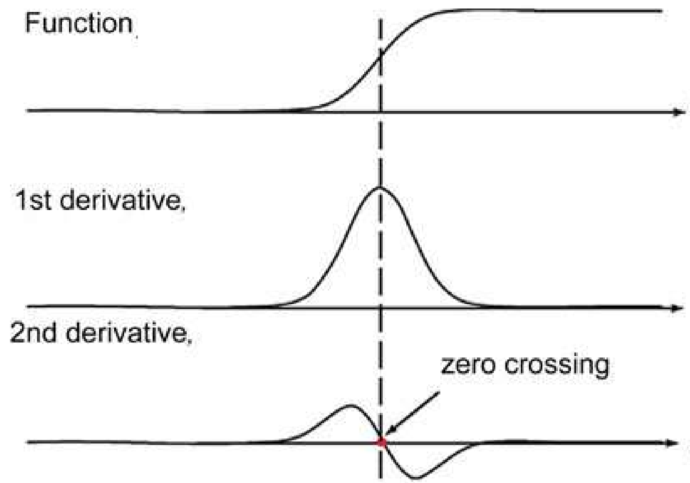
UiO : Department of Physics
University of Oslo

Non-integer order derivatives: from Niels Henrik Abel to physics

Sverre Holm

Derivatives: Change vs Complexity

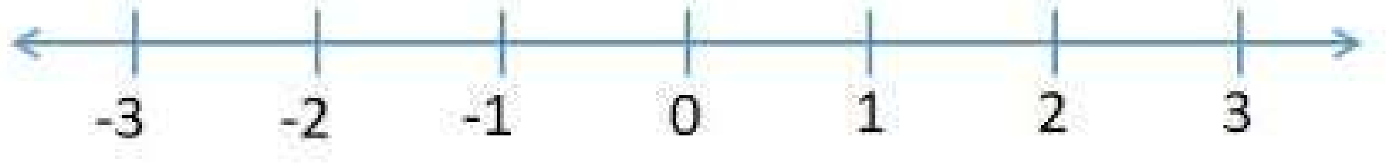


- Integer order:
  - Leibniz, Newton: $\frac{\mathrm{d}}{\mathrm{d}t} f(t),\ \dot{f}(t)$
  - **Change**: velocity, acceleration
  - Local characterization

- Fractional order:
  - $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t)$
  - **Complexity**: Power-laws in time and frequency
  - Global, memory

Differentation and integration

- Natural numbers:



$$\cdots, \iint f(t)\,\mathrm{d}t\,\mathrm{d}t, \int f(t)\,\mathrm{d}t, f(t), \frac{\mathrm{d}f(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}, \cdots$$

- Fourier transform → power laws:

$$\cdots, (\mathrm{i}\omega)^{-2}F(\omega), (\mathrm{i}\omega)^{-1}F(\omega), F(\omega), \mathrm{i}\omega F(\omega), (\mathrm{i}\omega)^{2}F(\omega), \cdots$$

- Generalization:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} \Rightarrow (\mathrm{i}\omega)^\alpha F(\omega)$$

Active community

International Conference on Fractional Differentation and its **Applications**:

- France ‘04, Portugal ‘06, Turkey ’08, Spain ‘10, **China ‘12**, Italy ‘14, Serbia ‘16, Jordan ‘18, Virtual (Poland) ’21, **UAE ‘23**

Applications

- Dielectrics, e.g. bioimpedance and electrochemistry:
  - Power-law: Curie-von Schweidler, 1889, 1907
  - Stretched exponential: Kohlrausch, 1854
  - Cole-Cole dielectric model, 1941: $\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + (\mathrm{i}\omega\tau)^\alpha}$
- Turbulence:
  - Viscous boundary layer thickness: $\delta_{BL} = \sqrt{\frac{2\eta}{\rho\omega}}$
  - Half-order derivative
- Acoustics, mechanics of complex media:
  - Medical ultrasound, seismics: power-law attenuation
  - Relaxation modulus, time: not exponential, but power law

L'Hôpital to Leibniz, 1695

"What if *n* should be 1/2?"

- "It looks like we will one day draw some very useful consequences from these paradoxes, because there are hardly any paradoxes without utility."
- «Il y a de l'apparence qu'on tirera un jour des consequences bien utilies de ces paradoxes, car il n'y a gueres de paradoxes sans utilite»

- Leibniz, G W 1695, Mathematische Schriften 1849, reprinted 1962, Hildesheim, Germany, pp. 301-302

XXIV.

Leibniz an de l'Hospital.

Hanover 30 Sept. st. n. 1695.

Vous voyés par là, Monsieur, qu'on peut exprimer par une serie infinie une grandeur comme $d^{\frac{1}{2}}\overline{xy}$, ou $d^{1:2}\overline{xy}$, quoyque cela paroisse eloigné de la Geometrie, qui ne connoist ordinairement

302

que les differences à exposans entiers affirmatifs, ou les negatifs à l'egard des sommes, et pas encor celles, dont les exposans sont rompus. Il est vray, qu'il s'agit encor de donner $d^{1:2}x$ pro illa serie; mais encor cela se peut expliquer en quelque façon. Car soyent les ordonnées x en progression Geometrique en sorte que prenant une constante $d\beta$ soit $dx = xd\beta : a$, ou (prenant a pour l'unité) $dx = x\,d\beta$, alors ddx sera $x . \overline{d\beta}^2$, et d^3x sera $= x . \overline{d\beta}^3$ etc. et $d^e x = x . \overline{d\beta}^e$. Et par cette adresse l'exposant differentiel est changé en exposant potentiel et remettant $dx : x$ pour $d\beta$, il y aura $d^e x = \overline{dx : x}^e . x$. Ainsi il s'ensuit que $d^{1:2}x$ sera egal à $x . \sqrt{dx : x}$. Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité. Vous es-

Lacroix, 1819: ½-order derivative

$$\frac{\mathrm{d}x^m}{\mathrm{d}x} = mx^{m-1}$$

$$\frac{\mathrm{d}^n x^m}{\mathrm{d}x^n} = \frac{m!}{(m-n)!}x^{m-n}$$

Arbitrary m, n:

$$\frac{\mathrm{d}^n x^m}{\mathrm{d}x^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

m=1, n=1/2:

$$\frac{\mathrm{d}^{1/2} x}{\mathrm{d}x^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = \frac{2}{\sqrt{\pi}}\sqrt{x}$$

410 CHAP. V. APPLICATION DU CALCUL INTÉGRAL

Si l'on fait $m = 1$, $n = \frac{1}{2}$, il viendra

$$d^{\frac{1}{2}}v = \sqrt{v\,dv}\,\frac{\int dx\,l\frac{1}{x}}{\int dx\left(l\frac{1}{x}\right)^{\frac{1}{2}}} = \frac{\sqrt{v\,dv}}{\frac{1}{2}\sqrt{\pi}},$$

en observant qu'entre les limites 0 et 1,

$$\int dx\,l\frac{1}{x} = 1, \quad \int dx\left(l\frac{1}{x}\right)^{\frac{1}{2}} = \left[\tfrac{1}{2}\right]^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{\pi},$$

π étant la demi-circonférence du cercle dont le rayon est 1 (1160).

C'est ainsi que l'on parviendrait à l'équation primitive de la courbe correspondante à l'équation différentielle

$$y\,d^{\frac{1}{2}}v = v\sqrt{dy},$$

Lacroix, S F, 1819, Traité du Calcul Différentiel et du Calcul Intégral, Paris (Courcier) 3, second edition, 409-410

Sinusoid

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} \sin kx = k^\alpha \sin\left(kx + \frac{\pi}{2}\alpha\right), \quad k \geq 0$$

