UiO : Department of Informatics
University of Oslo

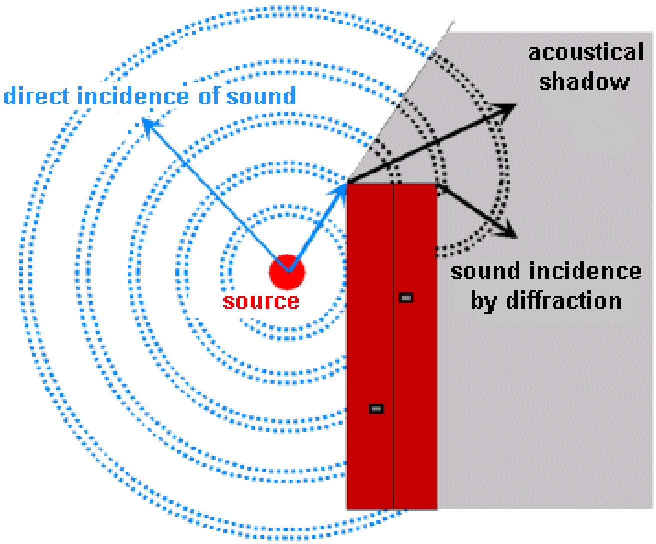
IN5450 Diffraction: from waves to signal processing

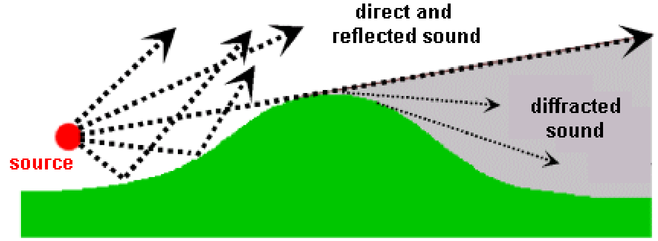
Sverre Holm

Diffraction

- Ray theory: Geometrical model of optics
- High-frequency – small wavelength model
- Diffraction:
  - Wavelength comparable to structure size
  - Edges of shadows are not perfectly sharp
  - Can hear around corners
- This course: only consequences of diffraction
- Johnson & Dudgeon, Array Signal Processing, Concepts and Techniques, ch. 2

Diffraction – (spredning)





Huygens' principle

- Christian Huygens, NL, 1629-1695
- Each point on a travelling wavefront can be considered as a secondary source of spherical radiation
- Also a model for an oscillating piston = acoustic source

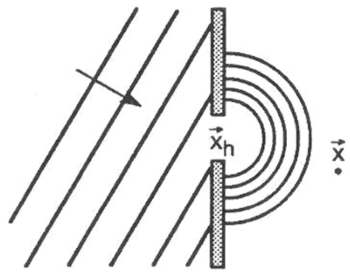


Figure 2.13 A wave is shown impinging on a hole in a planar screen. The Rayleigh-Sommerfeld diffraction formula tells us what the wavefield at the point $\vec{x}$ is in terms of the wavefield at the aperture.

Mathematical physics of diffraction

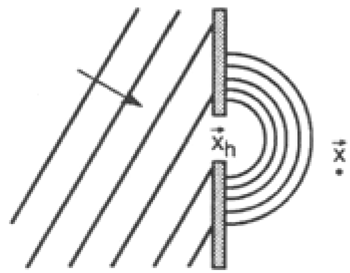
- Augustin Jean Fresnel (F) 1788 – 1827
- Gustav Robert Kirchhoff (D) 1824 – 1887
- Lord Rayleigh, John William Strutt (GB) 1842 – 1919, Nobel prize physics, 1904.
- Arnold Johannes Wilhelm Sommerfeld (D) 1868 – 1951
- Joseph von Fraunhofer (D) 1787 - 1826

