From Diffraction to the Fourier Transform

version 1.2

Sverre Holm
University of Oslo

07 February 2020

Contents

Chapter 1

Introduction

These notes are meant to complement chapter 2 “Signals in space and time” of [Johnson and Dudgeon, 1992]. The relation to Johnson & Dudgeon: Array Signal Processing, Concepts & Techniques is as follows:

- Chapter 2 on dispersion complements section 2.3.1, Dispersion, of [Johnson and Dudgeon, 1992].

- Chapter 3 on diffraction complements section 2.4.2, Diffraction, of [Johnson and Dudgeon, 1992]

1.1 Acknowledgement

This document has been typeset using LaTeX and the *BYUTextbook.cls* class file (modified for A4 paper size) used in the online textbook [Peatross and Ware, 2015] from the Optics Education Group, Department of Physics and Astronomy, Brigham Young University.

Chapter 2

Dispersion

2.1 Intrinsic and extrinsic dispersion

Intrinsic dispersion is a material property. Glass is a good example as the index of refraction varies with wavelength. In a glass prism, the dispersion in combination with refraction leads to the spreading of white light into its components from red to violet as shown in Fig. (2.1).

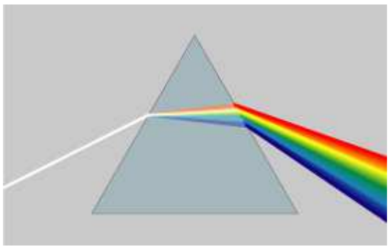

Figure 2.1 Dispersion in combination with refraction in a glass prism [By User: Joanjoc - CC BY-SA 3.0], via Wikipedia Commons.

Ocean waves are also intrincially dispersive as

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh(2\pi\frac{H}{\lambda})} \approx \sqrt{gH}, \qquad (2.1)$$

where g is the acceleration due to gravity, approximately 9.8 m/s^2, and H is the water depth. Thus large wavelengths travel faster than shorter ones except when the approximation holds, i.e. for shallow water.

Extrinsic dispersion occurs when it is the geometric boundary conditions of the propagation medium rather than the material which determines the dispersion. The prime example is a waveguide.

Chapter 3

From diffraction to the Fourier transform

3.1 Huygen's principle

Huygen's principle says that each point on a travelling wavefront can be considered as a secondary source of spherical radiation. Each such source will therefore spread according to the spatial part of the equation for spherical spreading:

$$s(r,t) = \frac{A}{r}\exp\{i(\omega t - kr)\}. \tag{3.1}$$

Adding up contributions, $s(\overrightarrow{x}_h)$, over an aperture, A, then results in:

$$s(\overrightarrow{x}) \propto \int\int_A s(\overrightarrow{x}_h)\frac{\exp\{ikr\}}{r}dA. \tag{3.2}$$

This principle is due to Christian Huygens, (1629-1695, Netherlands). The principle also expresses how an acoustic source like an oscillating piston is formed.

Other contributors to this field, which originally was concerned with optics, were:

- Joseph von Fraunhofer (D) 1787 - 1826
- Augustin Jean Fresnel (F) 1788 - 1827
- Gustav Robert Kirchhoff (D) 1824 –1887
- Lord Rayleigh, John William Strutt (GB) 1842 - 1919, Nobel prize physics, 1904.
- Arnold Johannes Wilhelm Sommerfeld (D) 1868 - 1951

3.2 Rayleigh–Sommerfeld diffraction formula

Eq. (3.2) captures the most essential part of diffraction, but the accurate formulation for the field from an aperture A is expressed in the Rayleigh-Sommerfeld diffraction formula:

$$s(\overrightarrow{x}) = \frac{1}{j\lambda}\int\int_A s(\overrightarrow{x}_h)\frac{\exp\{jkr\}}{r}\cos\theta dA. \tag{3.3}$$

It says that the wave at position $\overrightarrow{x}$ is a superposition of fields from the hole, due to the linearity of the wave equation as expressed in Huygen's principle. Each contribution is weighted by a spherical spreading function $\exp jkr/r$. There is also weighting by $1/\lambda$. In addition there is an obliquity factor $\cos\theta$ and finally a phase shift of $\pi/2$ due to the factor $1/j$.

There are two important approximations to the Rayleigh-Sommerfeld formula, the Fresnel approximation which is for the nearfield and small angles, and the Fraunhofer approximation which is valid in the farfield. The derivation leads to important estimates for the nearfield – farfield transition distance. It also leads to the important result that there is a Fourier relationship between the aperture excitation and the field in the farfield.