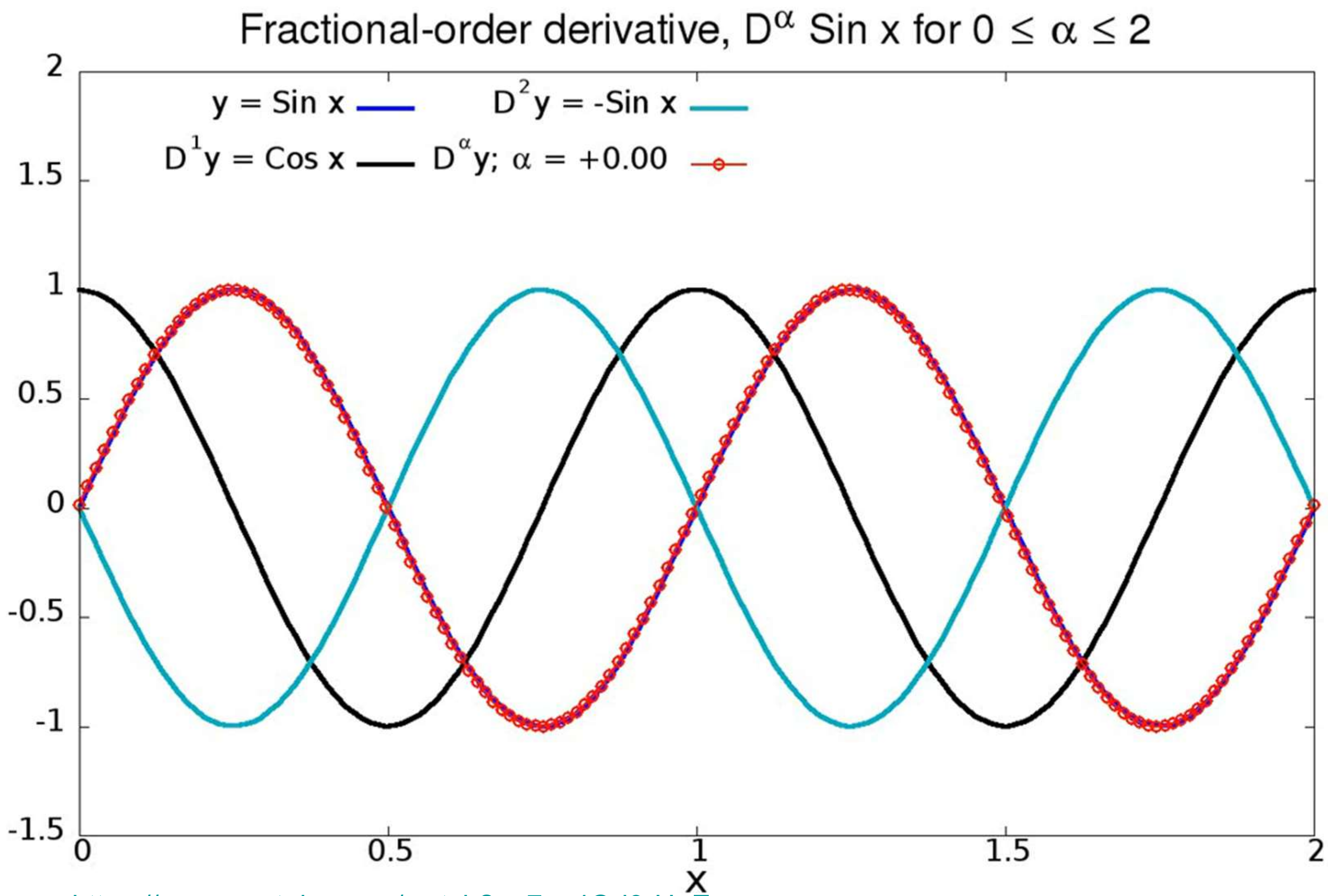


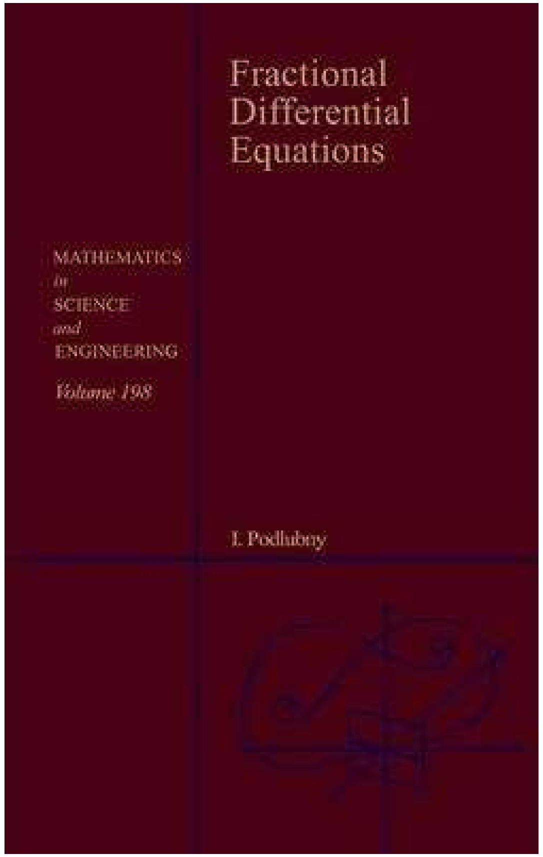
Fig:
Vikash Pandey,
YouTube

https://www.youtube.com/watch?v=7oc4Qd9-HnE

Niels Henrik Abel, 1823

“In his first paper on the generalization of the tautochrone problem, that was published in 1823, Niels Henrik Abel presented **a complete framework for fractional-order calculus** and used the clear and appropriate notation for fractional-order integration and differentiation.”

Podlubny, Magin, Trymorush, (2017). Niels Henrik Abel and the birth of fractional calculus, Fract. Calc. Appl. Anal., 1068-1075: https://doi.org/10.1515/fca-2017-0057



Abel Prize page: Abel’s work

1. Solution of algebraic equations by radicals;
2. New transcendental functions, in particular elliptic integrals, elliptic functions, abelian integrals;
3. Functional equations;
4. **Integral transforms**;
5. Theory of series treated in a rigorous way.

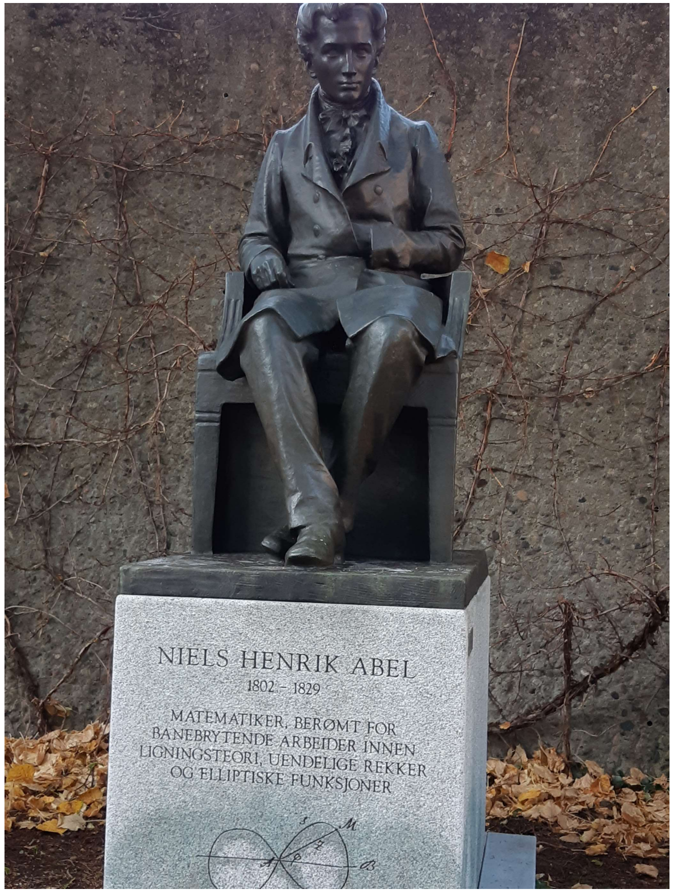
https://www.abelprize.no/c53681/artikkel/vis.html?tid=53889

Differentieres Værdien for s nGange, saa faaer man

$$\frac{d^n s}{dx^n} = \frac{1}{\Gamma(1-n)} \cdot \psi x \,,$$

altsaa naar s sættes $= \varphi x$

$$\frac{d^n \varphi a}{da^n} = \frac{1}{\Gamma(1-n)} \cdot \int \frac{\varphi^1 x . dx}{(a-x)^n} \; (x=0,\; x=a)$$



1. Fundamental theorem of calculus

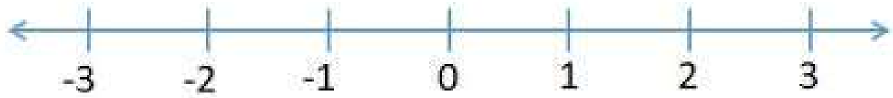
- Differentiation and integration cancel:

$$\int^{n} (\cdot) d\tau^{n} = \frac{\mathrm{d}^{-n}}{\mathrm{d}t^{-n}}(\cdot), \quad \mathrm{I}^{n}(\cdot) = \mathrm{D}^{-n}(\cdot)$$

[Abel’s and Liouville’s notation for repeated integration]