Diffraction: deviation from geometrical model

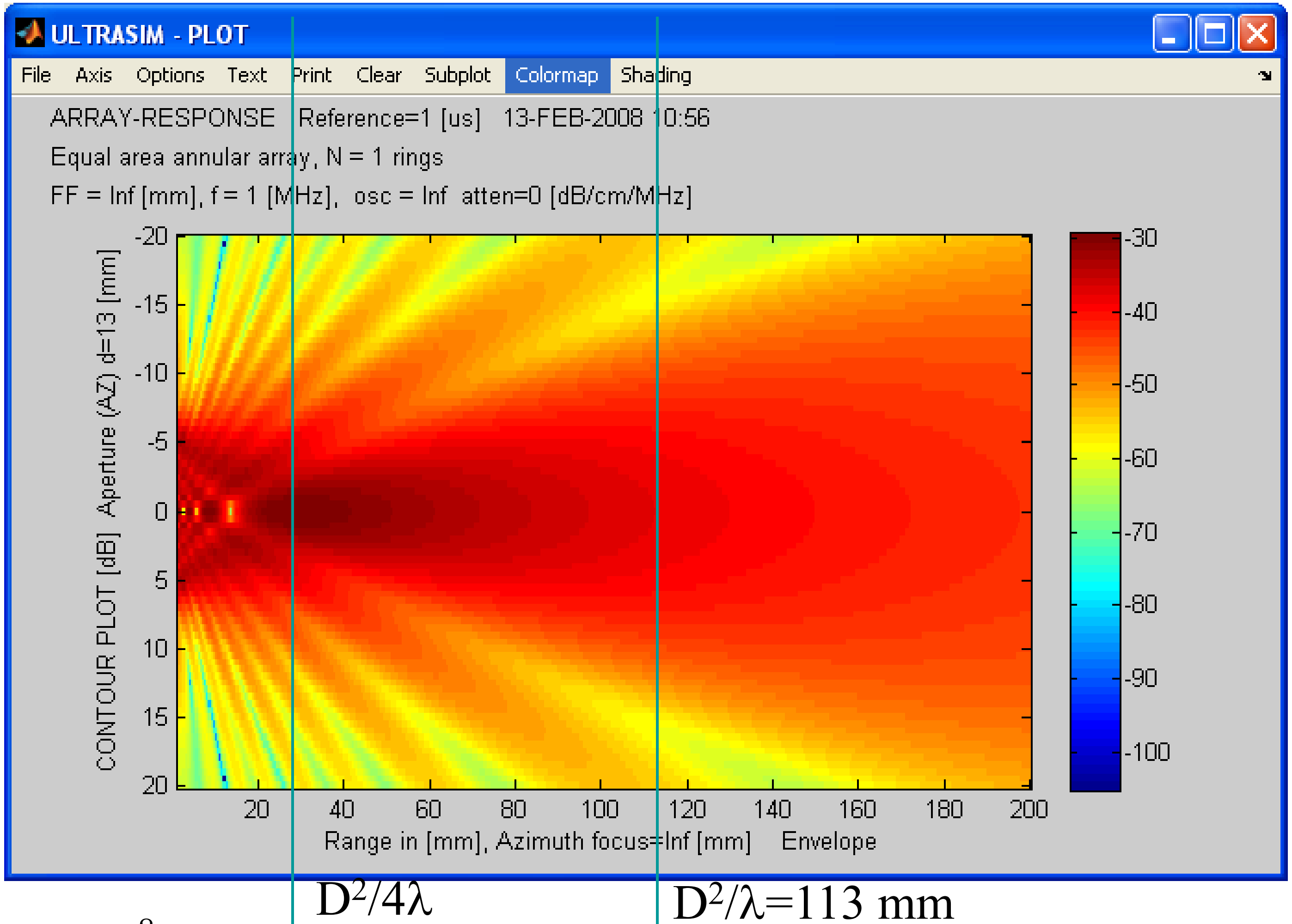
- Rayleigh-Sommerfeld diffraction formula from a hole with aperture A:

$$s(\overrightarrow{x}) = \frac{1}{j\lambda} \int\int_A s(\overrightarrow{x}_h) \frac{\exp\{jkr\}}{r} \cos\theta dA$$

  - Wave at x: superposition of fields from the hole, due to linearity of wave equation
  - Weighted by spherical spreading function e^{jkr}/r
  - Also weighted by $1/\lambda$
  - Obliquity factor $\cos\theta$
  - Phase shift of $\pi/2$ due to $1/j$



1 MHz 13 mm, unfocused xdcr



$D^2/4\lambda$ $D^2/\lambda = 113$ mm

Olympus-Panametrics A303S
(in our lab)

Simulation: http://www.ifi.uio.no/~ultrasim

Two approximations

- Fresnel, nearfield, (but not quite near)
- Fraunhofer, farfield
- Leads to
  - important estimates for nearfield – farfield transition distance
  - Fourier relationship between aperture excitation and field