3.3 The Fresnel approximation

Eq. (3.3) is approximated for small angles by letting $\cos\theta \approx 1$ and $r \approx d$. This is substituted for the amplitude factor, but not in the phase of the complex exponential.

For the phase the spherical surfaces are instead approximated by a quadratic function, therefore this is called a parabolic approximation:

$$r = [(x-\tilde{x})^2 + (y-\tilde{y})^2 + d^2]^{1/2} = d[1 + \frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{d^2}]^{1/2} \tag{3.4}$$
$$\approx d + \frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{2d}.$$

This leads to:

$$s(x,y) \approx \frac{\exp\{jkd\}}{j\lambda d} \cdot \int\int_A s(\tilde{x},\tilde{y}) \exp\{\frac{jk[(x-\tilde{x})^2 + (y-\tilde{y})^2]}{2d}\} d\tilde{x} d\tilde{y}. \tag{3.5}$$

The result is a nearfield approximation which is fine within $\approx 15°$ of the axis perpendicular to the aperture, the z-axis. It is also called the paraxial approximation. It can only be used when the distance to position $\overrightarrow{x}$ doesn't vary too much over the aperture.

The Fresnel approximation expresses a 2D convolution between the field in the field in the original aperture and and transfer function $h(x,y)$:

$$h(x,y) = \frac{exp\{jkd\}}{j\lambda d} \exp\{\frac{jk(x^2+y^2)}{2d}\} \tag{3.6}$$

This is a quadratic phase function which is the phase shift that a secondary wave encounters during propagation

3.4 Fraunhofer approximation

Here the phase term of the Fresnel approximation in Eq. (3.5) is expanded and the quadratic phase term variation over the aperture is neglected:

$$(x-\tilde{x})^2 + (y-\tilde{y})^2 = x^2 + y^2 - 2x\tilde{x} - 2y\tilde{y} + \tilde{x}^2 + \tilde{y}^2 \approx x^2 + y^2 - 2x\tilde{x} - 2y\tilde{y}. \tag{3.7}$$

If D is the maximum linear dimension of the aperture and d is the distance from source, then this is equivalent to assuming:

$$\frac{\tilde{x}^2}{2d} \leq \frac{(D/2)^2}{2d} << \lambda/2 => d >> \frac{D^2}{4\lambda}. \tag{3.8}$$

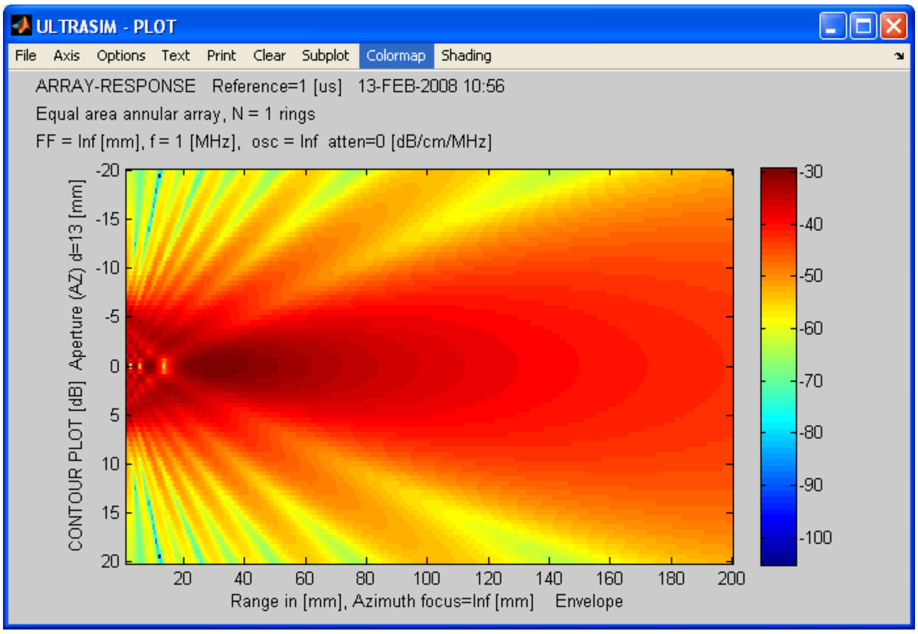

Figure 3.1 Acoustic field from a 1 MHz D =13 mm aperture unfocused transducer. Observe how the field starts transitioning to one with a uniform angle at depth $D^2/(4\lambda) \approx 28$ mm and that the transition is more or less complete at $\pi D^2/(4\lambda) \approx 88$ mm. Simulated with Ultrasim [Holm, 2001]

This is the definition of the Fresnel limit.

