Differential arrays –
from cardioid microphones to Yagi antennas

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Seemingly dissimilar applications

- End-fire arrays
- Echo sounder: Split beam for direction finding
- Microphones
- Yagi antennas
N-element end-fire vs broadside array

• Broad-side:
  – Element distance: $\sim \lambda/2$
  – Array gain: $N$
  – Beamwidth inverse proportional with frequency

• End-fire:
  – Element distance: $<< \lambda/2$
  – Array gain: $N^2$ (theoretical maximum)
    • Super-directive or supergain
  – Almost frequency-independent beam pattern
Literature

  - Sect. 9.4.2 Differential Sensor Array
Two-element array

• Conventional:
  – Sum: FIR Low-pass beamforming: **steers peak**

• Differential, end-fire
  – Simplest case: $\tau=0$, $w_0=w_1$
  – Difference: FIR High-pass beamforming: **steers null**

Uncini, 2015, Fig. 9.24
\[ R(\omega, \theta) = 1 - \omega e^{-j\omega(\tau + \tau_d \cos \theta)}, \quad \tau_d = d/c \]

Weight will be used to model propagation effects due to spherical spreading, 1/r-effect, in the near-field.
Beampattern, effect of delay at $f = f_c$, $w = 1$

Sharp null, good for direction finding

Dipole (figure eight), $\tau/\tau_d = 0$,

Hypercardioid, $\tau/\tau_d = 1/3$,

Cardioid, $\tau/\tau_d = 1$,

Almost omni, $\tau/\tau_d = 3$

[Most common mic]

Split beam echo sounder --- Microphone / Yagi antenna ---
Cardioid, $\tau = \tau_d$:
Dependency of angle of incidence

Uncini, Fig 9.29a, d=2.5 cm. Max gain = $10\log 2^2 = 6$ dB

$$R(\omega, \theta) = 1 - w e^{-j\omega(\tau + \tau_d \cos \theta)}, \quad \tau_d = d/c$$
Cardioid, \( \tau = \tau_d \): Dependency of element distance

Uncini, Fig 9.29b

\[
R(\omega, \theta) = 1 - w e^{-j\omega(\tau + \tau_d \cos \theta)}, \quad \tau_d = d/c
\]
Dependency of different patterns

Zeroes, maxima

Uncini, Fig 9.29a, d=2.5 cm

\[ R(\omega, \theta) = 1 - \omega e^{-j\omega(\tau + \tau_d \cos \theta)}, \quad \tau = 0, \quad 0.33d/c, \quad d/c \]

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Cut-off frequency and angularity

• Zeroes of $R(\omega, \theta)$: $\omega(\tau + \tau_d \cos \theta) = 0$, $2\pi$

• Maximum of $R(\omega, \theta)$ gives cut-off frequency:

$$\omega_c(\tau + \tau_d \cos \theta) = \pi \quad \Rightarrow \quad \omega_c = \pi / (\tau + \tau_d) \quad [\theta = 0]$$

• Well below cut-off, phase is small:

$$R(\omega, \theta) = 1 - e^{-j\omega(\tau + \tau_d \cos \theta)} \approx j\omega(\tau + \tau_d \cos \theta)$$

  – Gain proportional with frequency: compensated
  – Same angle dependency for all frequencies
Cut-off frequency

\[ f_c = \frac{\omega_c}{2\pi} = \frac{0.5}{(\tau + \tau_d)} = \frac{0.5}{(\tau + \frac{d}{c})} \]

- Cardioid: \( f_c = \frac{c}{4d} \)

- Uncini examples, \( d=2.5 \) cm: \( f_c = 3.4 \) kHz

- Next examples, \( d=8.5 \) mm: \( f_c = 10 \) kHz
Relatively insensitive to frequency: Cardioid below $f_c$, breaks up above $f_c$
Nothing like the large frequency variation for the Uniform Linear Array

$$\theta_{BW} \approx \frac{\lambda}{D} = \frac{c}{Df}$$

- Seven-element uniformly spaced array, \( d=8 \), unsteered
- \( d = \lambda @ f=4250 \text{ Hz} \) (\( c=340 \text{ m/s} \))