- Sequence of n-fold integrals and derivatives:

$$\cdots, \int_a^t \mathrm{d}\tau_2 \int_a^{\tau_2} f(\tau_1)\mathrm{d}\tau_1, \int_a^t f(\tau_1)\mathrm{d}\tau_1, f(t), \frac{\mathrm{d}f(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}, \cdots$$

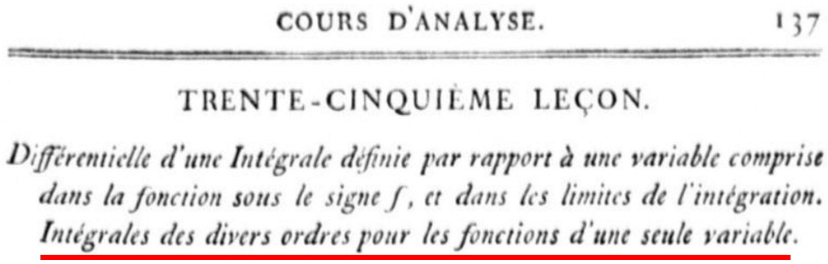


2. Cauchy formula for repeated integration, 1823(?)

Convolution with power law!

$$\mathrm{I}^n[f(t)] = \int^n f(t)\mathrm{d}t^n = \frac{1}{(n-1)!}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}}\mathrm{d}\tau, \qquad n = 1, 2, \ldots$$

Augustin Cauchy
(1789 – 1857)



COURS D'ANALYSE. 137

TRENTE-CINQUIÈME LEÇON.

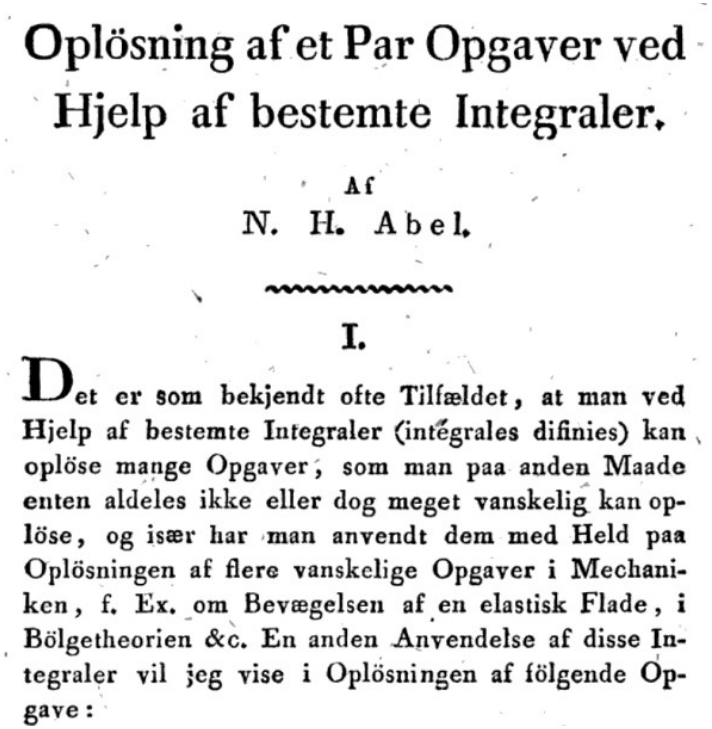
Différentielle d'une Intégrale définie par rapport à une variable comprise dans la fonction sous le signe ∫, et dans les limites de l'intégration. Intégrales des divers ordres pour les fonctions d'une seule variable.

- Not:
  - Cauchy's integral theorem: $\oint_C f(z)dz = 0,$
  - Cauchy's integral formula: $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz,$

- Wikipedia
- Augustin Louis Cauchy: Trente-Cinquième Leçon. In: Résumé des leçons données à l'Ecole royale polytechnique sur le calcul infinitésimal. Imprimerie Royale, Paris 1823.

Abel 1823

- N. H. Abel "Oplösning af et par opgaver ved hjelp af bestemte integraler." Magazin for natur-videnskaberne, 1823, pp. 55-68
- N.H. Abel, “Solution de quelques problems à l’aide d’intégrales définies,” 1823 (1881).
  - “Œuvres complètes de Niels Henrik Abel. Nouvelle edition”, L. Sylow and S. Lie, Grøndahl & Søn, Christiania, 1881, Ch. II, pp. 11–27. https://www.abelprize.no/nedlastning/verker/oeuvres_1839/oeuvres_completes_de_abel_1_kap04_opt.pdf
- N. H. Abel, “Solution of some problems using definite integrals,” 1823 (2017).
  - Podlubny, Magin, Trymorush, (2017). Niels Henrik Abel and the birth of fractional calculus. Fractional Calculus and Applied Analysis, https://doi.org/10.1515/fca-2017-0057
  - Supplement to arXiv version has an English translation of Abel’s paper https://arxiv.org/pdf/1802.05441.pdf



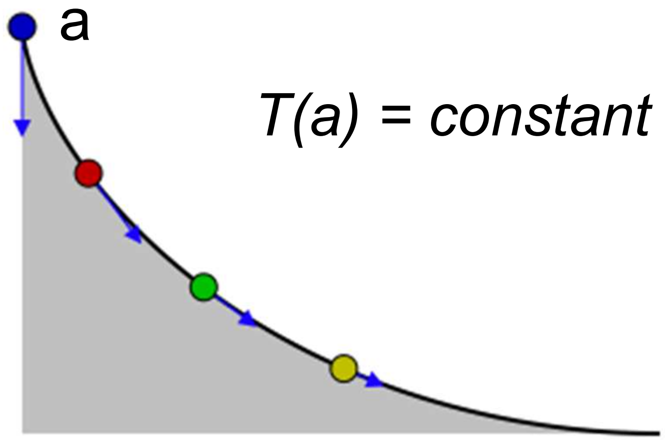
Oplösning af et Par Opgaver ved Hjelp af bestemte Integraler.

Af
N. H. Abel.

I.

Det er som bekjendt ofte Tilfældet, at man ved Hjelp af bestemte Integraler (intégrales difinies) kan oplöse mange Opgaver, som man paa anden Maade enten aldeles ikke eller dog meget vanskelig kan oplöse, og især har man anvendt dem med Held paa Oplösningen af flere vanskelige Opgaver i Mechaniken, f. Ex. om Bevægelsen af en elastisk Flade, i Bölgetheorien &c. En anden Anvendelse af disse Integraler vil jeg vise i Oplösningen af fölgende Opgave:

Abel's mechanical problem



- Generalization of tautochrone curve. How does time to descent vary with height, a, and curve shape *s(x)*:

$$T(a) = \frac{1}{\sqrt{2g}} \int_0^a \frac{1}{\sqrt{a-x}} \frac{\mathrm{d}s}{\mathrm{d}x} \mathrm{d}x$$

- ~Caputo derivative, $D^{0.5}(s)$
- Also solves for *s(x)* which is ~$D^{-0.5}(T)$
  - Abel, Auflösung einer mechanischen Aufgabe, J. Reine u. Angew. Math, 1826, pp. 153-157.
  - B. Holmboe, "Abel: Œuvres complètes", 1839, IV Résolution d'un problème mécanique:
  - https://www.degruyter.com/view/j/crll.1826.1826.issue-1/issue-files/crll.1826.1826.issue-1.xml
  - https://www.abelprize.no/nedlastning/verker/oeuvres_1839/oeuvres_completes_de_abel_1_kap04_opt.pdf

Energy balance, frictionsless

$$\frac{1}{2}mv^2 = mg(a-x) \Rightarrow v = \sqrt{2g(a-x)}$$

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \Rightarrow \mathrm{d}t = \frac{\mathrm{d}s}{(2g(a-x))^{1/2}}$$

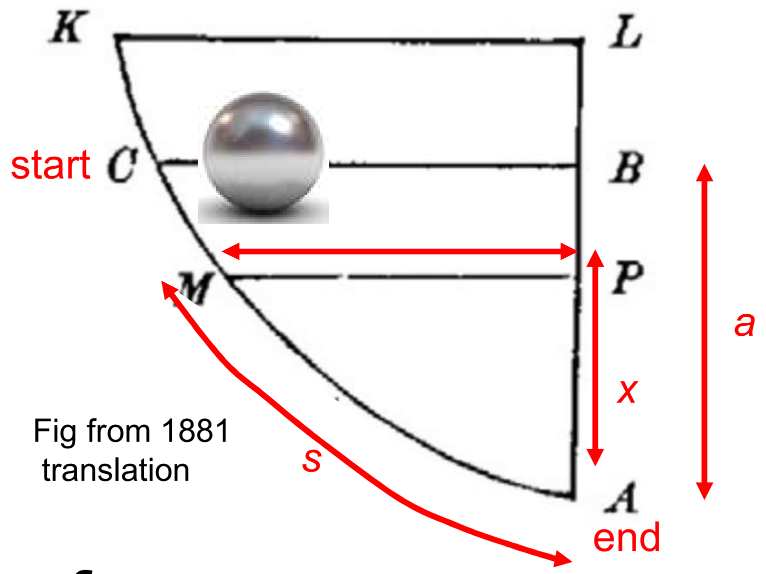


Fig from 1881 translation

Time of descent as a function of arc *s*:

$$T = \int \mathrm{d}t = \frac{1}{\sqrt{2g}} \int_0^a \frac{\mathrm{d}s}{(a-x)^{1/2}} = \frac{1}{\sqrt{2g}} \int_0^a \frac{s'}{(a-x)^{1/2}} \mathrm{d}x$$

