E}_i$ is the filtered incompressible kinetic energy. It should be noted that the exact values of ΔE_i are sensitive to the choice of threshold, however thresholds up to $0.1\mu_0/\tau$ do give values of ΔE on the same scale as shown in Figure 5.33.

The change in incompressible kinetic energy due to the collision event is less than $|\Delta E_i| = 0.25\mu_0$ when the solitary wave has propagated more than 20ξ . This is true for waves with larger Δr as well. Compared to the change in incompressible kinetic energy per vortex found in Table 5.1 however, this is negligible. This discrepancy indicates that most of the change in incompressible kinetic energy during the annihilation process does

not happen in the wave-vortex collision, but takes place when the dipole collapses. The decrease in E_i leading to dipole collapse does not need to occur as a singular event, but might happen gradually through a series of exchange events. This is something we observe in our simulations of decaying quantum turbulence, presented in Section 5.4. For a chain of collision events small changes in E_i can accumulate. It is possible that the change in incompressible energy per vortex is accounted for by a series of collisions between dipoles and vortices, or solitary waves and vortices.

If we look at the change in incompressible kinetic energy during the topological exchange itself we get the result shown in Figure E.4. The effect from the formation of the defects is irrelevant compared to the overall trend for the change in E_i when the wave is imprinted more than 20ξ from the vortex so it fluctuates around $\Delta E_i = 0$.

5.3.11 Magnitude of the pseudo-vorticity

We see how the peak magnitude of D in the wave and vortex change as the wave approaches the vortex in Figure 5.34. In the cases where the phase slips are clearly discernible (above $\Delta r = 1.4\xi$) the magnitude of D in the negative phase slip increase until the exchange takes place. In the figure, the time of exchange is set to $t = 0$. For $\Delta r = 1.69\xi$ the negative phase slips (green) crosses $t = 0$ at approximately $|D| = 0.45$, while for $\Delta r = 1.36$, when the phase slips are no longer discernible, the negative vorticity region in the wave vanishes during the time of exchange.

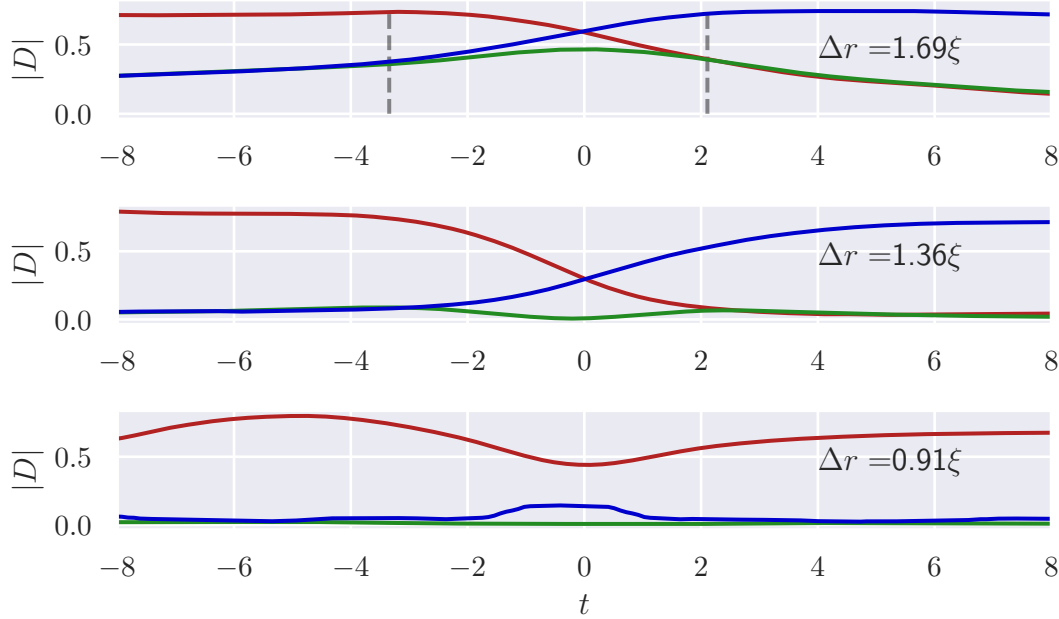


Figure 5.34: The peak magnitude of D in the wave and vortex over time. The time scale has been shifted to centre the exchange event at $t = 0$. The red lines represent the initial vortex. The blue lines represent the positive peak of D in the wave, while the green line is magnitude of the minimum of D in the negative region. In the top panel the grey lines indicate the time of formation and annihilation of the phase slips. For high Δr the pseudo-vorticity in the negative region of the wave increases during the approach, $t < 0$, and is maximal during the exchange, $t \approx 0$. Below $\Delta r = 1.4\xi$ however the region of negative vorticity vanishes at $t \approx 0$. For even lower Δr the magnitude of D in the vortex is reduced as the wave passes, but there is no exchange.

At $\Delta r = 1.36\xi$ there is no exchange of phase slips, but we see from Figure 5.34 how $|D|$ in the vortex (red) decreases as the wave approaches. The pseudo-vorticity in the wave (blue) increases as it approaches the vortex and takes over as the stable vortex after the collision event. During the collision itself the negative pseudo-vorticity vanishes. D is a conserved quantity [32], and without a region where D is negative, the positive pseudo-vorticity is simply distributed between the wave and vortex during the collision. This is why the magnitude of D at $t = 0$ is lower for $\Delta r = 1.36\xi$ than for $\Delta r = 1.69\xi$.

At $\Delta r = 1.36\xi$, the D field has two separate positive regions during the collision, the regions are not connected and the magnitude of D between them is very close to zero. As the phase slip moves from one region to the other it necessarily passes through the area where $D \approx 0$. This demonstrates that the phase slip of the condensate must not coincide with the peaks in the D -field, nor the centre of regions where $D \neq 0$. This is something we have observed in dipoles as well. The phase slips are generally significantly closer together than the peak of the D -field, or the centre of the regions of pseudo-vorticity. The D -field functions as a defect density field and tracks the position of the defects when they have a stable core and are well separated, but the connection between the location of phase singularities and the magnitude of the D -field does not hold for the collisions between solitary waves and vortices.

For even lower Δr , in the bottom panel in Figure 5.34, the process bears no characteristics of an exchange, the magnitude of D in the vortex is perturbed by the passing wave, but the pseudo-vorticity in the wave does not increase to the magnitude comparable to the defect.

The formation of phase slips is related to an increase in the D -field. We might expect there to be a critical value of the pseudo-vorticity at which the phase slips appear. The peak magnitude of the pseudo-vorticity around each phase slip at the moment of nucleation and annihilation, for various Δr , is shown in Figure 5.35. At $\Delta r = 1.24\xi$ no quasi-defects form, and we only pick up a signal from the vortex. Below $\Delta r = 1.36\xi$ we have used the time where E_i is minimal (Figure 5.29) as the moment of collision. The time of nucleation and annihilation both approach this value as Δr decreases. For the highest values of Δr the negative quasi-defect (green) has the same magnitude in the D -field as the positive quasi-defect (blue) when they form. As they annihilate the negative quasi-defect and the vortex (red) also have the same magnitude. Before the collision the peak magnitude of $|D|$ in the vortex is stable as long as the quasi-defects have the same magnitude in $|D|$. When the magnitude of positive pseudo-vorticity is higher than the negative one the pseudo-vorticity of the vortex decreases, since the overall value of D is conserved. As the initial dipole moment of the wave decreases the value of $|D|$ near the phase slip also decreases. The wave still has a non-zero value of $|D|$ even below $\Delta r = 1.2\xi$, but this is not shown here as there are no phase slips present in the wave. Below $\Delta r = 1.2\xi$ we see how the transfer of pseudo-vorticity between the vortex and wave is less effective for waves with lower dipole moments.

In Figure 5.36 the peak magnitude of D is plotted at the time of formation (top panel) and time of annihilation (bottom panel) for each of the three quasi-defects at different initial distances between the wave and vortex, r_0 . At formation, the D -field around the vortex (red) is close to the theoretical value of $D \approx 0.7$ when the wave is imprinted far from the vortex, or very close to the vortex. There is an in-between region where the influence from the wave on the pseudo-vorticity in the vortex is stronger. After the annihilation the positive quasi-defect which originated in the wave (blue) is left behind to form a defect.