The Fraunhofer approximation is the result of inserting Eq. (3.7) in Eq. (3.5):

$$s(x,y) \approx \frac{\exp\{jkd\}}{j\lambda d} \exp\{\frac{jk(x^2+y^2)}{2d}\} \cdot \int\int_A s(\tilde{x},\tilde{y}) \exp\{\frac{jk(x\tilde{x}+y\tilde{y})}{d}\} d\tilde{x} d\tilde{y}. \tag{3.9}$$

The result is a far-field approximation which is valid far away from aperture. It should now be evident that $s(x,y)$ is the 2D Fourier transform of field in hole, $s(\tilde{x},\tilde{y})$.

This result links the physics and the signal processing via the important Fourier transform. It is also the basis for a simplified expression like angular resolution $\theta \approx \lambda/D$, which says that a small aperture leads to wide beam and vice versa just like a short time-function has a wide spectrum in temporal signal processing.

3.5 Practical consequences

3.5.1 Nearfield–farfield limit

The transition from the near-field to the farfield is gradual in practice and there is no clear point where it takes place. Therefore there exists several limits which are used. Here they are listed according to increasing distance from the aperture or source:

- $d_F = D^2/(4\lambda)$: Fresnel limit

- $d = \pi r^2/\lambda = \pi/4 \cdot D^2/\lambda$: Diffraction limit
- $d_R = 2D^2/\lambda$: Rayleigh distance, maximum path length difference $\lambda/16$

In all cases, the transition point distance is proportional to D^2/λ. The difference between the various criteria is the multiplication factor, which can be 0.25, 0.79, or 2. This is illustrated in Fig. 3.1 which is a simulation of one of the 1 MHz ultrasound probes in our lab.

☞ Rule-of-thumb

A simple interpretation and rule-of-thumb is that whenever the aperture tries to resolve objects that are smaller than the size of the aperture itself, the imaging takes place in the near field. This principle can be formulated by starting with the resolution of an aperture at a distance d_e which is θd_e, where the angular resolution is $\theta \approx \lambda/D$. Setting this equal to the size of the aperture gives:

$$D = \theta d = d_e \lambda/D \Rightarrow d_e = D^2/\lambda \qquad (3.10)$$

3.5.2 Hard and soft baffle in acoustics

The validity of the Rayleigh Sommerfeld diffraction formula of Eq. (3.3) was tested against measurements for common ultrasound transducer elements in [Selfridge et al., 1980]. It was found that the $\cos\theta$ term indeed was important. The result was that the common formula for the farfield pressure radiation pattern of a strip element of length d must be multiplied by the same factor:

$$p = p_0 \frac{\sin(\pi d/\lambda \sin\theta)}{\pi d/\lambda \sin\theta} X(\theta) \qquad (3.11)$$

where the obliquity factor is $X(\theta) = \cos\theta$.

This is actually the soft baffle case in acoustics, where the surroundings of the acoustic element is such that the pressure is zero. The other case is a hard baffle, where the boundary condition outside the source has zero particle velocity, and then the obliquity factor is just $X(\theta) = 1$. In [Szabo, 2014] (chapter 7) it is argued that in some cases even an average of the two, $X(\theta) = (1 + \cos\theta)/2$ describes realistic transducer elements.

3.5.3 Lower limit for range

In ultrasound imaging, one way to take the consequence of the small angle limit of the Fresnel approximation and the resulting approximation $r \approx d$, is by the ratio of the distance and the aperture $f_\# = d/D$. This is called the f-number. The angle, θ, will be given by $\tan\theta/2 = d/(D/2) = 0.5/f_\#$. The rule-of-thumb is that the f-number should not be lower than 1 to 2, otherwise the image quality may actually deteriorate with a larger aperture. This corresponds to angles $\theta = 28\ldots53°$ degrees which is a bit higher than that corresponding to the angle in the Fresnel theory above of 15°. That angle would limit the f-number to 3.8.