Generalizes, just for the fun of it …

Da nu T er lig Ψa, saa bliver altsaa Ligningen

$$\Psi a = \int \frac{ds}{\sqrt{a-x}} \text{ (fra } x=o \text{ til } x=a)$$

Istedetfor at oplöse denne Ligning, vil jeg i Almindelighed vise, hvorledes man kan finde s af Ligningen:

$$\Psi a = \int \frac{ds}{(a-x)^n} \text{ (fra } x=o \text{ til } x=a)$$

hvor n er mindre end 1, for at ikke Integralet skal blive uendeligt mellem de givne Grændser; Ψa er en hvilkensomhelst Funktion, der ikke bliver uendelig naar $a=o$.

$$\psi(a) = \int_0^t \frac{s'(x)\mathrm{d}x}{(a-x)^n}$$

$$0 \leq n < 1$$

~ Caputo fractional derivative, 1967

also in Liouville, 1832

Abel, 1823, side 56

Af det Foregaaende flyder fölgende mærkværdige Theorem:

Vi have seet at naar

$$\psi a = \int \frac{ds}{(a-x)^n} \quad (x=o,\ x=a)$$

saa er

$$s = \frac{\sin n\pi}{\pi} \cdot x^n \cdot \int \frac{\psi(xt)\,.\,dt}{(1-t)^{1-n}} \quad (t=o,\ t=1)$$

Man kan ogsaa udtrykke s paa en anden Maade, som jeg for dens Besynderligheds Skyld vil anföre, nemlig

$$s = \frac{1}{\Gamma(1-n)} \int^n \psi x \,.\, dx^n = \frac{1}{\Gamma(1-n)} \cdot \frac{d^{-n}\psi x}{dx^{-n}}$$

hvor n er mindre end 1

s = arc shape
T = $\Psi\alpha$ = time

~ Caputo fractional derivative

Cauchy formula for repeated integration for non-integer order =>

n'th order integral of Liouville 1832
= derivative of negative order

Even non-integer order derivatives and integrals are inverse operations
⬄ Assumes the fundamental theorem of calculus for non-integer orders

Abel, 1823, side 59, 61

Extension to non-integer derivatives appears to be almost trivial to Abel

Differentieres Værdien for *s* *n*Gange, saa faaer man

$$\frac{d^n s}{dx^n} = \frac{1}{\Gamma(1-n)} \cdot \psi x \, ,$$

altsaa naar *s* sættes $= \varphi x$

$$\frac{d^n \varphi a}{da^n} = \frac{1}{\Gamma(1-n)} \cdot \int \frac{\varphi^1 x . dx}{(a-x)^n} \; (x=o, \; x=a)$$

“Differentiating the value of *s, n* times, one obtains ...”,

- Caputo derivative of order 0…1

Abel, 1823, side 62

Caputo (1967)

530 **Michele Caputo**

These perfectly elastic fields are insufficient models for the description of many physical phenomena because they do not allow us to explain the dissipation of energy. A more complete description of the actual elastic fields is obtained by introducing stress–strain relations which include also linear combinations of time derivatives of the strain and the stress. The numerical coefficients appearing in the general linear combinations of higher order derivatives are called viscoelastic constants of higher order.

Elastic fields described by elastic constants of higher order have been discussed by many authors (e.g. see Knopoff (1954), Caputo (1966a). For a description of the various models of losses see Knopoff (1964).) Knopoff studied the case in which the stress–strain relations are of the type

$$\tau_{rs} = \lambda g^{hi} g_{rs} e_{hi} + 2\mu e_{rs} + \frac{d^m}{dt^m} [\lambda_m g^{hi} g_{rs} e_{hi} + 2\mu_m e_{rs}], \tag{1}$$

where λ_m and μ_m are constant; he obtained a condition for these viscoelastic constants of higher order analogous to those existing for the perfectly elastic fields and also proved that in order to have a dissipative elastic field the stress–strain relations should contain a time derivative of odd order.

A generalization of the relation (1) is

$$\tau_{rs} = \sum_{m=0}^{p} \frac{d^m}{dt^m} [\lambda_m g^{hi} g_{rs} e_{hi} + 2\mu_m e_{rs}], \tag{2}$$

where one can also consider λ_m and μ_m as functions of position.

We can generalize (2) to the case when the operation d^m/dt^m is performed with m as a real number z and also further by replacing the summation with an integral as follows:

$$\tau_{rs} = \int_{a_1}^{b_1} f_1(r,z) \frac{d^z}{dt^z} (g^{hi} g_{rs} e_{hi}) dz + 2 \int_{a_2}^{b_2} f_2(r,z) \frac{d^z}{dt^z} (e_{rs}) dz. \tag{3}$$

Relations (1) and (2) are a special case of (3). They are obtained by setting

$$\left. \begin{aligned} f_1(r,z) &= \sum_{m=q}^{p} \delta(z-m)\lambda_m, \\ f_2(r,z) &= \sum_{m=q}^{p} \delta(z-m)\mu_m, \end{aligned} \right\} \tag{4}$$

where $\delta(z-m)$ are unitary delta functions.

If $a_i = p = q = 0$ then we have the case of a perfectly elastic field; if $p = q = 1$ then we have a perfectly viscous field; if $a_i = q = 0$ and $p = 1$ then we have a viscoelastic field.

We have now to establish a few relations which we shall have to use later.

Let $f(t)$ and its ith order derivatives $(i = 1, 2, \ldots, m+1)$ be continuous in the interval $(0, +\infty)$ and also let z be a real number $(0 < z < 1)$.

We shall assume the definition of the derivative of $f(t)$ of order $m+z$ as follows:

$$\frac{d^{m+z}}{dt^{m+z}} f(t) = \frac{1}{\Gamma(1-z)} \int_0^t (t-\xi)^{-z} f^{(m+1)}(\xi) d\xi. \tag{5}$$

We want to prove that if $|f^{(i+1)}(t)| e^{-pt}$, $(p > 0)$ $(i = 0, 1, \ldots, m)$, is integrable in $(0, +\infty)$, then

$$\int_0^\infty \left[\frac{d^{m+z}}{dt^{m+z}} f(t)\right] e^{-pt} dt = p^z \left\{ p^m \int_0^\infty f(\xi) e^{-p\xi} d\xi - p^{m-1} f(0) - p^{m-2} \ldots f(0) \ldots \right.$$
$$\left. \ldots - f^{(m-1)}(0) + p^{-1} f^{(m)}(0) \right\}. \tag{6}$$

Abel (1823)

62 **Abel, Opgavers Opløsning**

nu er

$$\frac{\Gamma(m+1)}{\Gamma(m-k+1)} = \frac{1}{\Gamma(-k)} \int \frac{t^m . dt}{(1-t)^{1+k}} (t=o, t=1)$$

altsaa:

$$\frac{d^k \psi x}{dx^k} = \frac{1}{x^k . \Gamma(-k)} \int \frac{\Sigma a^{(m)} . (xt)^m . dt}{(1-t)^{1+k}} (t=o, t=1)$$

men $\Sigma a^{(m)} (xt)^m = \psi(xt)$, altsaa

$$\frac{d^k \psi x}{dx^k} = \frac{1}{x^k \Gamma(-k)} . \int \frac{\psi(xt) . dt}{(1-t)^{1+k}} (t=o, t=1)$$

Sættes $k = -n$ saa faaer man

$$\frac{x^n}{\Gamma(n)} . \int \frac{\psi(xt) . dt}{(1-t)^{1-n}} (t=o, t=1) = \frac{d^{-n} \psi x}{dx^{-n}}$$

Men vi have seet at

$$s = \frac{x^n}{\Gamma(n) . \Gamma(1-n)} \int \frac{\psi(xt) . dt}{(1-t)^{1-n}} (t=o, t=1)$$

altsaa

$$s = \frac{1}{\Gamma(1-n)} . \frac{d^{-n} \psi x}{dx^{-n}}, \text{naar } \psi a = \int \frac{ds}{(a-x)^n} (x=o, x=a),$$

q. e. d.