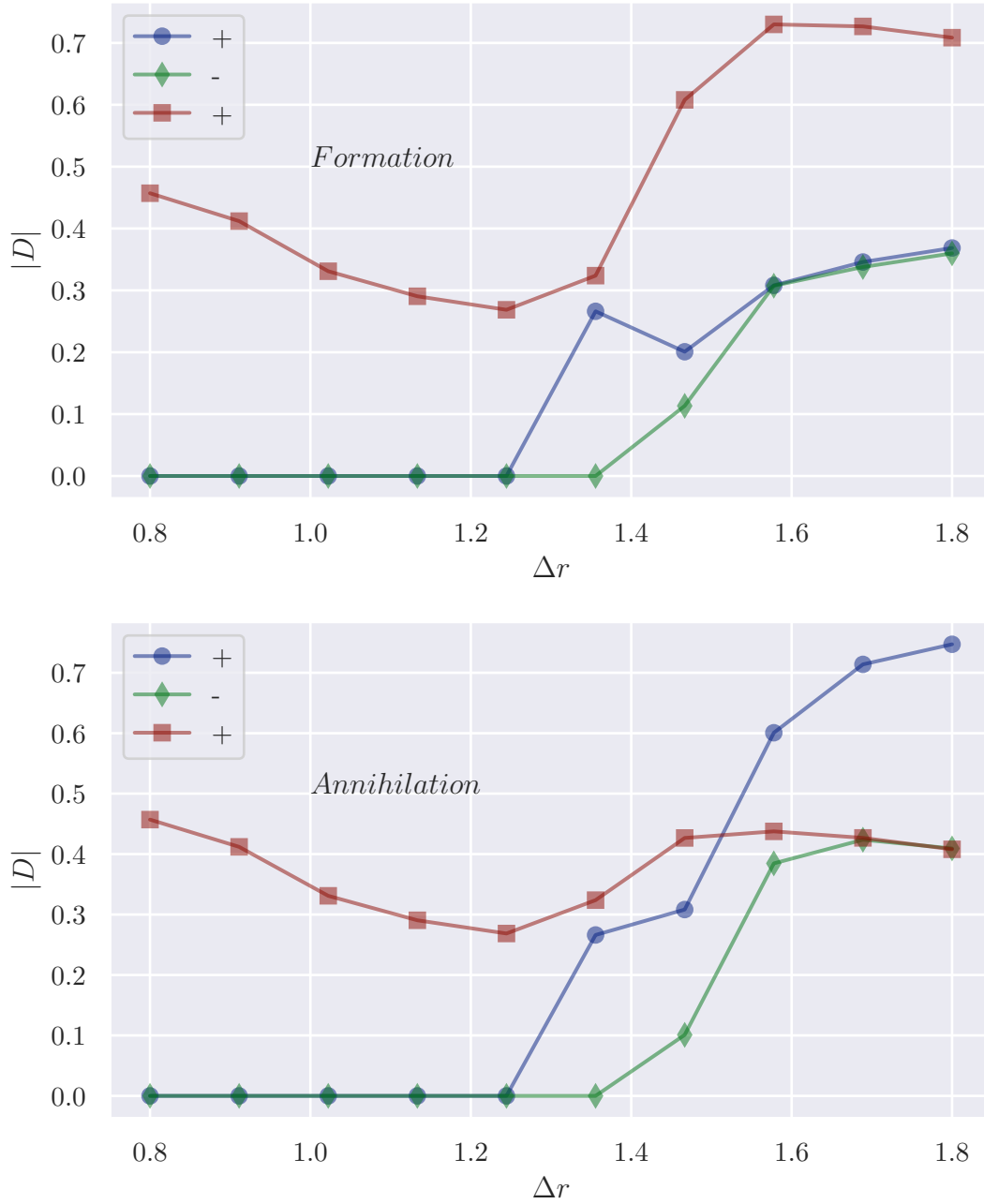


Figure 5.35: The peak magnitude of the pseudo-vorticity during the formation and annihilation of the phase slips as we vary the initial dipole moment of the wave, Δr . In the top panel is the magnitude as the phase slips form, and in the bottom panel as they annihilate. Below $\Delta r = 1.36$ there are no phase slips forming and we instead use the trough in E_i to indicate the time of collision. This is identical in the top and bottom panels. The red squares indicate the vortex, the blue circles the positive quasi-defect and the green rhombuses the negative quasi-defect.

At the time of the annihilation the density profile around the defect has not yet fully formed, and the magnitude of the D -field is overall lower than 0.7. The new phase slips form when the positive pseudo-vorticity (blue) peaks at around 0.3. At the time of annihilation the positive pseudo-vorticity in the annihilating defect (red) peaks at roughly 0.4. The negative phase slip (green) varies at formation. The phase slips do not appear or disappear at a specific value of $|D|$.

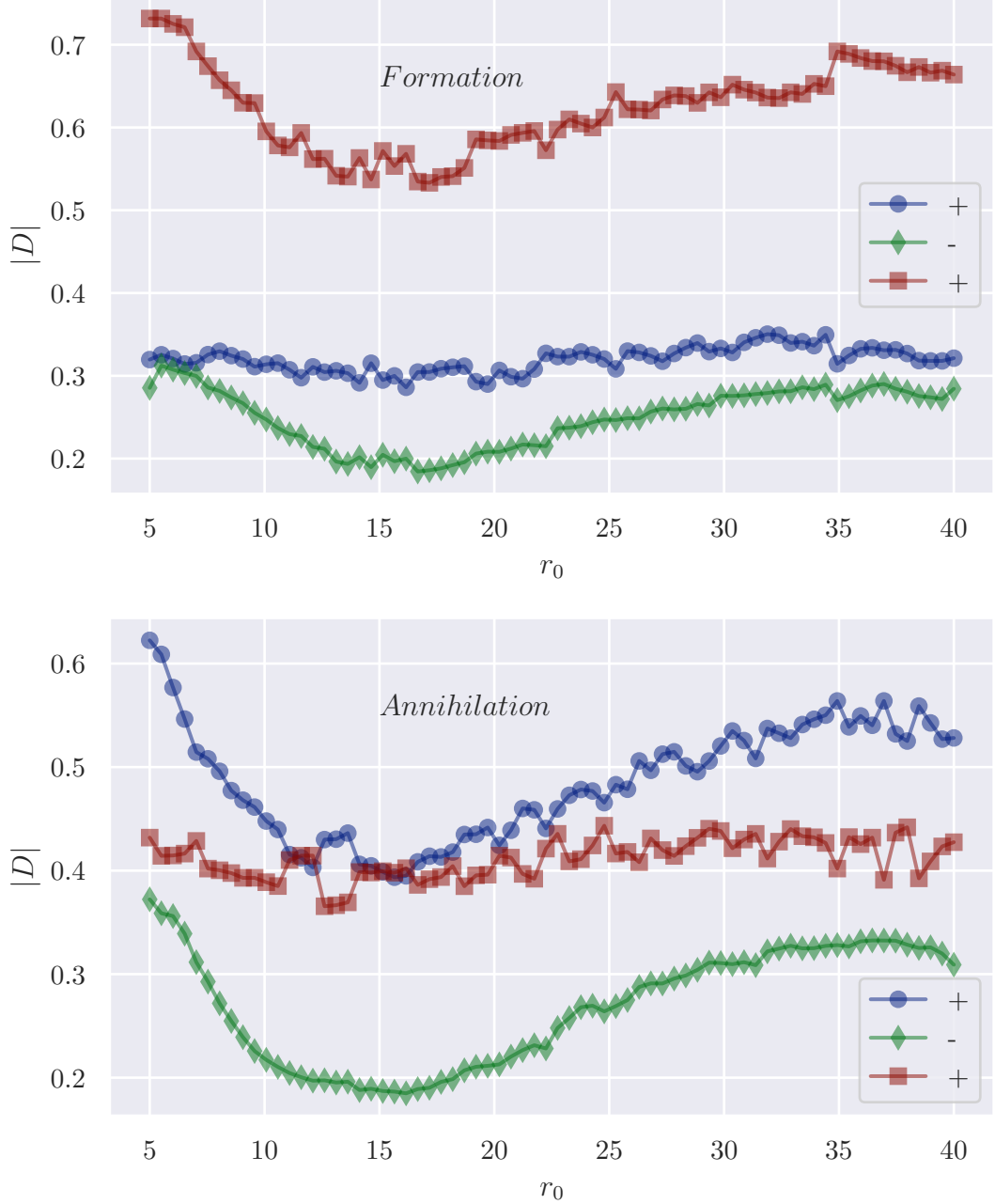


Figure 5.36: The peak absolute value of the D -field in the vicinity of the three phase slips during their formation (upper panel), and at the time of annihilation (lower panel). Again, red indicates the vortex, blue is the positive quasi-defect and green the negative quasi-defect.

To better understand how the magnitude of the D -field varies with r_0 we checked whether $|D(r_0)|$ is correlated to other properties of the incoming wave. We found that

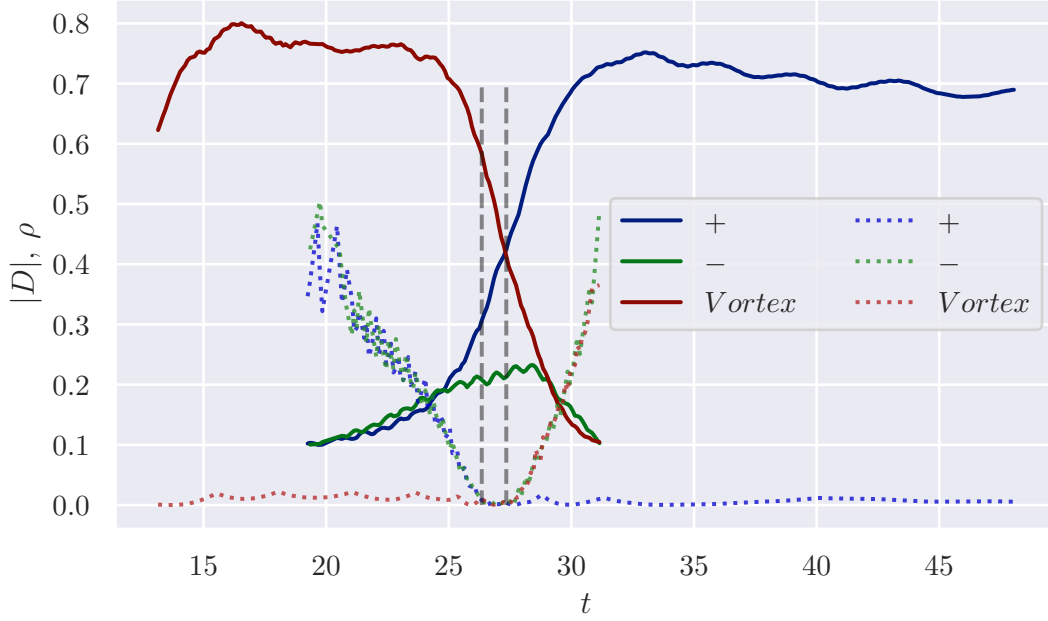


Figure 5.37: The peak magnitude of the pseudo-vorticity field, D , over time in a simulation of a wave imprinted at $r_0 = -20.2\xi$, with $\Delta r = 1.5\xi$. The grey dashed lines indicate the time of formation and annihilation of the phase slips. In this collision the positive phase slip forms at $D = 0.3$ and the vortex annihilates at $D = 0.4$. The dotted lines show the minima of the density within the regions of rotational flow. The phase slips are present as long as the density of the condensate goes to zero.

the D -field upon nucleation in the negative phase vorticity region and the vortex, were anti-correlated with the area of the trench. Recently after the wave is formed it spreads out and its area increases. After having propagated for some distance however, it starts shrinking again due to the influence of the vortex. Imprinting a wave without the central vortex we only observe the wave spreading, and no localisation. The overall area of the trench varies, but the area of the core is stable around $0.56\xi^2$. Since the core is of approximately the same size while the overall area of the trench varies the gradients in the condensate, and hence the D -field, varies. We can then understand the change in $|D|$ at different r_0 from the balance between the waves inherent tendency to spread out and its tendency to localise under influence of the vortex.

We see from Figure 5.37 how the magnitude of $|D|$ in the wave increases during the approach. The phase slips form and annihilate at different magnitudes of $|D|$, indicating that this is not the key parameter for determining when they are present or not. In this case we can see that the phase slip forms as the density of the condensate is very close to zero. When looking at the minimum density at formation and annihilation (Figure E.1) we find that the highest density we see when the phase slips are present is slightly below $\rho = 0.02\rho_0$. We use this as threshold for the core region of the defects.