Elko and Meyer. «Microphone arrays» 2008
Pressure gradient microphone: compensated for 6 dB/octave

Compensated by the mass of the diaphragm

d = 8.5mm, \( \tau / \tau_d = 1, \theta = 0 \)
Difference array: Quite sensitive to parameter variations

- Broadband array needs an equalizer to boost low frequencies ⇐ large sensitivity to low-frequency self-noise
- Element distance, $d \ll \lambda$
  - But not too small, otherwise sensitivity to noise increases
- Higher order differential arrays are even more sensitive
Figure-of-eight microphone

If the mic diaphragm is open to the air on one side but closed at the other, it is considered to be pressure-operated: although it reacts to air pressure, it is not sensitive to direction, resulting in an omnidirectional mic pattern.

Where the diaphragm is open on both sides, as in this diagram, it responds to the pressure-gradient (the difference between the pressure at the front and the back of the diaphragm).

In this case, sound from the side results in even pressure on both sides of the diaphragm, which is why figure-of-eight mics reject sound from the side, but are responsive to both the front and rear.

https://www.soundonsound.com/techniques/using-microphone-polar-patterns-effectively
Cardioid microphone

Most cardioid mics now incorporate a vented 'labyrinth' in a single-capsule design that manipulates the phase of sounds hitting the rear, to produce the desired cardioid pattern. The supercardioid and hypercardioid designs use the same principle to create a more focused pattern to the front, at the expense of reducing the rear rejection.

If you notice vents at the side of the mic head, the mic probably has a cardioid pattern (or a variation on it).

https://www.soundonsound.com/techniques/using-microphone-polar-patterns-effectively
Proximity effect of cardioid: bass boost when close-in

Difference due to $1/r$:

- $d=10$ cm:
  - 1. element: 10 cm
  - 2. el.: 10+0.85 cm
  - Effective $w = \frac{10}{10.85} = 0.92$

- $d=2.5$ cm:
  - $w = \frac{10}{12.5} = 0.8$

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Exact differential array

- Plane wave (far-field) pressure field:
  \[ p(r, t) = A_0 e^{i(\omega_0 t - kr \cos \theta)} \]

- Spatial derivative (dropping time):
  \[ \frac{d}{dr} p(k, r) = jk \cos \theta A_0 e^{-jkr \cos \theta} \]

- Beam pattern shape
  \[ \propto k \cos \theta = \frac{\omega}{c} \cos \theta \]
  
  – ~no-delay difference array (\(\tau=0\))
Differential array (cont)

- \( jk \cos \theta = \frac{j\omega}{c} \cos \theta \)
  - high-pass 6 dB/octave

- Pressure gradient from conservation of linear momentum (Euler):
  \[
  \rho_0 \frac{\partial v}{\partial t} = -\nabla p \Rightarrow \nabla p \propto j\omega|v|
  \]

- Therefore called pressure gradient or velocity microphone
Far-field: n’th order differential array

\[
\frac{d^n}{dr^n} \rho(k, r) = A_0(jk \cos \theta)^n e^{-jr \cos \theta}
\]

- Beampattern \( \propto \cos^n \theta \)
- Frequency response: \( \propto \omega^n \): 6n dB/octave
Near-field

Pressure: \[ p(r, t) = A_0 e^{i(\omega_0 t)} \frac{e^{-jk_0 r \cos \theta}}{r} \]

\[ \frac{d^n}{dr^n} p(k, r, \theta) = A_0 \frac{n!}{r^{n+1}} e^{-jkr \cos \theta} (-1)^n \sum_{m=0}^{n} \frac{(jkr \cos \theta)^m}{m!} \]

- Sum of dipole-like terms of type \( \cos^m \theta \)
- May optimize coefficients for desirable properties
- Compare to differential array, \( n=1 \):

\[ R(\omega, \theta) = 1 - e^{-j\omega (\tau + \tau_d \cos \theta)} \approx j\omega (\tau + \tau_d \cos \theta) \]
Differential array in practice