Differentieres Værdien for s n Gange, saa faaer man

$$\frac{d^n s}{dx^n} = \frac{1}{\Gamma(1-n)} . \psi x,$$

altsaa naar s sættes $= \phi x$

$$\frac{d^n \phi a}{da^n} = \frac{1}{\Gamma(1-n)} . \int \frac{\phi^1 x . dx}{(a-x)^n} (x=o, x=a)$$

Man maa lægge Mærke til, at i det Foregaaende n altid maa være mindre end 1.

Sættes $n = \frac{1}{2}$ saa har man

$$\psi a = \int \frac{ds}{\sqrt{a-x}} (x=o, x=a)$$

og $$s = \frac{1}{\sqrt{\pi}} . \frac{d^{-\frac{1}{2}} \psi x}{dx^{-\frac{1}{2}}} = \frac{1}{\sqrt{\pi}} \int^{\frac{1}{2}} \psi x . dx^{\frac{1}{2}}$$

Podlubny, Magin, Trymorush, (2017)

Conclusion

“It is not clear why Niels Henrik Abel abandoned the direction of research so nicely formed in his 1823 paper, and one can only guess the reasons.

Abel had all the elements of the fractional-order calculus there:

- the idea of fractional-order integration and differentiation,
- the mutually inverse relationship between them,
- the understanding that fractional-order differentiation and integration can be considered as the same generalized operation,
- and even the unified notation for differentiation and integration of arbitrary real order.”

THE ABEL PRIZE

Literature on Abel and his mathematics

I. Podlubny, R.L. Magin and I. Trymorush: *Niels Henrik Abel and the birth of fractional calculus.* Fractional Calculus and Applied Analysis, 20(5), (2017) 1068-1075.

B. Riemann *Theorie der Abel'schen Funktionen*, Journal für die reine und angewandte Mathematik 54 (1857) 101- 155.

M. Rosen *Abel's theorem on the lemniscate*, The American Mathematical Monthly 88 (1981) 387-395. [Permission required]

M. Rosen *Niels Hendrik Abel and equations of the fifth degree*, The American Mathematical Monthly 102 (1995) 495-505. (Chauvenet Prize of the Mathematical Association of America 1999) [Permission required]

R. C. Rowe and A. Cayley *Memoir on Abel's Theorem*, Philosophical Transactions of the Royal Society of London, Vol. 172. (1881), pp. 713-758. [Permission required]

C. Skau *Gjensyn med Abels og Ruffinis bevis for umuligheten av å løse den generelle n'te gradsligningen algebraisk når n≥5*, Normat 38 (1990) 53-84.

C. Størmer *Abels oppdagelser*, Norsk Matematisk Tidsskrift 11 (1929) 85-96 and 125-138.

L. Sylow *Abels studier og hans opdagelser*, in Festskrift ved hundreaarsjubilæet for Niels Henrik Abels fødsel. Kristiania 1902. (French edition 1902: *Les etudes d'Abel et ses decouvertes*.)

$$F(\omega) = \mathcal{F}\left(f(t)\right) = \int_{\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

Temporal power-law ⇔ Frequency power-law ⇔ Fractional derivative

- Important Fourier transform:

$$f(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0 \quad \leftrightarrow \quad F(\omega) = (\mathrm{i}\omega)^{\beta-1}$$

- Order α differentiation:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) \quad \leftrightarrow \quad (\mathrm{i}\omega)^\alpha F(\omega)$$

Power-laws ⇔ Caputo and Riemann-Liouville fractional derivatives

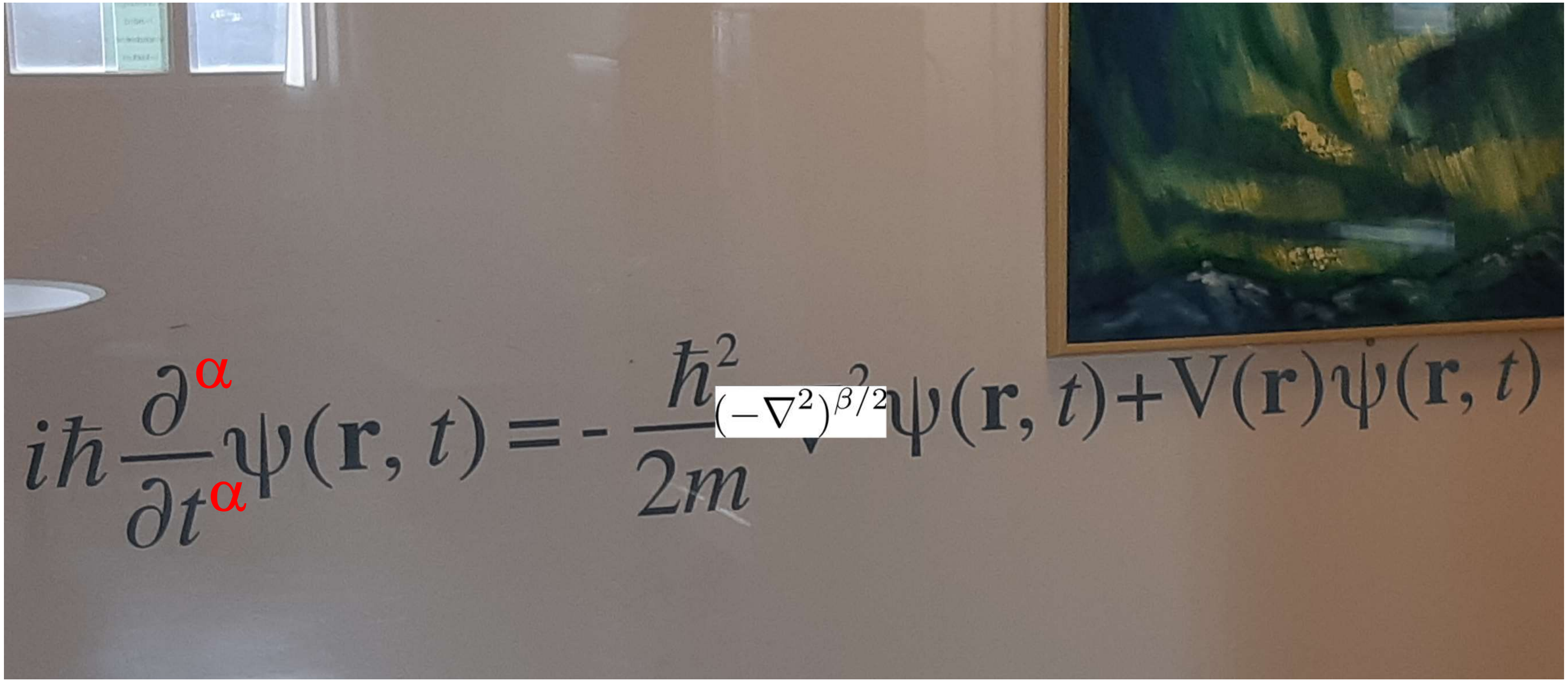
- Order α differentiation: $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) \quad \leftrightarrow \quad (\mathrm{i}\omega)^\alpha F(\omega)$
- Fractional differentiation: convolution with power law:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m} * \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, \quad m = \lceil \alpha \rceil$$

- Two ways to differentiate a convolution:

$$\left(\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) \right) * g(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(f(t) * g(t) \right)$$

Caputo Riemann-Liouville


$i\hbar \frac{\partial^{\alpha}}{\partial t^{\alpha}} \psi(\mathbf{r}, t) = - \frac{\hbar^2}{2m} (-\nabla^2)^{\beta/2} \psi(\mathbf{r}, t) + V(\mathbf{r}) \psi(\mathbf{r}, t)$