Having looked at two of the processes that constitute vortex decay in isolation, we will now proceed to look at 2D decaying quantum turbulence directly.

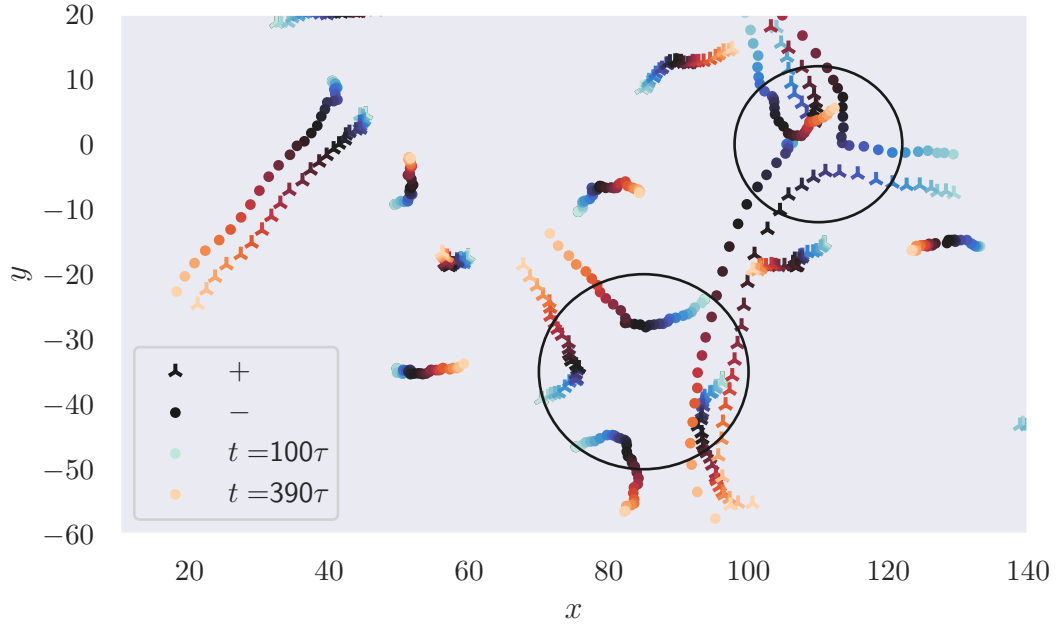


Figure 5.38: An illustration of the trajectories of a many-vortex configuration from one of the GPE simulations with periodic boundary conditions at $\gamma = 0$. The stars indicate the positions of positive vortices, while the circles indicate the positions of negative vortices. The blue-ish spots show the vortices at earlier times, while the red-ish spots show later times. The trajectories are shown between $t = 100\tau$ and $t = 390\tau$. Two exchange events are highlighted by the black circles.

5.4 Decaying quantum turbulence

We do pairs of simulations at $\gamma \in \{0, 5 \times 10^{-4}, 10^{-2}\}$ for both periodic boundary conditions and in disk traps. We have also done a simulation using the PVM at $\gamma = 0$ using periodic boundary conditions and a critical distance for vortex annihilation of 2ξ . The condensate is initialised with a random phase, and then relaxed until $it = 125i\tau$. At this point there are on average 106 vortices in the periodic system and 45 vortices in the trapped system. The numerical domain is 512×512 and the system size is $264\xi \times 264\xi$. This yields a similar initial vortex density as used by Baggaley and Barenghi [3]. For each of the set-ups we perform 8 simulations with different initial conditions and average over the results. Baggaley and Barenghi [3] do an ensemble average over 10 configurations, since we are using a computational node with 32 cores, it makes more sense resource wise for us to do an ensemble average over 8 configurations.

In Figure 5.38 we see the trajectories for the vortices in one of the GPE simulations in periodic boundary conditions for $\gamma = 0$ from $t = 100\tau$ to $t = 390\tau$. The vortices start out at the blue dots and move towards the red dots. Two circles mark two exchange events. The lower one shows a simple four-vortex exchange similar to what we studied in Section 5.2. The top vortex pair leave the collision relatively undisturbed, for this vortex pair the outgoing dipole moment is lower than the incoming dipole moments. The other process is a more complicated five-vortex exchange process. A dipole comes in from the above and slows down near the positive vortex already present within the circle. Next a dipole comes in from the right, the negative vortex in this dipole pairs up with the positive dipole which was initially within the circle, and leaves down to the left, while the positive vortex pairs up with the negative vortex which makes a complete

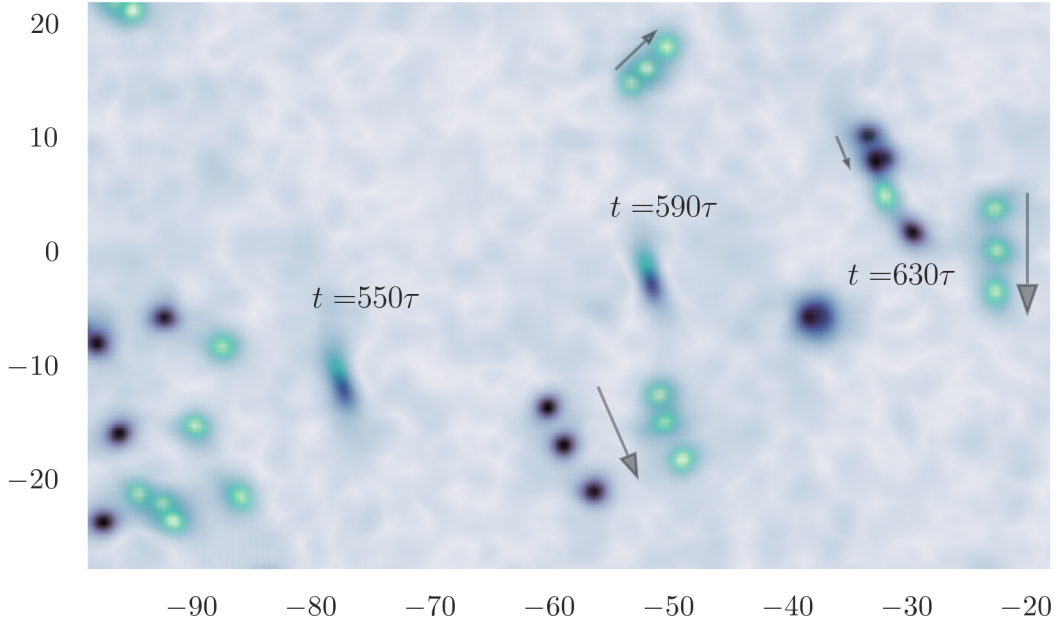


Figure 5.39: We have overlayed three images of the D -field of the condensate at $t = 550\tau$, $t = 590\tau$ and $t = 630\tau$. The regions where $|\Psi|^2 = 1$ are set to be transparent, while waves and vortices remain visible. At $t = 550\tau$ and $t = 590\tau$ we see a solitary wave moving from left to right across the image. At $t = 630\tau$ the wave has split up into a dipole due to interaction with the surrounding vortices. The motion of the surrounding vortices is indicated by arrows.

turn to travel upwards out of the circle. Exchange processes of this sort are typical in our simulations at $\gamma = 0$ and $\gamma = 5 \times 10^{-4}$.

In Figure 5.39 we see an example of a solitary wave splitting up into a dipole due to interaction with the surrounding vortices. The solitary wave is moving from left to right across the figure and splitting up at $t = 630\tau$. After the process shown in the figure, the dipole changes partners with one of the neighbouring vortices and keeps propagating through the system. Four subsequent partner exchanges later a new solitary wave is formed which collides with a vortex and disperses. Since the phase slips disappear and reappear this is a process which cannot be captured by the PVM, where the vortices are simply removed. In general the decay process happens through a series of exchange events, which more often than not reduce the dipole moment of the dipole or wave, until the wave disperses. We have seen how solitary waves with large dipole moments can retain their identity after a collision with a vortex. It is possible that the cascade of exchange events continues for several steps after the phase slips have annihilated.

The vortex decay in the turbulence simulations is shown in Figure 5.40. The dotted lines indicate that the condensate is trapped, while solid lines are periodic. The green dashed line shows the decay rate of the PVM. In comparing the yellow and teal lines we see the impact of the trap on a damped condensate. In a uniform condensate (yellow) the decay of the vortices decay as $N_v \sim t^{-1}$. Since $\alpha = -1/(n-1)$ this implies that the vortex decay is dominated by a two-body process. The central attraction due to damping cause vortices and anti-vortices to attract each other and annihilate without the need to transfer energy to surrounding vortices. With the wall present however, this mechanism is not the sole driver of annihilation. None of the values we found for α in Figure 5.40 suggest that a four-body annihilation process dominates the vortex decay.

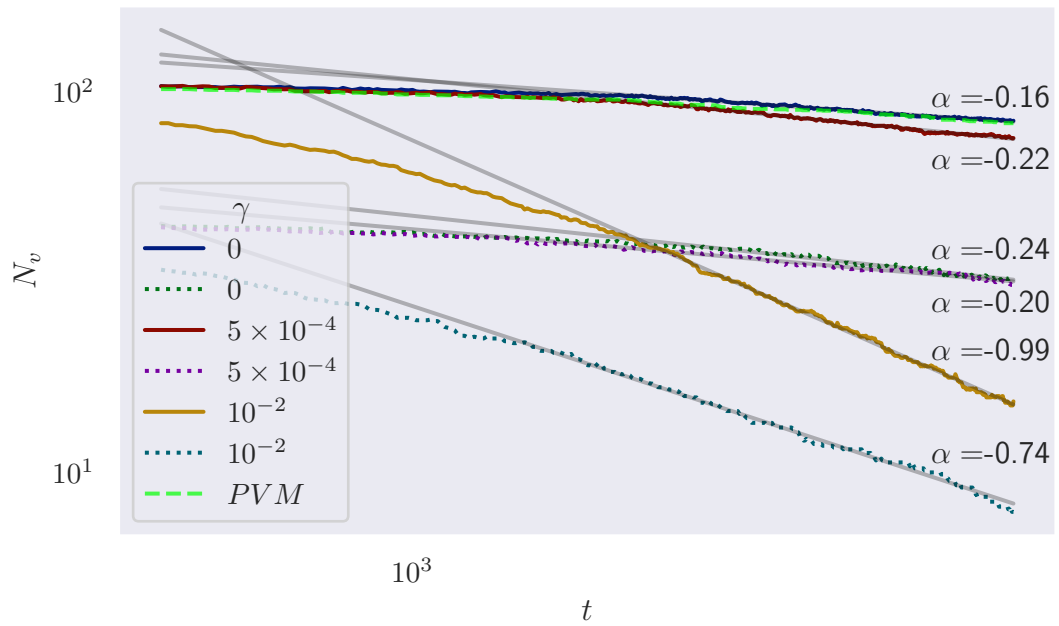


Figure 5.40: The vortex number over time for five different turbulence simulations. Dotted lines are simulations done in a disk trap, while solid lines are performed with periodic boundary conditions. The damping coefficient varies from 0 to 10^{-2} . We have fitted $N_v(t)$ to a power law, t^β , on the interval $[2000\tau, 5000\tau]$. The coefficient β is shown for each of the simulations. The trapped simulation with $\gamma = 0$ has $\alpha = -0.24$, while at $\gamma = 5 \times 10^{-4}$ we have $\alpha = -0.20$. The decay exponent for the PVM-simulation is not shown, it is $\alpha = 0.13$. The x -axis ends at $t = 5000\tau$.