1. Fresnel approximation

$$s(\overrightarrow{x}) = \frac{1}{j\lambda}\int\int_A s(\overrightarrow{x}_h)\frac{\exp\{jkr\}}{r}\cos\theta dA$$

- cos θ ≈ 1, r ≈ d for amplitude (not for phase!)
- Phase:
  - spherical surfaces ≈ quadratic
  - parabolic approximation

Fresnel derivation

- Point in the hole $(\tilde{x}, \tilde{y}, 0)$, in observation plane **(x,y,z=d)**
- Distance: $r = [(x-\tilde{x})^2 + (y-\tilde{y})^2 + d^2]^{1/2}$

$$r = d[1 + \frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{d^2}]^{1/2}$$

- Approximate $(1+x)^{1/2} \approx 1+x/2$, i.e. small x/d ⇔ small angles

$$r \approx d + \frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{2d}$$

- Use the above expression for the phase and *r=d* for the amplitude in the Rayleigh-Sommerfeld integral

Fresnel approximation

$$s(x,y) = \frac{\text{exp}\,\{jkd\}}{j\lambda d} \int\int_A s(\tilde{x},\tilde{y})\,\text{exp}\,\{\frac{jk[(x-\tilde{x})^2 + (y-\tilde{y})^2]}{2d}\} d\tilde{x} d\tilde{y}$$

- Nearfield approximation & within ≈ 15º of z-axis
- Also called paraxial approximation
- **2D convolution** between field in hole and h(x,y):

$$h(x,y) = \frac{exp\{jkd\}}{j\lambda d}\,\text{exp}\,\{\frac{jk(x^2+y^2)}{2d}\}$$

- This is a quadratic phase function = the phase shift that a secondary wave encounters during propagation

2. Fraunhofer approximation

- Expand phase term of Fresnel approximation

$$s(x,y) = \frac{\exp\{jkd\}}{j\lambda d} \int\int_A s(\tilde{x},\tilde{y}) \exp\{\frac{jk[(x-\tilde{x})^2 + (y-\tilde{y})^2]}{2d}\} d\tilde{x} d\tilde{y}$$

- and neglect quadratic phase term variation over hole

$$(x-\tilde{x})^2 + (y-\tilde{y})^2 = x^2 + y^2 - 2x\tilde{x} - 2y\tilde{y} + \tilde{x}^2 + \tilde{y}^2 \approx x^2 + y^2 - 2x\tilde{x} - 2y\tilde{y}$$

- If D = max linear dimension of hole, this is equivalent to assuming (d=dist. from source):

$$\frac{\tilde{x}^2}{2d} \leq \frac{(D/2)^2}{2d} << \lambda/2 => d >> \frac{D^2}{4\lambda}$$ Fresnel limit

Fraunhofer approximation

$$s(x,y) = \frac{\exp\{jkd\}}{j\lambda d} \exp\{\frac{jk(x^2+y^2)}{2d}\} \int\int_A s(\tilde{x},\tilde{y}) \exp\{\frac{jk(x\tilde{x}+y\tilde{y})}{d}\} d\tilde{x} d\tilde{y}$$

- Far-field approximation: valid far away from hole
- s(x,y) = **2D Fourier transform** of field in hole

Linking physics & signal processing

- Fourier transform relationship is a very important result
- *Link* between the physics and the signal processing!
- Basis for simplified expressions like angular resolution ≈ λ/D etc
- Small hole leads to wide beam and vice versa just like a short time-function has a wide spectrum etc