1. Dielectrics: Power-laws

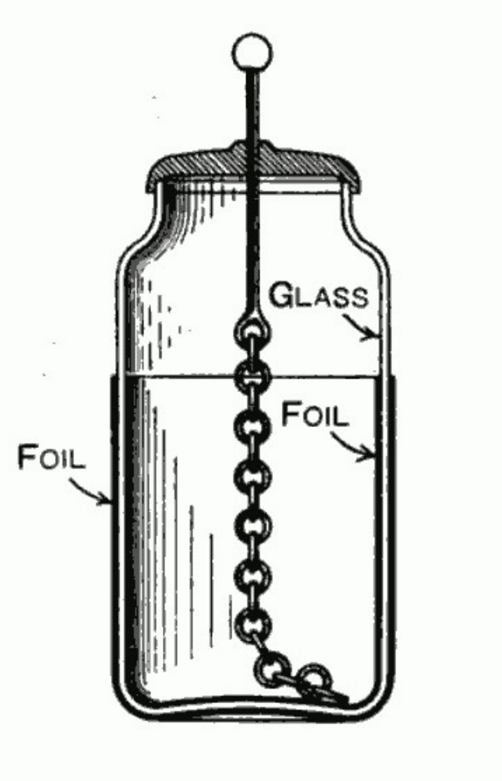
- Current response to a step voltage, Curie-von Schweidler law, 1889, 1907:

$$I(t) \propto t^{\alpha-1}$$

- Kohlrausch, 1854: Discharge of a capacitor:

$$Q(t) \propto \exp\left[-(t/\tau)^{\alpha}\right]$$

  – Stretched exponential: power-law inside exponential
  – Stretches the decay rel. to exp. function



Leyden jar

Dielectricity & Constant Phase Element

- Displacement field, D vs electric field E:

$$D = \varepsilon_0 \varepsilon_r E$$

- Debye, 1912, e.g. liquids, solids:

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + \mathrm{i}\omega\tau}$$

- Cole-Cole, 1941:

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + (\mathrm{i}\omega\tau)^\alpha}$$

- Let $\varepsilon_\infty \to 0, \;\; (\omega\tau)^\alpha \gg 1$:

$$C \approx \frac{A}{d} \frac{\varepsilon_0 \varepsilon_s}{(\mathrm{i}\omega\tau)^\alpha}$$

Capacitance:

$$C = \frac{A}{d} \varepsilon_0 \varepsilon_r$$

CPE - Constant Phase Element: bioimpedance electrochemistry, …

Fractional model for dielectrics

$$D = \epsilon_0 \epsilon_r E$$

- Cole-Cole model, 1941:

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + (\mathrm{i}\omega\tau)^\alpha} \Rightarrow \underline{D(t) + \tau^\alpha \frac{\mathrm{d}^\alpha D(t)}{\mathrm{d}t^\alpha} = \varepsilon_0 \varepsilon_s E(t) + \tau^\alpha \varepsilon_0 \varepsilon_s \frac{\mathrm{e}^\alpha D(t)}{\mathrm{e}t^\alpha}}$$

- Current response to a voltage step:

$$I_{step}(\omega) = \frac{U(\omega)}{Z(\omega)} = \frac{\varepsilon_0 A}{d} \varepsilon_r(\omega) = \frac{\varepsilon_0 A}{d} \mathcal{F}\{\phi(t)\}$$

- Charge response to a voltage step:

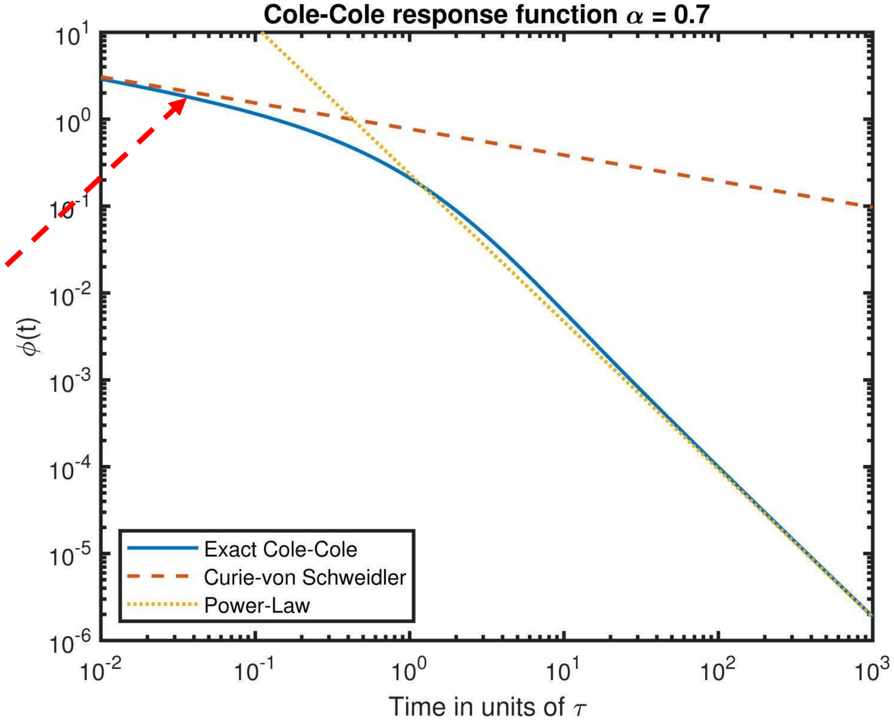
$$Q_{step}(\omega) = \frac{I_{step}(\omega)}{\mathrm{i}\omega A} = \frac{\varepsilon_0}{d} \frac{\varepsilon_r(\omega)}{\mathrm{i}\omega} = \frac{\varepsilon_0}{d} \mathcal{F}\{\Psi(t)\}$$

Cole-Cole ➔ Curie-von Schweidler law

- Current response to a voltage step:

$$\phi_{CC}(t) = \frac{1}{\tau}(t/\tau)^{\alpha-1}\,\mathrm{E}_{\alpha,\alpha}\left(-(t/\tau)^{\alpha}\right) \sim \begin{cases} \frac{1}{\tau\Gamma(\alpha)}(t/\tau)^{\alpha-1}, & t \ll \tau \\ \frac{1}{\tau\Gamma(-\alpha)}(t/\tau)^{-\alpha-1}, & t \gg \tau \end{cases}$$

- $E_{\alpha,\alpha}$: two-parameter Mittag-Leffler function
  - Generalization of exponential
  - Garrappa, Mainardi, Maione, Models of dielectric relaxation based on completely monotone functions, Fract. Calc. Appl. Anal. 2016
- Curie-von Schweidler law, 1889!
  - Holm, "Time domain characterization of the Cole-Cole dielectric model" J. Electr. Bioimpedance, 2020.
  - Also asymptote of Havrili-Negami and Cole-Davidson models

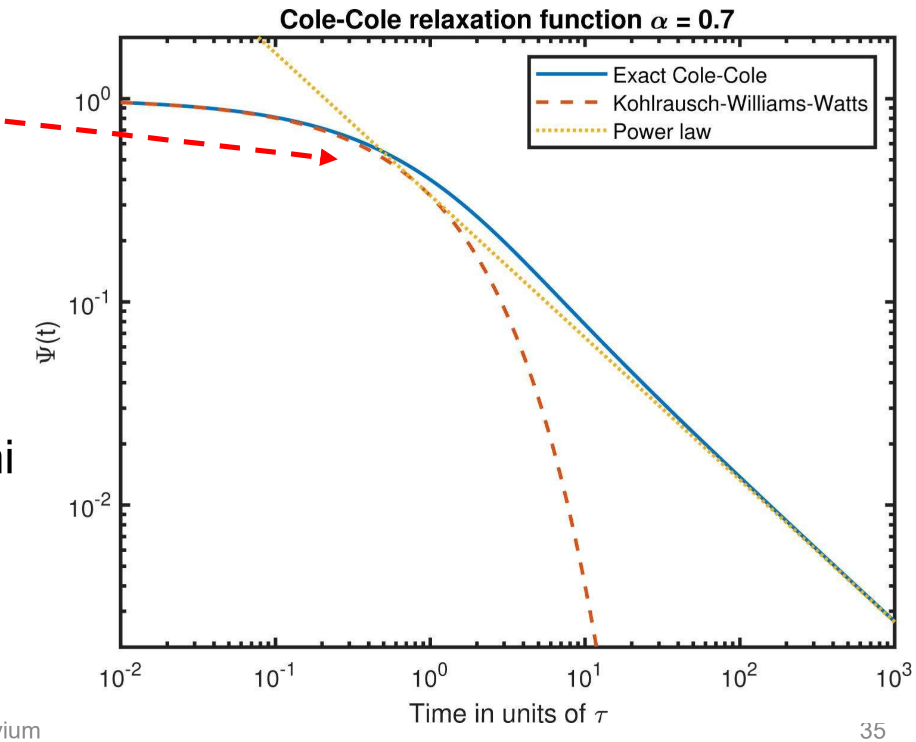


Cole-Cole ➔ Kohlrausch response

- Charge response to a voltage step:

$$\Psi_{CC}(t) = \mathrm{E}_\alpha\left(-(t/\tau)^\alpha\right) \sim \begin{cases} \exp\left[\frac{-(t/\tau)^\alpha}{\Gamma(\alpha+1)}\right], & t \ll \tau \\ \frac{(t/\tau)^{-\alpha}}{\Gamma(1-\alpha)}, & t \gg \tau. \end{cases}$$