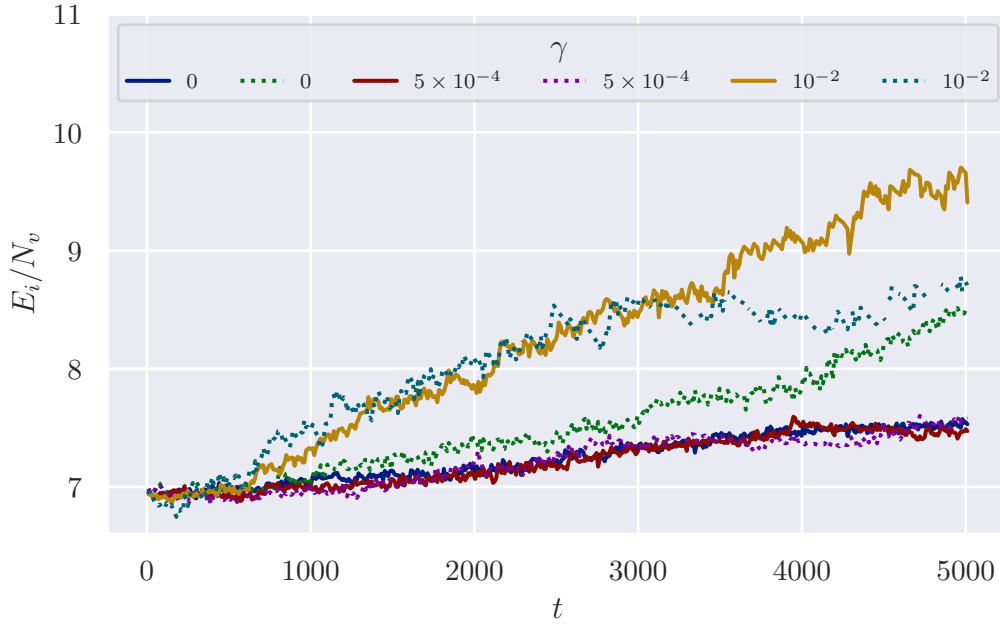


Figure 5.41: Incompressible kinetic energy per vortex. Dotted lines are for trapped condensates, solid lines are simulations in a condensate with uniform background density. The results for $\gamma = 0$ and $\gamma = 5 \times 10^{-4}$ in is very similar in the uniform condensate. E_i/N_v increase more rapidly for $\gamma = 10^{-2}$ due to the higher rate of vortex decay. In this case E_i/N_v in the trapped condensate flatten out at $t = 3000\tau$. At around $t = 4000\tau$ there is an increase in the steepness of the E_i/N_v for the trapped condensate at $\gamma = 0$, this is probably associated with instabilities in this particular set of simulations, causing the incompressible kinetic energy to blow up (Figure E.5).

For a trapped system where $\gamma = 0$ (green dotted line), we have $\alpha = -0.24$, which gives $n \approx 5$. This would suggest that the vortex decay is dominated by a five-body process. It is possible however that a combination of multiple types of processes are present.

At $\gamma = 0$ in a periodic system we would expect to find $\alpha = -1/3$. Instead we find $\alpha = -0.16$. The four-vortex annihilation process might be present in the simulations, but the relatively large vortex density might cause processes with higher n to contribute. For larger systems which evolve for longer we might retrieve the result $\alpha = -1/3$. Groszek et al. [4] found a signature the four-vortex process in combination with a three-vortex process, with vortex numbers comparable to our simulations. Their vortex density however was significantly lower than ours. The periodic simulations done in Baggaley and Barenghi [3] are done for a system four times the size of ours with simulation times two orders of magnitude longer than ours. For the PVM the decay rate is given by $N_v \sim t^{-0.13}$, slightly lower than the GPE simulations at $\gamma = 0$. Despite the difference in the microscopic details of the annihilation mechanism we obtain a similar decay rate.

A central quantity for the spontaneous self-organisation seen in Simula, Davis and Helmerson [8] and Kanai and Guo [9], is the incompressible kinetic energy per vortex. For our simulations the results are shown in Figure 5.41. For $\gamma = 10^{-2}$ there is a steep increase in E_i/N_v because of the high annihilation rate. The vortex numbers at the end of the simulations with $\gamma = 10^{-2}$ are very low. This is the reason that E_i/N_v plateaus in the trapped case, the few vortices that remain are so far apart that they take a very long time to annihilate. Comparing the undamped trapped simulations (green dotted) to undamped simulations with periodic boundary conditions (blue solid), we see that the influence of the wall contributes to a steeper increase in the energy per vortex. A dipole

Table 5.1: Incompressible kinetic energy per vortex. r is here the Pearson correlation coefficient.

γ	r	$\Delta E_i / \Delta N_v$ [μ_0]	
0	0.998	4.56	
0	0.997	4.03	Trap
5×10^{-4}	0.998	5.29	
5×10^{-4}	0.997	5.26	Trap
10^{-2}	1.000	6.41	
10^{-2}	0.998	6.43	Trap

annihilating with the wall can be interpreted as a similar four-vortex process as vortices annihilating in the bulk, since the vortices have ghosts on the far side of the wall. The phonons produced however, cannot propagate in all directions. This limits the energy transfer $E_i \rightarrow E_c$, and cause the steeper increase in E_i/N_v that we observe. This is also supported by the change in incompressible kinetic energy per annihilation event seen in Table 5.1. At $\gamma = 0$ the value of $\Delta E_i / \Delta N_v$ is lower in the trapped system than in the periodic system, causing the vortices remaining after an annihilation event to have a higher E_i . The increase in E_i/N_v indicates that the evaporative heating mechanism is present. The steepness of increase in E_i/N_v however is not sufficient to judge how effective it is, since it depends on annihilation rate. The required value of E_i/N_v to observe clustering also depends on the steepness of the trap walls [4], and E_i/N_v does not supersede the necessary value to observe clustering in the trap we are using.

If we compare the trapped simulations at $\gamma = 0$ (green dotted) and $\gamma = 5 \times 10^{-4}$ (pink dotted) we see that there is a significant difference in the steepness of E_i/N_v . Considering that they have very similar decay rates (Figure 5.40) the difference must come from the effectiveness of energy transfer in the annihilation processes. Table 5.1 confirms that this is the case. The energy transfer during the annihilation process in the presence of a wall is sensitive to γ , while it makes no difference in the periodic case. This supports the conclusion that the limited production of sound waves during annihilation with the wall are central for the evaporative heating mechanism at zero temperature.

The change in incompressible kinetic energy per vortex presented in Table 5.1 is taken from the slope of a linear regression of the collection of points $[N_v(t), E_i(t)]$. The correlation coefficients between E_i and N_v confirm that these are correlated quantities in all our simulations. For the trapped simulation with $\gamma = 0$ we only use points up until 4000τ since the relative error in the total energy of the system exceeds 0.5% after this point (Figure E.5). With a higher damping coefficient E_i decreases more significantly per annihilation event, this makes the evaporative heating mechanism less efficient. In Section 5.3.10 we found that the change in incompressible kinetic energy due to the vortex interaction with the solitary wave is a lot smaller than the change in energy per vortex found in Table 5.1. The change in E_i during annihilation must either come at an earlier stage of the process, or through a long series of such interactions.

The spectra of the incompressible kinetic energy at $t = 4000\tau$ is shown in Figure 5.42. In the near field the spectra follow the expected near-field solution for a single vortex, k^{-3} . At intermediate k the spectra follow the far-field scaling, k^{-1} , for a single vortex. At the largest length scales the spectra deviate from this law, as $E_i(k)$ drops for the lowest k . This is indicative of an even distribution of vortices and anti-vortices in the condensate. A clustered state would exacerbate the energy at large scales, while for an evenly distributed state the vortices screen each other diminishing the energies on large

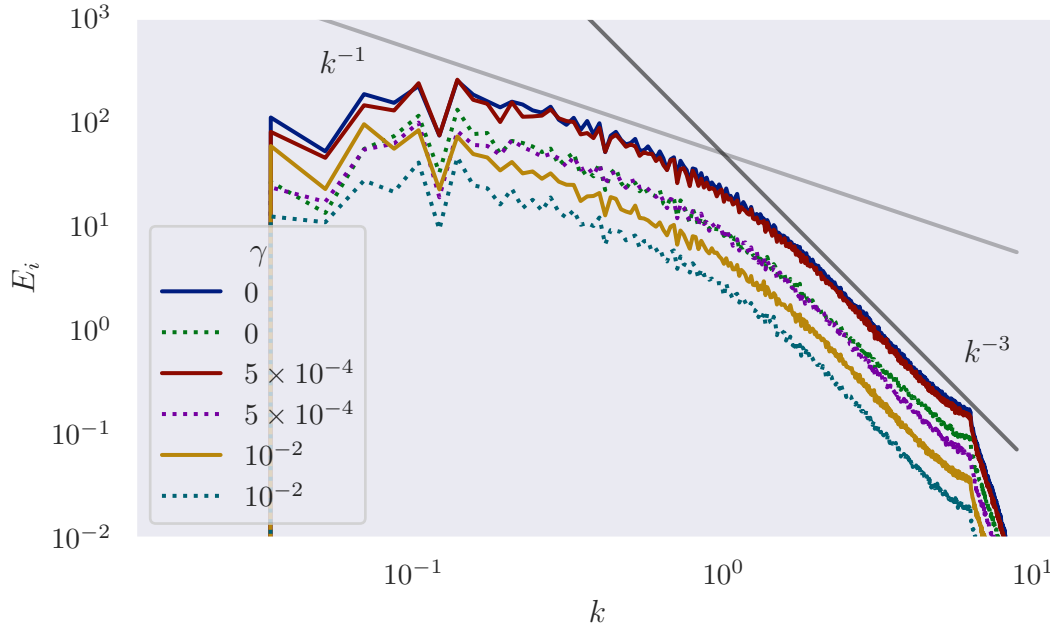


Figure 5.42: The spectra of incompressible kinetic energy for the different simulations at $t = 4000\tau$. Again, the dotted lines indicate a condensate in a disk trap, while the solid lines show the results for a uniform condensate. The grey lines show the scaling laws, k^{-3} , for the core of the vortices, and k^{-1} for the far field of the single vortex solution. At low k all the spectra flatten out or dip.

scale. If we compare the spectrum of the periodic simulations with $\gamma = 0$ at different points in time (Figure E.6) we see that there is hardly any change. However there is a small increase in the large scale energy from $t = 0$ to $t = 4000\tau$. This indicates that there is an energy flux towards larger scales in the system, but it is tiny.