- Approx. by finite differences, d << \( \lambda \).
- Response: \( \propto \omega^n (a_0 + a_1 \cos \theta + a_2 \cos^2 \theta + \ldots + a_n \cos^n \theta) \)
- \( n = 1 \):
  - Figure-of-8: \( a_0 = 0, \quad a_1 = 1 \)
  - Hypercardioid: \( a_0 = 1/4, \quad a_1 = 3/4 \)
  - Cardioid: \( a_0 = 1/2, \quad a_1 = 1/2 \)
First, second order differential array

0: \(m_0\)
1: \(m_0 - m_{1,1}\)
2: \((m_0 - m_{1,1}) - (m_1 - m_{2,1})_2\)

Uncini, 2015, Fig. 9.24
Maximum directional gain, n=1…4

n=1: hypercardioid

n=2: Narrower beam than that of \( \cos^2 \)

Elko and Meyer. «Microphone arrays» 2008
Performance metrics:
Beamwidth, Gain, Front/Back

More gain less beamwidth

More beamwidth less gain

Front to Back Ratio

https://www.everythingrf.com/community/what-is-front-to-back-ratio-in-an-antenna
Optimization, first order

Maximum gain
- $n=1$: hypercardioid
- Array gain: $20 \log(n+1) = 10 \log N^2$
  - $N$ is no of elements

Best front-back ratio
- $n=1$: supercardioid
  $$E_{SC_1}(\theta) = \frac{\sqrt{3} - 1 + (3 - \sqrt{3}) \cos \theta}{2}.$$ 

Elko, “Differential microphone arrays”, 2004
Hyper- vs super-cardioid
Max gain vs best front/back-ratio

Theory: 0.366, 0.634

Quaranta, Dimino, D’altrui, General guidelines for acoustic antenna designed for beamforming noise source localization, 2007
Maximum theoretical array gain, end-fire vs uniform linear array

Array gain of ULA (lower) and differential array

\[ AG = 10 \log N^2, \quad BW \propto \frac{1}{\sqrt{N}} \]

\[ AG = 10 \log N, \quad BW \propto \frac{\lambda}{D} = \frac{2}{N} \]
Yagi-Uda antenna: Parasitic elements

• Single active element
  - Length $\lambda/2$ => narrowband

• Passive elements:
  - Reflector
    • typ 5% longer: Capacitive reactance - voltage phase lags that of the current
  - Directors:
    • typ 5% shorter: Inductive reactance - current phase lags phase of the voltage
    • Ex. 3 or 17 directors

why only 1 reflector?
Phasing animation (from Wikipedia)

- Time delays due to element distance
- Reradiation from passive elements (parasitic)
  - Field behind first reflector is nearly canceled
- Inherently narrow-band

Uda, S., 1925, "On the Wireless Beam of Short Electric Waves". Journ. Institute of Electrical Engineers of Japan

Physics department, UiO

• CubeSTAR:
  – 437.465 MHz, \(\lambda=0.686\) m
  – 432-438 MHz amateur band

• Circularly polarized
  – 4 x 436CP30, each with:
  – 2 x (13 directors+1 reflector+1 driven element) = 30 elements

• 4 stacked together
  – 1.143 m=1.67 \(\lambda\), gain=20.5 dB,
    -3 dB beamwidth=16 deg

Eirik Vikan, UiO Satellite Ground Station: Simulation, Implementation and Verification, MSc, 2011

https://www.duo.uio.no/bitstream/handle/10852/11067/Eirik_Vikan_UiO_Satellite_Ground_Station_Simulation_Implementation_and_Verification.pdf
https://www.m2inc.com/FG436CP30
Broadband? Log periodic array

- VHF/UHF, 50-1300 MHz
- 21 elements, 2 m boom
- Forward gain: 10 to 12 dBi (rel isotropic)
  - Like a 4-5-element Yagi
- Complex as several or all elements are active

- Create CLP-5130-1N
Broadband? Adjustable element lengths

Multi-frequency

3-element adjustable Yagi, Russian Antarctica Base: RI1ANR 14-52 MHz
Differential arrays

- Array gain up to $N^2$ vs $N$ for ULA
- Frequency-independent beampattern
- Sensitive designs
  - commonly $N=2$

- Microphone: proximity effect
- Yagi antenna: narrowband