- E_α: Mittag-Leffler function
- Kohlrausch 1854!
- Williams, Watts, 1970
  - Holm, "Time domain characterization of the Cole-Cole dielectric model" J. Electr. Bioimpedance, 2020
  - Garrappa, Mainardi, Maione, Models of dielectric relaxation based on completely monotone functions, Fract. Calc. Appl. Anal. 2016
  - Also asymptote of Havrili-Negami and Cole-Davidson models



2. Rheology, complex media: Power-laws

- Nutting, 1921: Relaxation modulus: stress, σ, response to a step in strain, ε:
  - paint, oil

$$G(t) \propto t^{-\alpha}, \quad 0 < \alpha < 1$$

- Scott Blair, 1940/50s:
  - cheese, clay

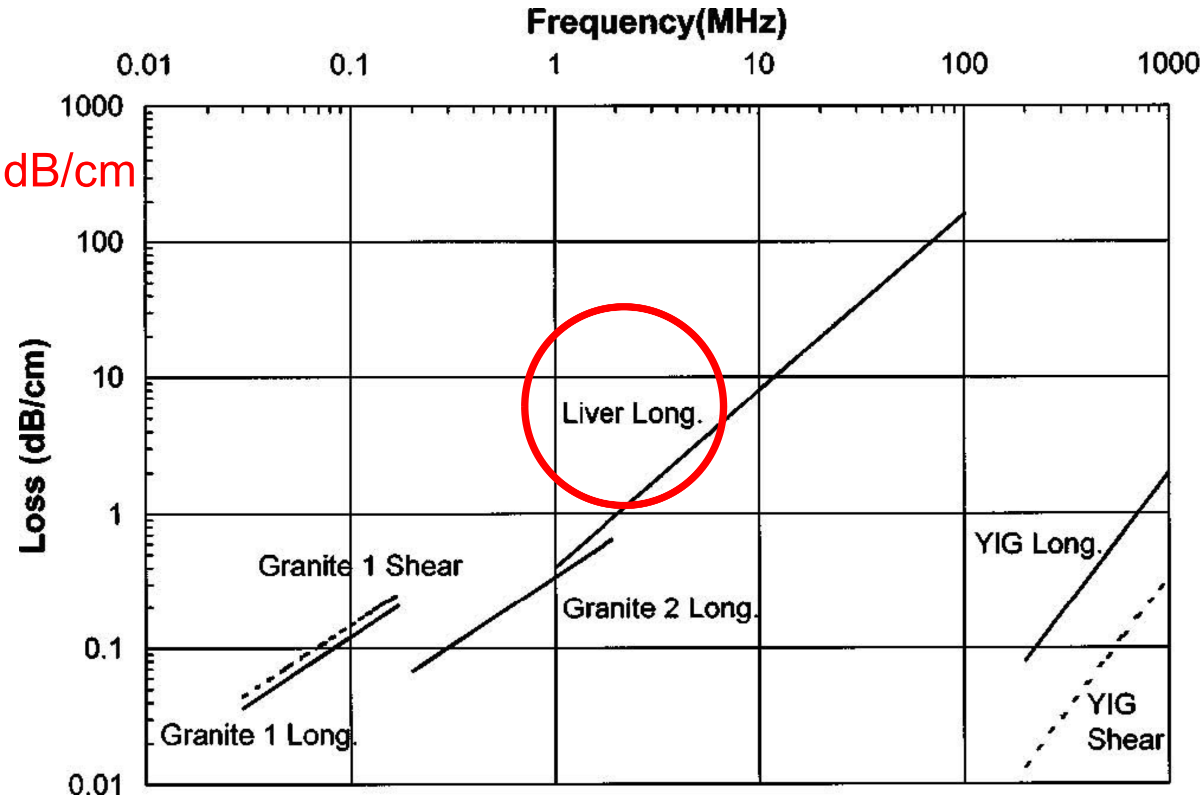
$$\sigma(t) = \eta \frac{\partial^\alpha \epsilon(t)}{\partial t^\alpha}$$

  - "*I gave up the work eventually, mainly because I could not find a definition of a fractional differential that would satisfy the mathematicians*."

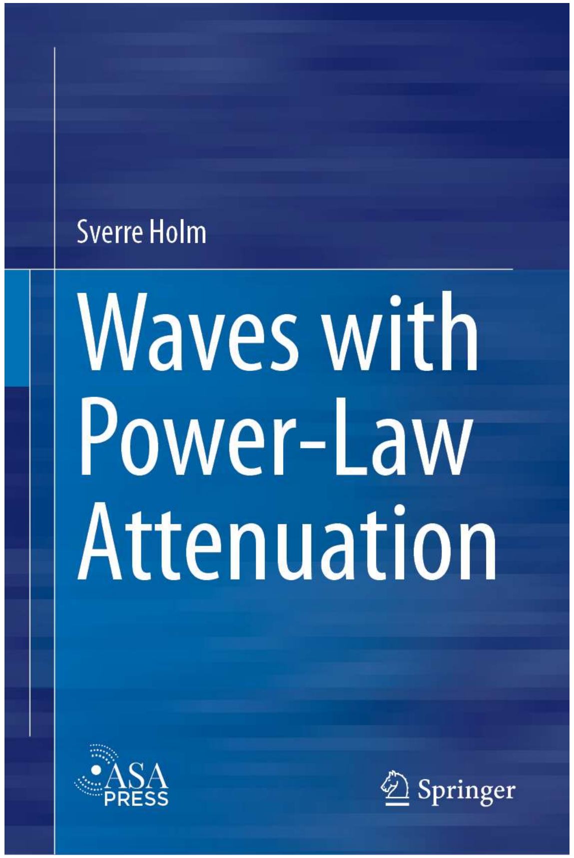
Power laws: loss ω^y

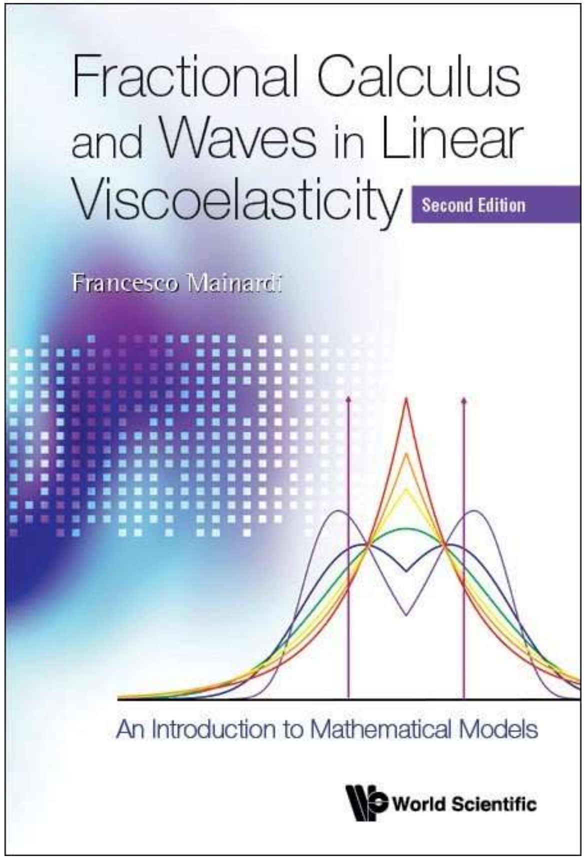
- Medicine, geophysics, ...
- Longitudinal, pressure:
  - Granite: y ≈ 1
  - Liver: y ≈ 1.3
- Shear:
  - YIG: y = 2
    (Yttrium indium garnet)
  - Granite: y ≈ 1

- Szabo and Wu, "A model for longitudinal and shear wave propagation in viscoelastic media", JASA (2000).



Data for shear and longitudinal wave loss which show power-law dependence over four decades of frequency.