The spectra of E_c at $t = 4000\tau$ is shown in Figure 5.43. At $\gamma = 0$ we see that there is an equipartition of the compressible modes (blue and green), the energy E_c is equally distributed across all k . The thermal modes equilibrate and there is no net flux of energy across length scales. When we introduce finite damping the overall compressible kinetic energy drops. The drop is more significant for E_c than it is for E_i . For $\gamma = 5 \times 10^{-4}$ (red) there is an equipartition of energy at high length scales, and a drop off as k increases. At $\gamma = 10^{-2}$ (purple and yellow) E_c is lower at low k , peaks in the region where $k \approx 1/\xi$ and drops off as k increases further. Since we expect the compressible modes to dissipate evenly across k , we suspect that this peak is related to vortex motion. At $t = 0$ more of the compressible kinetic energy is concentrated at larger length scales, and there is a flux to higher k during the simulation. As the spectra evolve the $\gamma = 0$ simulation approaches the equipartition state over time (Figure E.7). For $\gamma = 10^{-2}$ the compressible kinetic energy drops across all length scales, and peak becomes sharper and shifts towards smaller k .

Figure 5.44 shows the spectra of quantum energy at $t = 4000\tau$. For the uniform configurations there is a peak in the energy distribution just above $k \approx 1/\xi$. The vortices are the main contributions to E_q in this case. If we compare $\gamma = 0$ (blue) to $\gamma = 5 \times 10^{-4}$ we see that there is more energy at large wave numbers in the former case. The presence of dissipation flattens out high k perturbations more effectively. This implies that features with steep density gradients, such as solitary waves, are more

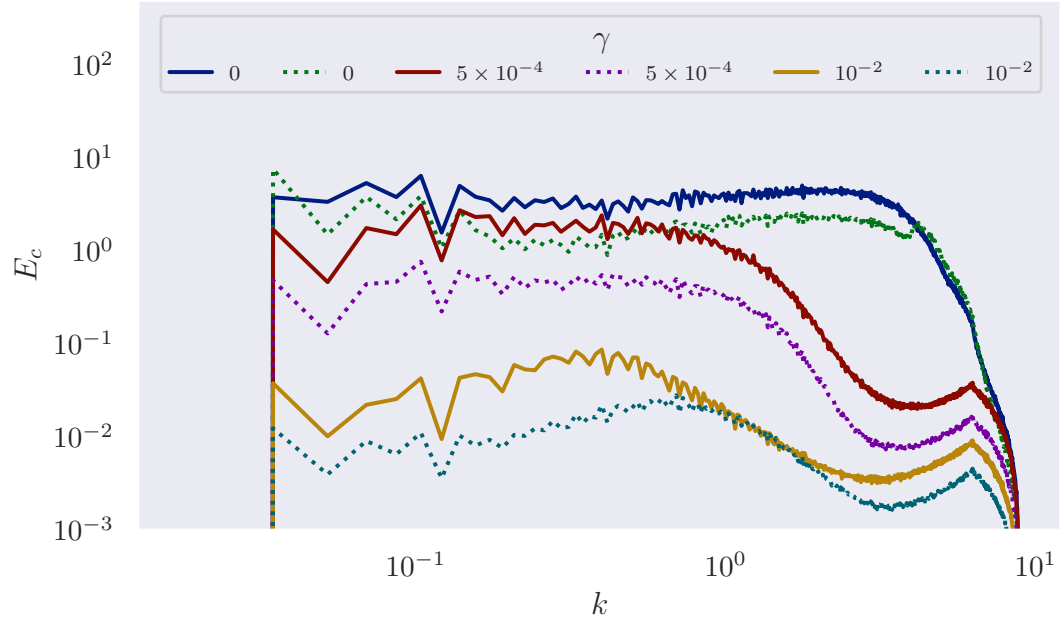


Figure 5.43: The spectra of the compressible kinetic energy at $t = 4000\tau$. The dotted lines indicate a trapped condensate, while the solid lines show the spectra of uniform condensates.

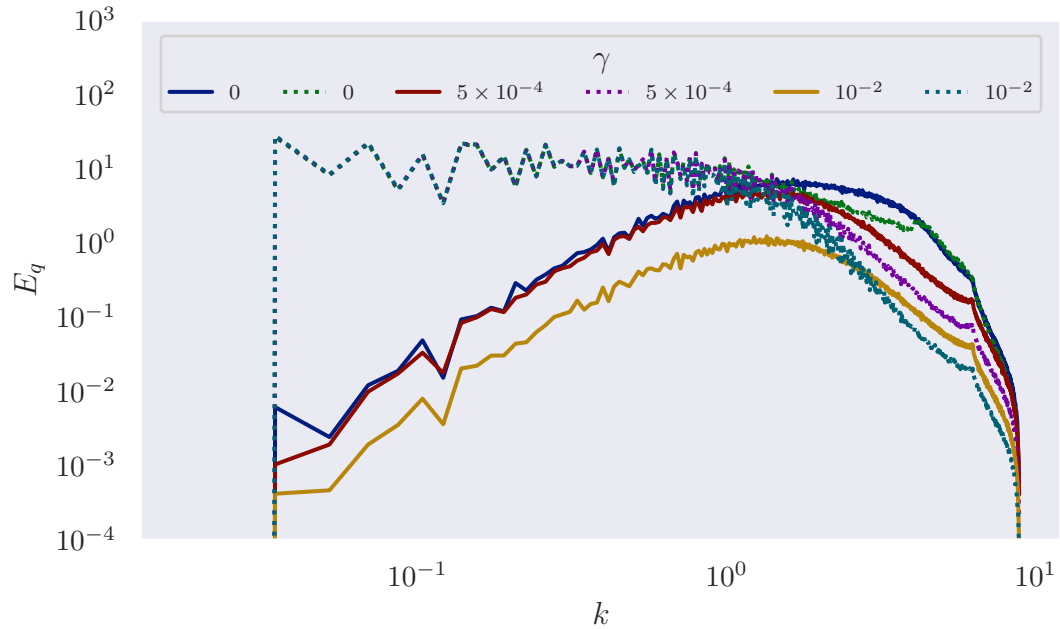


Figure 5.44: The spectra of E_q at $t = 4000\tau$. For the simulations in a uniform condensate the spectra peak at roughly $k = 1/\xi$, while the trapped condensates have flat spectra above $k = 1/\xi$.

prevalent at $\gamma = 0$. At $\gamma = 10^{-2}$ (purple) the overall energy is lower, but the peak is at the same k as for $\gamma = 5 \times 10^{-4}$, supporting the interpretation that this peak is associated with vortices. In the trapped condensates (yellow and green) the steep density gradient near the trap wall is the dominant contribution to E_q , and the spectrum of E_q is largely determined by the shape of the trap. With the specific disk trap we have used the energy associated with the quantum pressure is equally distributed across k above the vortex length scale $k \approx 1/\xi$. In Figure E.8 we see how the energy distributions of E_q evolve in time. For $\gamma = 10^{-2}$ (red) the overall energy decreases with time. The decrease is even across all wave numbers. At $\gamma = 0$ (blue) the distribution of E_q evolve in time. The energy spectrum starts out with a peak associated with the vortices at $k \approx 1/\xi$. As time passes the quantum energy at high k increases. This is probably a result of the solitary waves, or other steep waves, formed due to vortex motion and annihilation.

Chapter 6

Summary and conclusions

We have studied two isolated processes that play central roles in 2D decaying quantum turbulence. In the dipole-dipole collisions we have found the critical vortex separation for vortex annihilation. Interestingly, despite the discontinuous transition from dipole to solitary wave, in terms of the phase slips, there is a gradual change in the dynamics from low energy dipoles to solitary waves. Without checking the phase explicitly the two look very similar and interact with the surrounding vortices and trap wall in a similar manner. During the annihilation event we found that the energy transfer from E_i to E_c is mediated by the quantum energy, E_q . When comparing the trajectories of the phase slips in the collision to the estimate from the PVM, we found that the PVM deviates from the GPE when the vortices are accelerating, i.e. when they emit radiation. The PVM also fails to replicate the phase slip trajectories when vortices are close together, due to effects from the interacting density profiles of the vortices. For dipoles with low dipole moments we observed an increase in the vortex separation associated with a decrease in the compressible kinetic energy supporting the conclusion that the vortices reabsorb radiation.

For the wave-vortex collision we have found that the collision event takes place as a topological exchange. The wave splits up and two new phase slips form, one of which annihilates with the vortex leaving its former partner behind to form a new defect. The change in incompressible kinetic energy during this event is minimal. The change in incompressible kinetic energy occurring during the four-vortex annihilation happens primarily in the first stage of the process – during the dipole collapse. It is possible however that the dipole collapse occurs through a series of exchange events where the dipole moment gradually decreases until the defects annihilate. Since solitary waves interact with vortices in a similar manner to dipoles it is possible for such a chain of exchange events to continue after the defects have annihilated.

From the decaying 2D quantum turbulence we saw a clear signature for a 2-vortex annihilation process for the uniform condensate when $\gamma = 10^{-2}$. At $\gamma = 5 \times 10^{-4}$ and $\gamma = 0$ we do not find the direct signature from the decay rate that the four-vortex annihilation process is the dominant mechanism for annihilation. Our results indicate that more vortices are involved in the annihilation process. We believe this is because of the vortex density in our simulations. The microscopic details of the decay process in the PVM and the GPE are different. Still, when comparing the PVM with to the GPE for the simulations in uniform condensate at $\gamma = 0$, they predict similar decay rates. When looking at the spectra of the incompressible kinetic energy of the turbulent states at $t = 4000\tau$, we see that the energy on the largest scales are smaller compared to the single-vortex prediction. This indicates that screening effects are more prominent than

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clustering effects in our simulations.