This limit is also justified by the mainlobe width of the resulting beam. It can for instance be found from Eq. (3.11) by finding the peak to zero distance which is $\sin\theta = \lambda/d$. Therefore the resolution at a distance D is $D\sin\theta \approx \lambda f_\#$. As $f_\#$ approaches 1, the resolution approaches λ and the geometric wave propagation theory underlying eq. (3.11) breaks down.

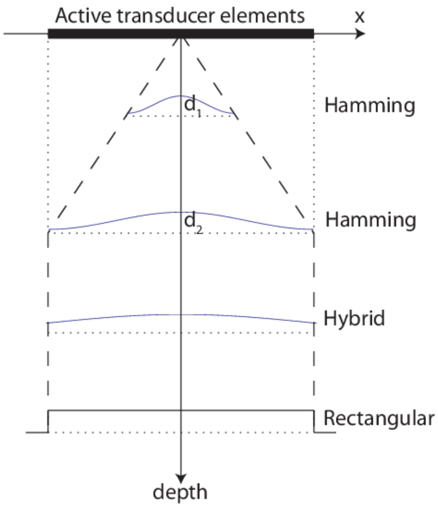

Figure 3.2 Dynamic aperture and dynamic weighting as typically used in medical ultrasound

This is the idea behind the concept of the expanding aperture often used in ultrasound imaging as shown in Fig. 3.2. It usually expands linearly with range or depth, maintaining a constant $f_{\#}$, out to the range where there the full aperture is taken into use and from then on the full aperture is used.

Appendices

Appendix A

Approximations

A.1 Power series approximation

One version of Newton's generalized binomial theorem is [Rottmann, 2003]:

$$(1+x)^{n/m} = 1 + \frac{n}{m}x - \frac{n(m-n)}{2!m^2}x^2 + \frac{n(m-n)(2m-n)}{3!m^3}x^3 + \cdots \tag{A.1}$$

Often used approximations are based on keeping only the first two or three terms and are valid when $x \ll 1$:

$$\frac{1}{1+x} = (1+x)^{-1} \approx 1 - x + x^2 + \cdots \tag{A.2}$$

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots \tag{A.3}$$

$$1/\sqrt{1+x} = (1+x)^{-1/2} \approx 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots \tag{A.4}$$

A.2 McLaurin series for trigonometric functions

The argument is always expressed in radians in these formulas:

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots \tag{A.5}$$

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots \tag{A.6}$$

$$\tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 - \cdots \tag{A.7}$$

The small angle approximations use only a single term:

$$\sin\theta \approx \tan\theta \approx \theta \tag{A.8}$$

$$\cos\theta \approx 1 \tag{A.9}$$

When the argument $\theta < 0.2$ radians $\approx 11.50°$ the error in $\sin\theta$ is less than 0.7 %, the error in $\tan\theta$ is less than 1.4 %, and the error in $\cos\theta$ is less than 2 %. In practice the approximate formulas are useful up to approximately 0.25 radians or 15°.

Appendix B

Norwegian terminology

- Bølgeligningen
- Planbølger, sfæriske bølger
- Propagerende bølger, bølgetall
- Sinking/sakking: $\vec{\alpha}$
- Dispersjon
- Attenuasjon eller demping
- Refraksjon
- Ikke-linearitet
- Diffraksjon; nærfelt, fjernfelt
- Gruppeantenne (= array)

Kilde: Bl.a. [Hovem, 1999].

Bibliography

[Holm, 2001] Holm, S. (2001). Ultrasim-a toolbox for ultrasound field simulation. In *Proc. Nordic Matlab Conference, Oslo, Norway*.

[Hovem, 1999] Hovem, J. M. (1999). *Marin akustikk*. NTNU, Trondheim, Norway.

[Johnson and Dudgeon, 1992] Johnson, D. H. and Dudgeon, D. E. (1992). *Array signal processing: Concepts and techniques*. Simon & Schuster.

[Peatross and Ware, 2015] Peatross, J. and Ware, M. (2015). *Physics of Light and Optics*. available at optics.byu.edu.

[Rottmann, 2003] Rottmann, K. (2003). *Matematisk formelsamling (Mathematical formulas)*. Spektrum forlag.

[Selfridge et al., 1980] Selfridge, A., Kino, G., and Khuri-Yakub, B. (1980). A theory for the radiation pattern of a narrow-strip acoustic transducer. *Appl. Phys. Lett.*, 37(1):35–36.

[Szabo, 2014] Szabo, T. L. (2014). *Diagnostic Ultrasound Imaging: Inside Out, Second Ed.* Academic Press.