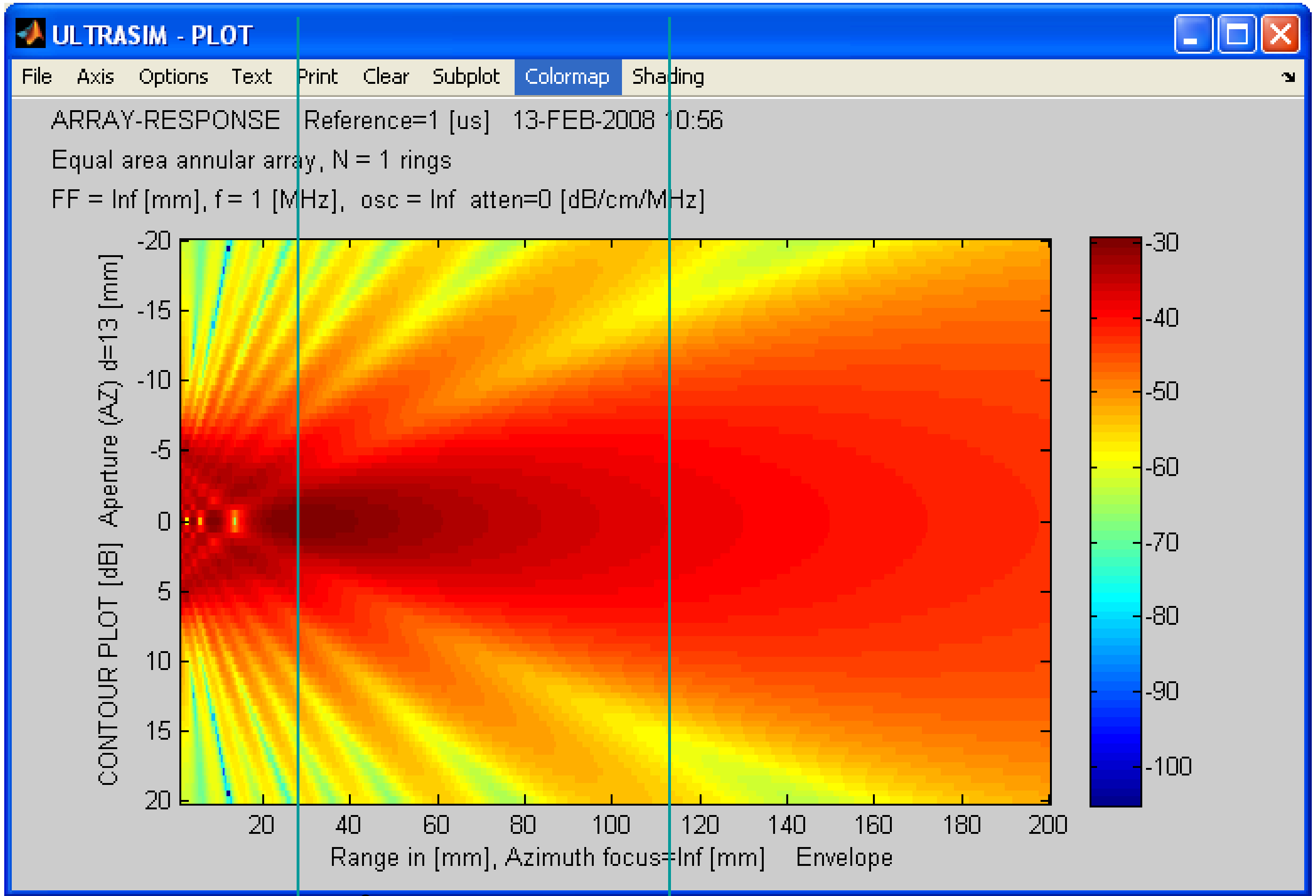
Nearfield-farfield limit

Not a clear transition, several limits are used, in increasing size:

- $d_F = D^2/4\lambda$: Fresnel limit
- $d = \pi r^2/\lambda = \pi D^2/4\lambda$: Diffraction limit
- $d = D^2/\lambda$: max path length difference $\lambda/8$
- $d_R = 2D^2/\lambda$: Rayleigh dist: Δ path = $\lambda/16$

- Proportional to D^2/λ, multiplied by 0.25, 0.79, 1, or 2

1 MHz 13 mm, unfocused xdcr



$D^2/4\lambda$ $D^2/\lambda = 113$ mm

Olympus-Panametrics A303S
(in our lab)