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Appendices

Appendix A

Calculating the radiation of energy in a head-on dipole-dipole collision

As vortex dynamics in a 2D superfluid is analogous to the dynamics of charges in 2D electromagnetism [48], we can calculate the energy radiated from accelerating vortices from the Larmor formula,

$$E_c = \int dt a^2.$$

For a head-on collision of two dipoles where the velocities after collision change by 90° , like in Figure 5.9, we can assume that the duration of the collision is approximately [49]

$$\Delta t \sim \frac{\Delta r}{v}.$$

The acceleration is given by

$$a \sim \frac{v}{\Delta t}.$$

From the PVM one can show that the dipole speed v is inverse proportional to the dipole separation, $v = 1/\Delta r$, as we have also confirmed in Section 5.1. Hence the collision duration scales as $\Delta t \sim \Delta r^2$, and the radiated energy during the collision is then

$$E_c \propto \int_0^{\Delta t} dt \frac{1}{\Delta r^2 \Delta t^2} = \frac{1}{\Delta r^4}.$$

In Figure 5.12 we compare the compressible kinetic energy before and after the collision. In addition to the theoretical prediction $\Delta E_c \sim 1/\Delta r^4$ we calculate the radiated energy from Larmors formula using the tracked trajectories of the vortices in the GPE simulation. Assuming that the acceleration is constant we can fit a second order polynomial in t to $x(t)$ and $y(t)$ during the acceleration stage. The coefficient of the second order term is then the acceleration. We can then find the radiated energy $\Delta E_c \propto \Delta t a^2$, where Δt is the duration of the acceleration stage.

Appendix B

Finding the time of nucleation and annihilation from the trajectories

The time of formation and annihilation of the phase slips during the wave-vortex collision are found from the trajectories of the minima in the condensate. To do this we must first sort the trajectories. The ρ -tracker produces a $N_T \times 3 \times 2$ array, where the first dimension is the number of time steps, the second dimension indexes the phase slip, and the third dimension gives the spatial coordinate of the phase slip position. The tracker looks for three minima at each time step, but is not always able to find a minima within the regions it is searching in. This shifts the order of the phase slip positions between the trajectories in the array. The data clearly follows three distinct trajectories, but the tracked trajectories appear to jump back and forth between the three phase slips. To sort the trajectories we start by calculating the difference along the three trajectories from one point in time to the next, and slice the trajectories into segments when the tracked trajectories jump between different phase slip trajectories. This gives a set of segments that can be patched together. The three longest continuous segments generally belong to the three different phase slips, so these are taken at the starting point of the phase slip trajectories. The next step is to run through the free segments, patching them to the partial trajectory where one of the ends of the free segment is closest to one of the ends of the partial trajectory. For easier comparison between all of the simulations we order the three trajectories according to their mean y -value, this places the trajectories of the phase slips in a consistent order across the simulations of the wave-vortex collisions in Section 5.3.

There is a kink in the trajectory where the phase slips form and disappear. This is found using change point detection. For each point along the trajectory the slope is calculated using linear regression over a frame of points centred on that point. This is done for the x and y coordinates independently. Next we filter the slopes using a Savitzky-Golay filter and find the most prominent extremum for each of the three phase slips. We assume that the first one corresponds to the time of nucleation of the phase slips t_s , and the last one to the time of annihilation t_e . The frame width used for finding the slopes is of order 1τ , and the filter uses a similar sized frame.

Appendix C

Vortex shift and incompressible kinetic energy

The vortex is not located at the centre of the trap after the collision between the solitary wave and the vortex in Section 5.3. This could in principle be the reason for the change in incompressible kinetic energy observed in Section 5.3.10. A vortex at the centre of a circular trap has a ghost vortex at infinity, a deviation from the centre brings the ghost vortex into finite distance, altering the velocity field in the trap. The vortex and its ghost effectively form a dipole velocity field. The closer the vortex is to the boundary the nearer the ghost vortex, the lower the incompressible kinetic energy. The same principle applies to our elliptical trap. The shift from the centre during the phase slip exchange is associated with a change in incompressible kinetic energy.

The position of the vortex at the time of the annihilation, and its position 10τ later are shown in Figure C.1. To find the change in E_i due to the shift we imprint a vortex at the positions corresponding to either t_e or $t_e + 10\tau$ in a condensate with no other disturbances and relax the system until $it = i\tau$. Then we compare the incompressible kinetic energy of the system to that of a system with a vortex at the centre. The change in E_i due to the shift shown in Figure C.2. The difference in incompressible kinetic energy due to the vortex being offset from the centre is $0.012\mu_0$ at its maximum. Comparable to the change in E_i during the collision event found in Section 5.3.10 this effect is negligible. The change in incompressible kinetic energy cannot be ascribed to the displacement of the vortex. At the distances relevant for this set-up, we can treat the incompressible kinetic energy associated with the vortex as independent of position.

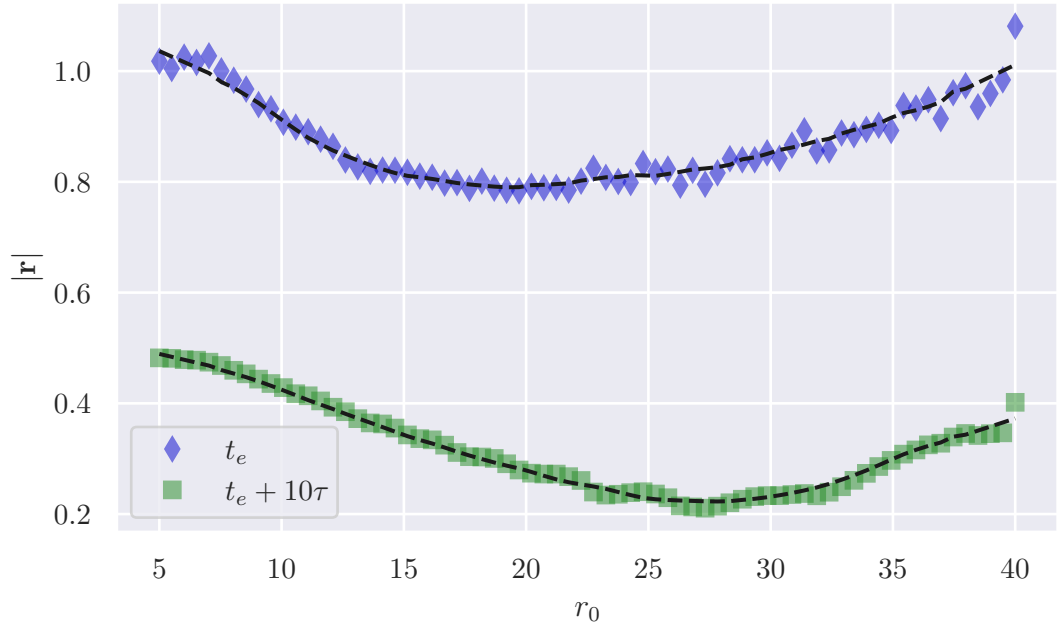


Figure C.1: The distance of the new vortex $|\mathbf{r}|$ from the origin at the end of the collision.

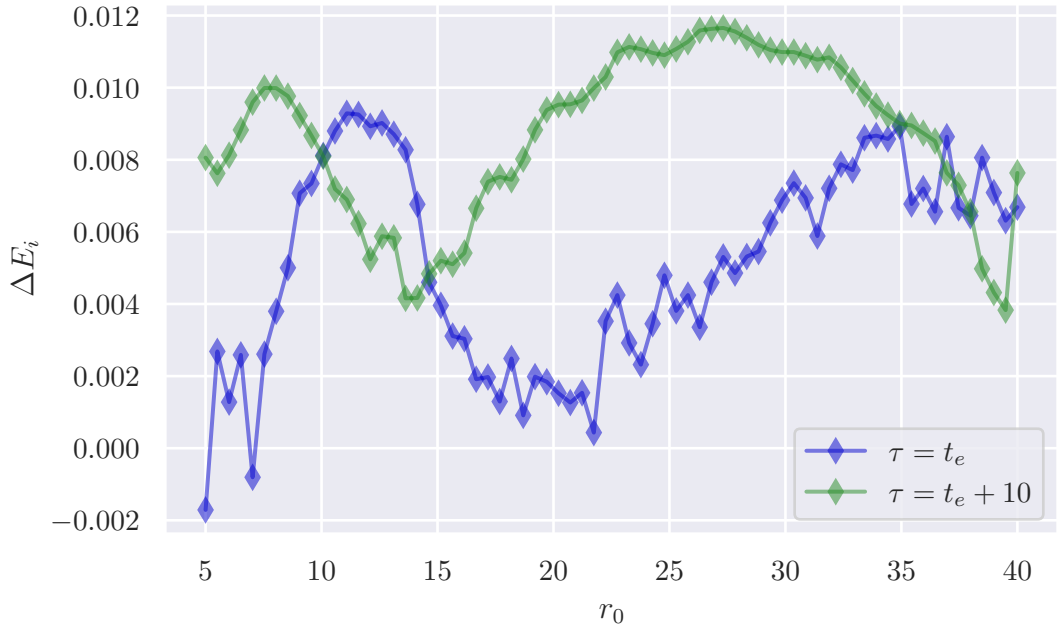


Figure C.2: The change in incompressible kinetic energy due to the shift of the vortex position.

Appendix D

Numerical implementation of the vortex velocity from Equation 3.4

First, we refine the integration domain by interpolating the Ψ -field using linear spline interpolation, and use this to evaluate the derived fields ρ , D and $\mathbf{J}^D$. Next, we mask out a region within the vortex core where $\rho < 0.1\rho_0$, and calculate the mean velocity from Equation 3.4 within this region. If the region contains less than 5 pixels, the threshold on ρ is increased by $0.01\rho_0$. To determine the vortex velocity, we evaluate the average of Equation 3.4 over the grid points within the vortex core. This average value is taken over a small number of points, so it is sensitive to fluctuations within the core. In Section 5.1, we only use data from the interval $[10\tau, 20\tau]$, to avoid the initial waves influencing the vortex velocity.