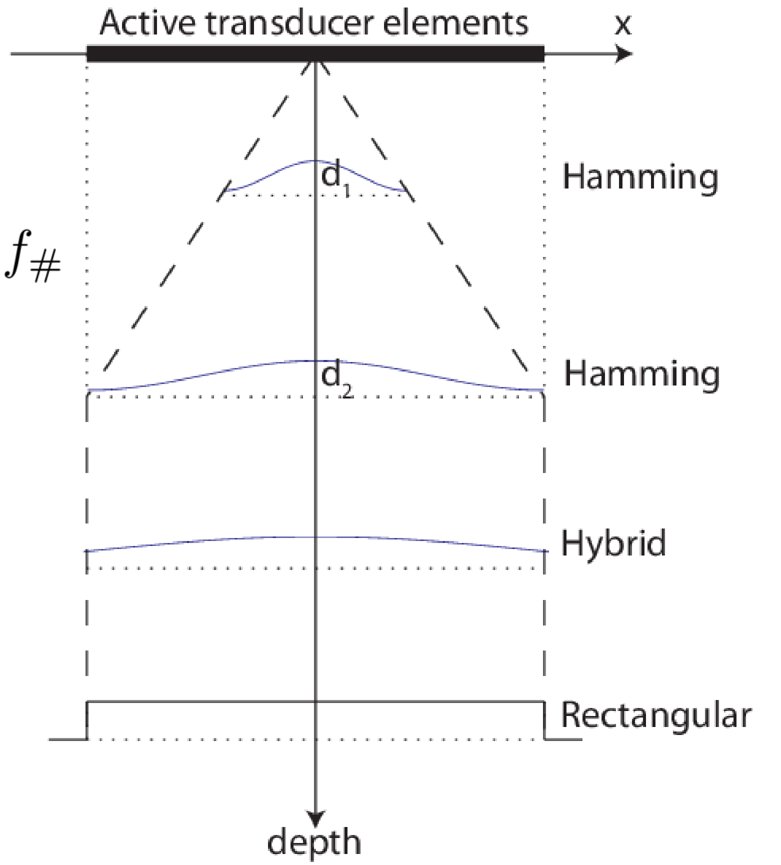
Simulation:
http://www.ifi.uio.no/~ultrasim

Expanding aperture

- f-number = distance/aperture:

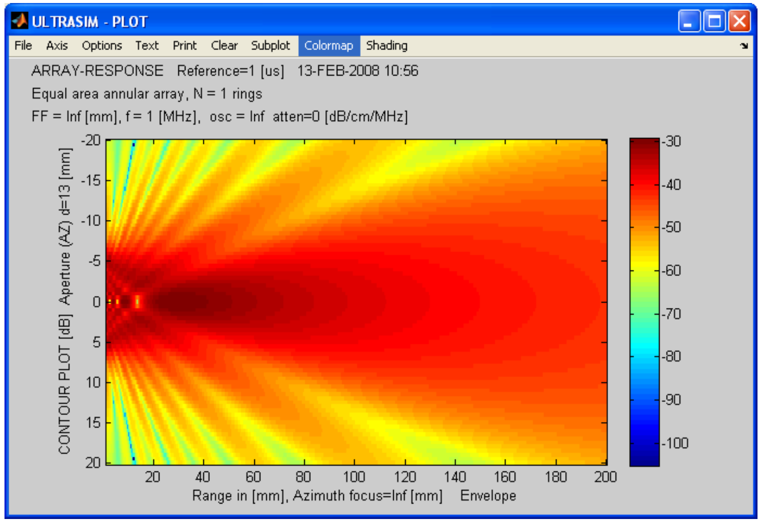
$$f_{\#} = d/D \Leftrightarrow \tan\theta/2 = d/(D/2) = 0.5/f_{\#}$$

- Fresnel theory: <15 deg ⇔ f# > 3.8
- Rule-of-thumb f# > 1-2 ⇔ 53…28 deg
- Ultrasound imaging



Field vs distance

1. Near: Collimated beam
   - Beamwidth ♡ source width
   - Laser, x-ray
2. Mid: Near/far transition
3. Far: Farfield
   - Fourier relationship
   - Constant angle
   - Sonar, radar, ...



Near | Far

Example of collimated beam: Laser pointer

- Aperture: D = 1mm(?), could be 2
- Wavelength (red): λ = 650 nm
- Near/far-transition:
  - $D^2/4\lambda$ = $(1 \ldots 2e\text{-}3)^2/650e\text{-}9$ ♡ 1.5 ... 6 m
  - D^2/λ ♡ 6 ... 24 m

Array Processing Implications

- Diffraction means that opaque objects located between the source and the array can induce complicated wavefields
  - Scattering theory:
    - Acoustics: Schools of fish
    - Electromagnetics: rain drops
    - Complicated, but important to understand

Norsk terminologi

- Bølgeligningen
- Planbølger, sfæriske bølger
- Propagerende bølger, bølgetall
- Sinking/sakking: $\overrightarrow{\alpha}$
- Dispersjon
- Attenuasjon eller demping
- Refraksjon
- Ikke-linearitet
- Diffraksjon; nærfelt, fjernfelt
- Gruppeantenne (= array)

Kilde: Bl.a. J. M. Hovem: ``Marin akustikk'', NTNU, 1999