Appendix E

Additional figures

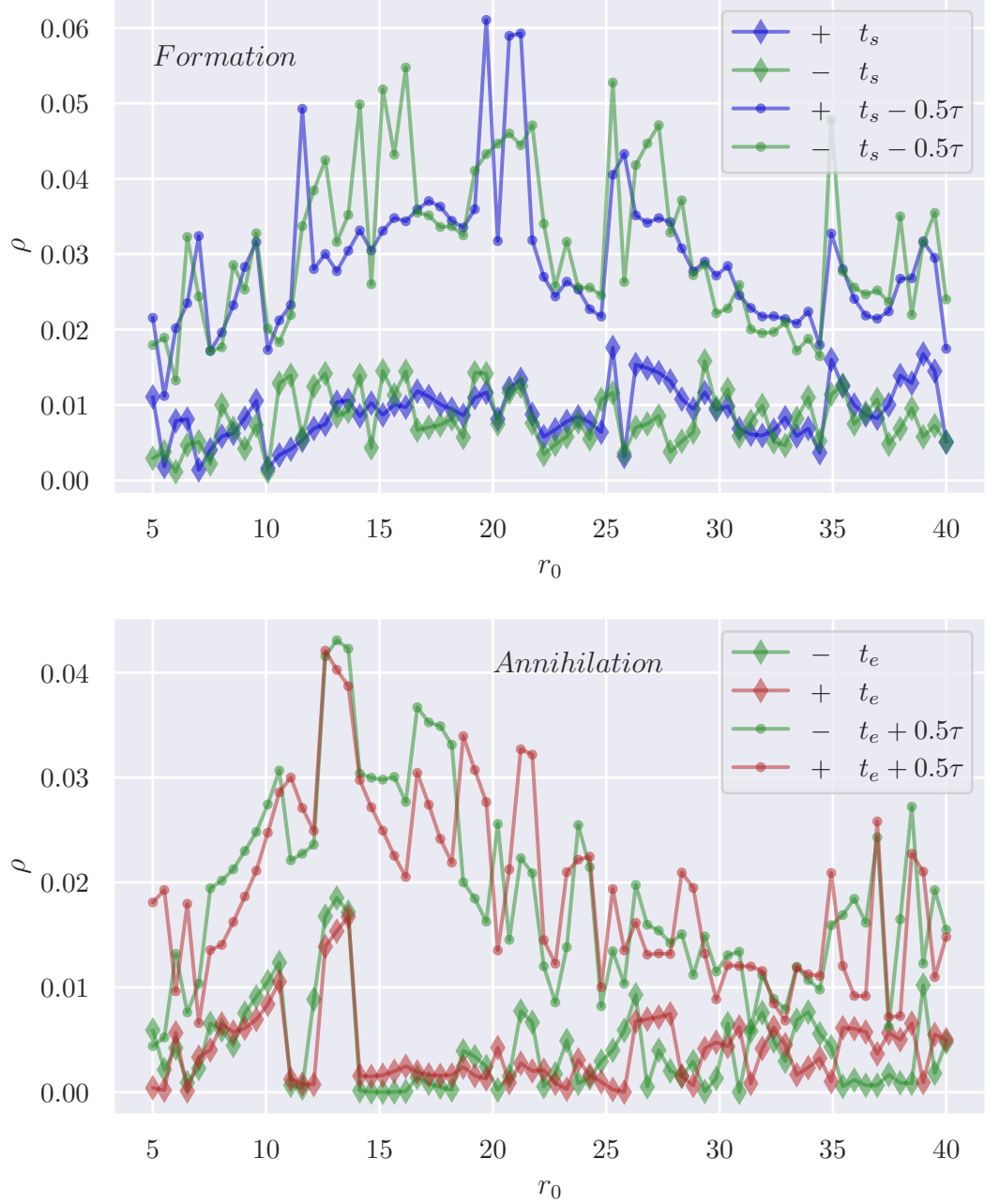


Figure E.1: The minima of the condensate density near the phase slips in a wave-vortex collision with $\Delta r = 1.5\xi$ at different r_0 . The top panel shows the minimum condensate density in the regions of positive and negative pseudo-vorticity in the incoming wave 0.5τ before the phase slips form, and at the time of formation. In the lower panel we see the minimum condensate density near the phase slip of the original vortex and the negative phase slip from the wave as they annihilate, and in the associated regions of positive and negative pseudo-vorticity 0.5τ after annihilation. The minimum density when the phase slips are present is generally lower than $\rho = 0.02\rho_0$. Since there can be phase slips below $\rho = 0.02\rho_0$ we take this as the threshold for the core region of the vortices and quasi-defects in our simulations. It is not a sufficient condition for the presence of phase slips, but serves to give an indication of the area of the core region.

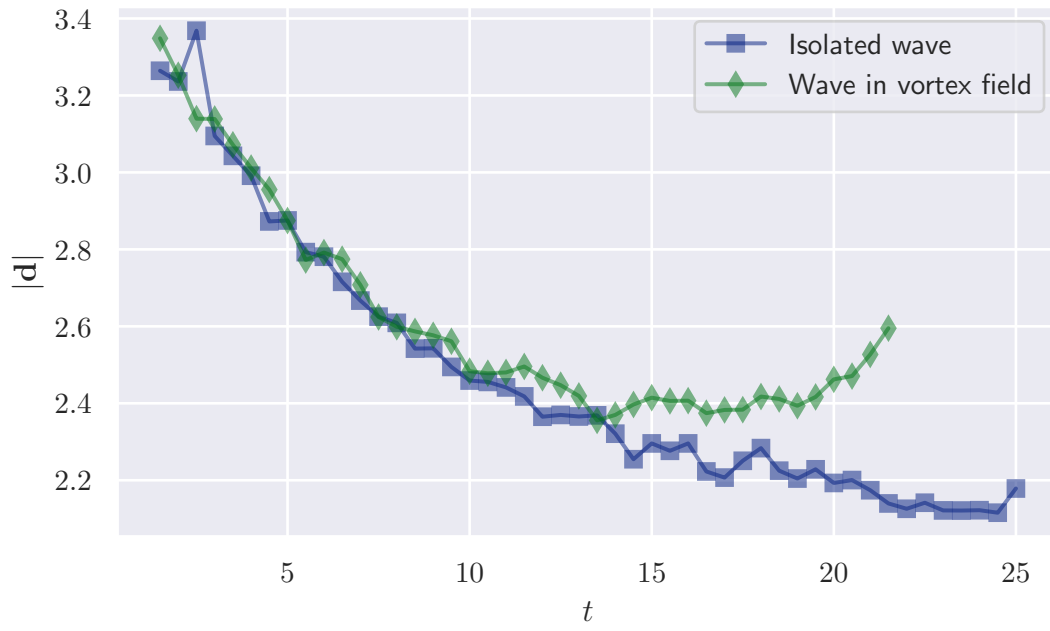


Figure E.2: The dipole moment of an isolated wave in a trap and a wave interacting with a vortex. The dipole moment in the wave interacting with a vortex starts deviating from the dipole moment of the isolated wave at around $t = 15\tau$. This is the same point that the incompressible kinetic energy starts deviating in Figure 5.21.

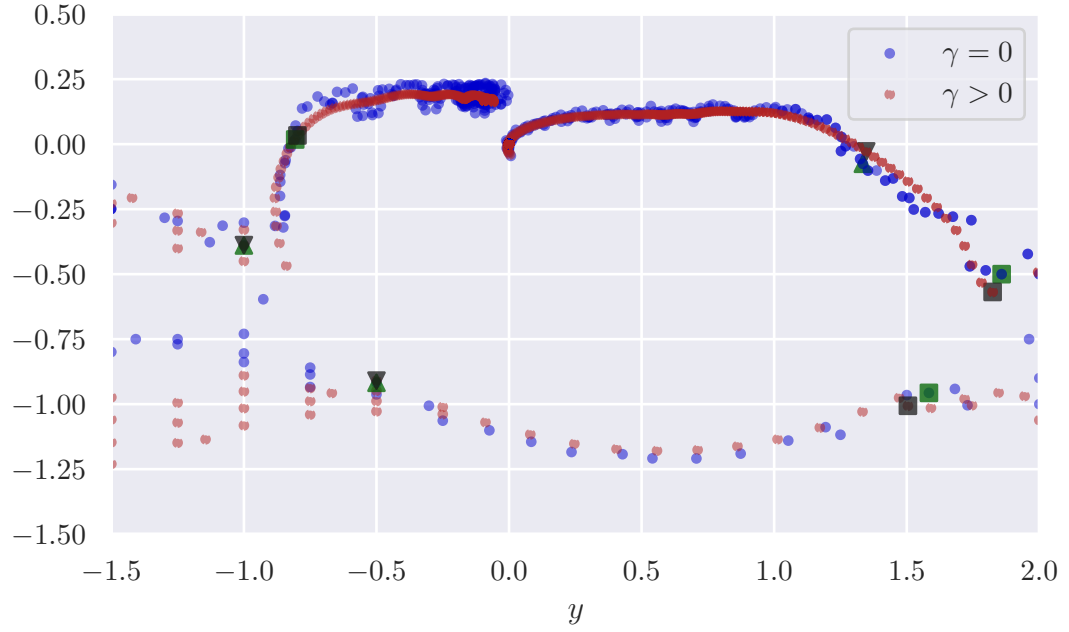


Figure E.3: The position of the quasi-defects in a simulation of a wave-vortex collision with dissipation along the boundary (red) along with a where no dissipation is present (blue). The black triangle and square indicate the time of nucleation+ t_s , and annihilation, t_e , of the defects as calculated from the dissipative simulation. The green triangle and square show the position of the phase slips in the non-dissipative case at t_s and t_e . The trajectories in the non-dissipative simulation are more noisy, but correspond well with the dissipative simulation. In both cases $\gamma = 0$ at the centre of the trap so we expect the dynamics of the wave-vortex interaction to be the same in both simulations. This is confirmed by the position along the trajectories at the time of formation and annihilation, which is the same for both simulations. We can use then the non-dissipative simulation to calculate the change in incompressible energy during the collision and assume that it is applicable to the process studied in the dissipative simulation.



Figure E.4: The change in incompressible kinetic during the exchange event in a wave-vortex collision. When the wave is imprinted close to the vortex there is some decrease in E_i during the event. At larger distances the change in E_i varies around 0.

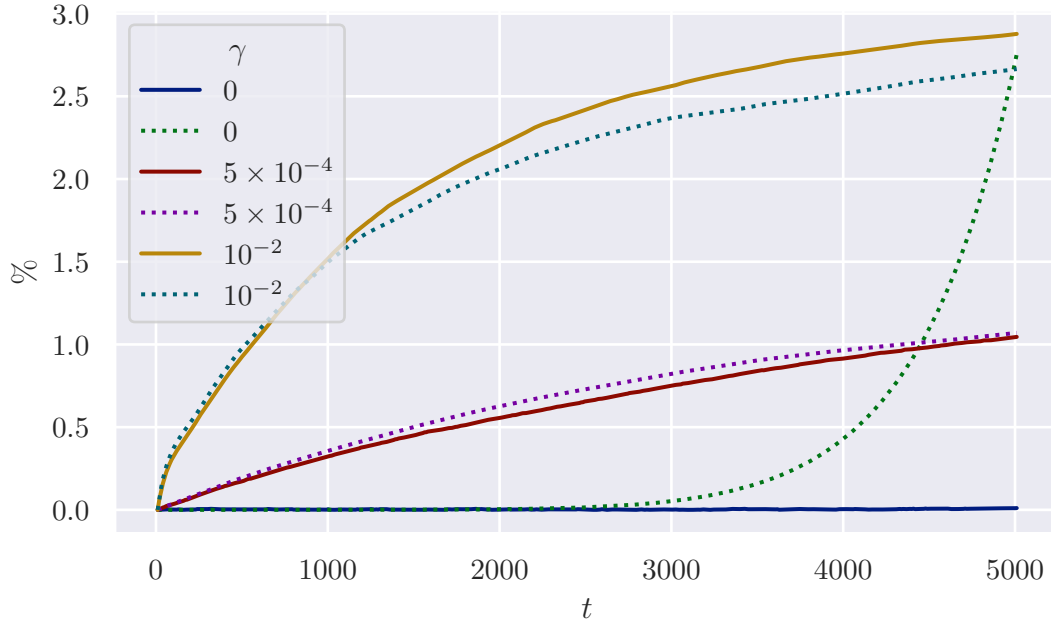


Figure E.5: The change in the total energy over time relative to $t = 0$ for the six sets of turbulence simulations. The relative change in percent is given by $\delta H = 100 \frac{|H - H(t_0)|}{|H(t_0)|}$. For the dissipative simulations the total energy over time deviates from $H(t_0)$ as expected. For the simulations at $\gamma = 0$ the periodic one is stable and conserves energy over time, while the trapped simulation is unstable. In this case the error passes 1% at around $t = 4500\tau$. The spectra presented in Section 5.4 are snapshots at $t = 4000\tau$, at which time the error in the trapped simulation at $\gamma = 0$ is less than 0.5%.

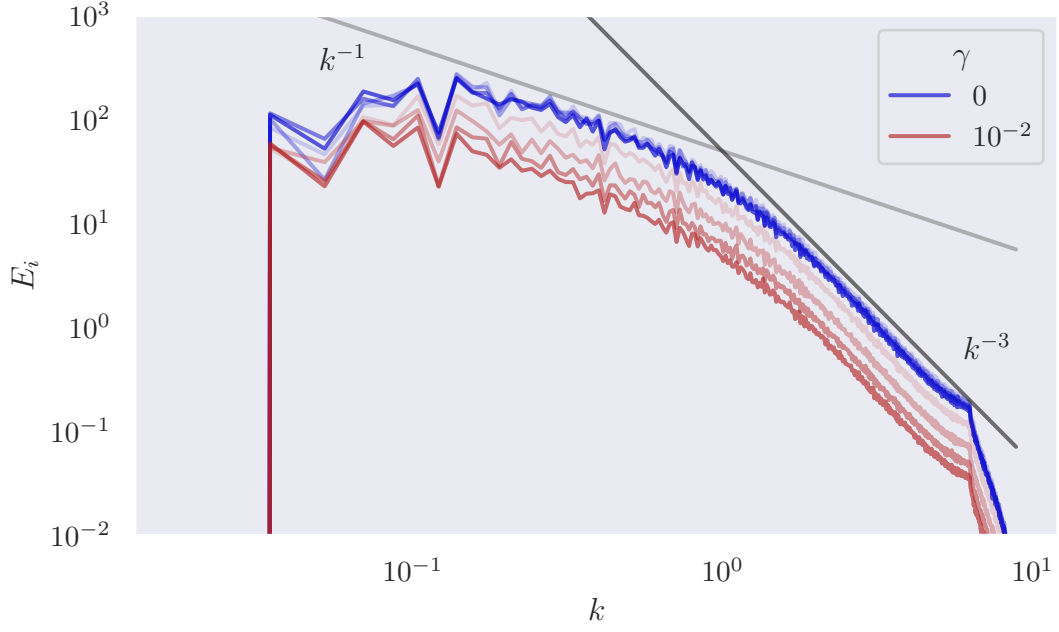


Figure E.6: The spectra of incompressible kinetic energy in uniform condensates. The intensity of colour indicate the time of the spectra. The lowest colour intensity is at $t = 0$, while the highest is at $t = 4000\tau$, with intervals of 1000τ displayed in between. At $\gamma = 0$ there is virtually no change in the spectrum apart from a small increase at low k . There is hardly any transport of energy across length scales apart from a small flux towards large length scale. At $\gamma = 10^{-2}$ there is a decrease at high k while the spectrum at low k is virtually static indicating a transport of energy towards small length scales.

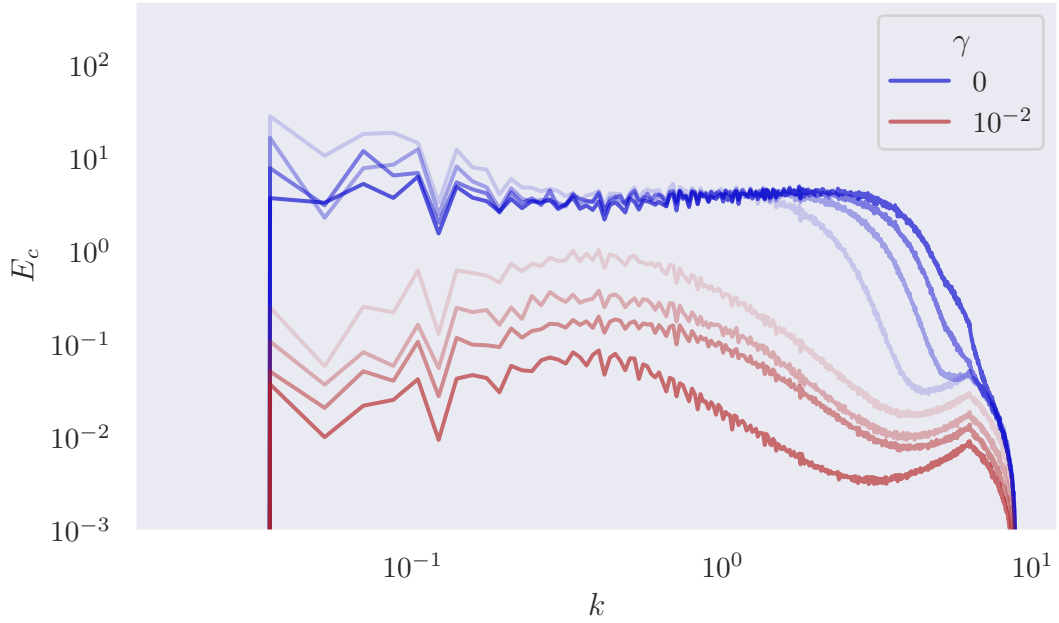


Figure E.7: The spectra of E_c at times $t = 0$ until $t = 4000\tau$ at intervals of 1000τ with increasing colour intensity indicating later times. The spectra are shown for a uniform condensate at two different levels of dissipation. At $\gamma = 0$ we see how the compressible modes approach equipartition of energy across k . The compressible modes at $\gamma = 10^{-2}$ drop, with a peak forming over time. As the waves dissipate quickly in this system we assume that this peak is related to vortex motion.

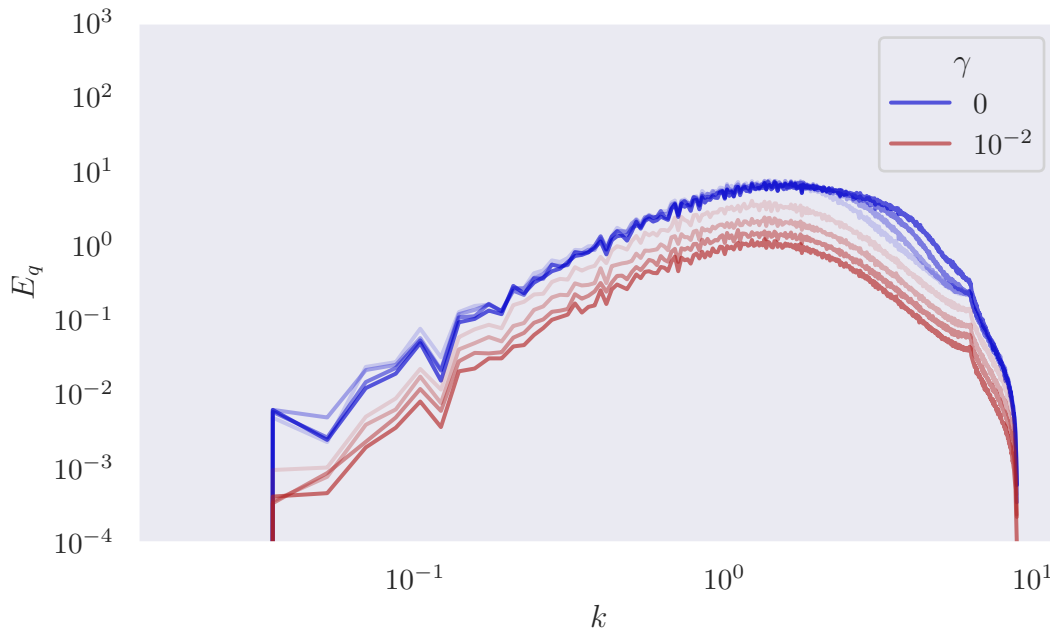


Figure E.8: The spectra of E_q for uniform condensates from $t = 0$ until $t = 4000\tau$ shown in 1000τ increments. The colour intensity increase with increasing time. At $\gamma = 0$ we see a peak at around $1k = / \xi$, this is associated with the density profile of the vortices. Over time the peak broadens. The high k contributions probably come from solitary waves or other kinds of waves with steep gradients which remain in the system over time. For $\gamma = 10^{-2}$ the quantum energy decreases. The drop in E_q is independent of k .