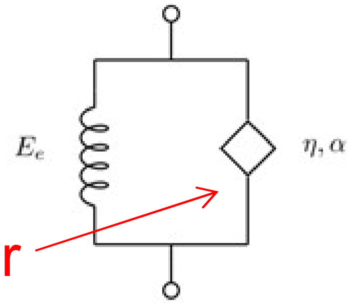


Power-law relaxation responses ⇔ power-law attenuation

May be described by wave equations with fractional operators

Viscous wave (Stokes) equation → Fractional Kelvin-Voigt wave equation

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u = 0$$



Fractional loss operator

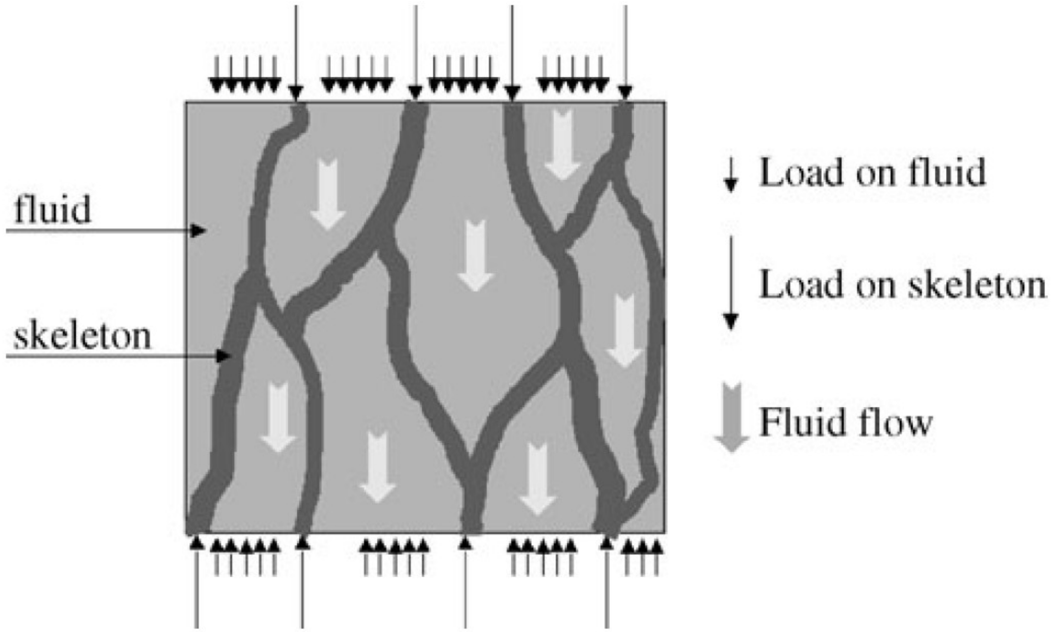
- *u* is displacement
- Attenuation increases with $\omega^{\alpha+1}$

- Parsimonous description: nice for simulation
- Fundamentally, relaxation processes:

$$\alpha_k(\omega) = \omega^2 \int A(\Omega)\frac{\Omega}{\omega^2+\Omega^2}\,\mathrm{d}\Omega$$

$$= AN\omega^2 \int g(E)\frac{\tau(E)}{1+\omega^2\tau^2(E)}\,\mathrm{d}E, \quad \tau = 1/\Omega$$

3. Poroelasticity, rigid frame: turbulence

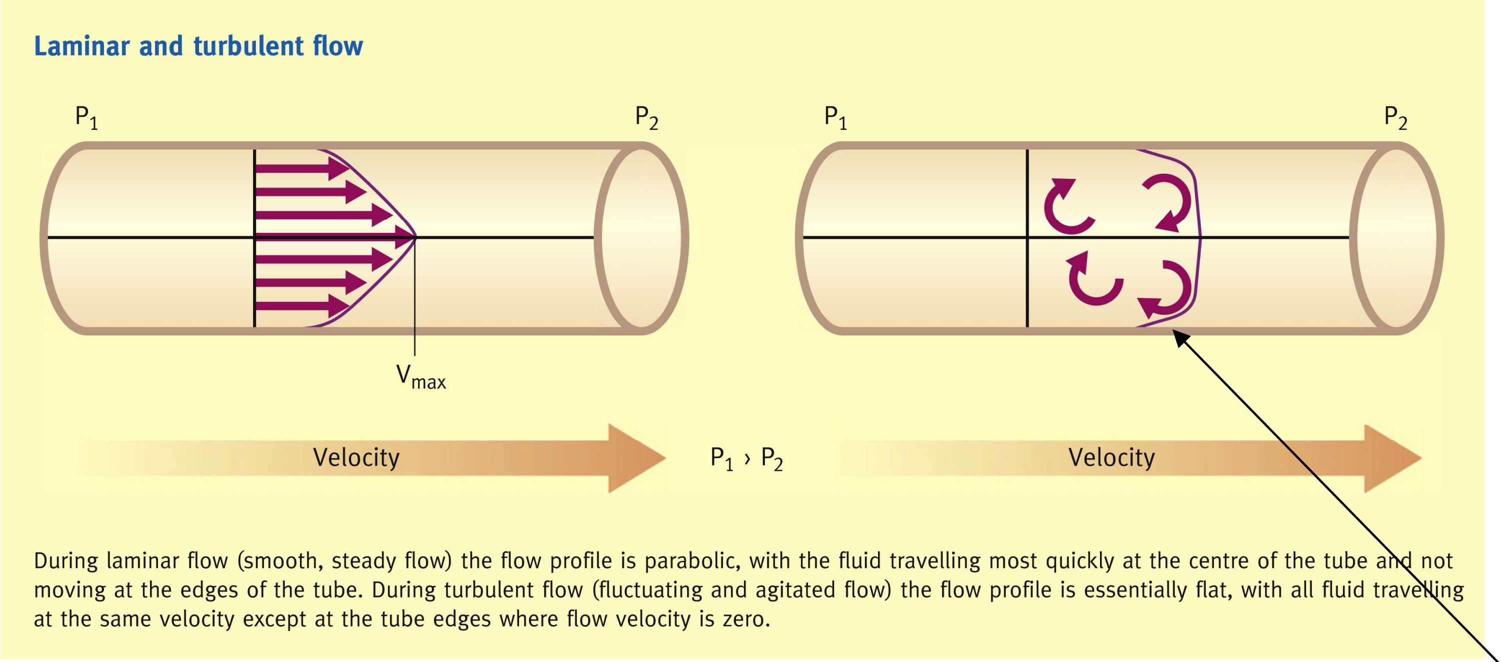


- Biot 1956, J Acoust Soc Am
- Bone acoustics, sediments, brain biomechanics
- Fast and slow compressional waves + shear wave

Bulk properties:			Sediment	Brain tissue
ϕ	[0 ... 1]	Porosity	0.44	0.2
ρ_s	[kg/m^3]	Solid density	2650	1000
ρ_f	[kg/m^3]	Fluid density	1000	1000
K_s	[Pa]	Bulk modulus, solid	36e9	0.5e6
K_f	[Pa]	Bulk modulus, fluid	2.25e9	2.2e9
Fluid parameters:				
η	[Pa s]	Viscosity	1e-3	1e-3
B	[m^2]	Permeability	17.5e-12	1e-10
α_p	[m]	Pore radius	28e-6	10e-6
α	[1 ... 3]	Tortuosity	1.24	1
Rigid frame response parameters:				
K_r	[Pa]	Bulk modulus	26.3e6	-
μ_r	[Pa]	Shear modulus	24e6	900

Pakulam, et al Poromechanical Models, ch 5 in P. Laugier and G. Haïat (eds.), Bone Quantitative Ultrasound, 2011

Viscodynamic operator – drag coefficient – correction of viscosity



Mitchell, Viki, and Kate Cheesman. "Gas, tubes and flow." *Anaesthesia & Intensive Care Med,* 2010

Biot, 1956:

$$F_{3D}(\omega) = \frac{z}{4}\frac{\mathrm{I}_1(z)}{\mathrm{I}_2(z)} \begin{cases} = 1, & \omega = 0 \\ \to \frac{1}{4}\sqrt{\mathrm{i}\omega\tau} & \end{cases} \qquad z = \sqrt{\mathrm{i}\omega\tau}, \quad \tau = \frac{\rho_f R^2}{\eta}$$

Inverse of boundary layer thickness

4. Klein–Gordon and Dirac equations

Derivation [edit]

The non-relativistic equation for the energy of a free particle is

$$\frac{\mathbf{p}^2}{2m} = E.$$

By quantizing this, we get the non-relativistic Schrödinger equation for a free particle:

$$\frac{\hat{\mathbf{p}}^2}{2m}\psi = \hat{E}\psi,$$

where

$$\hat{\mathbf{p}} = -i\hbar\nabla$$

is the momentum operator (∇ being the del operator), and

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

is the energy operator.

The Schrödinger equation suffers from not being relativistically invariant, meaning that it is inconsistent with special relativity.

It is natural to try to use the identity from special relativity describing the energy:

$$\sqrt{\mathbf{p}^2c^2 + m^2c^4} = E.$$

Then, just inserting the quantum-mechanical operators for momentum and energy yields the equation

$$\sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}\,\psi = i\hbar\frac{\partial}{\partial t}\psi.$$

Spatial memory?

The square root of a differential operator can be defined with the help of Fourier transformations, but due to the asymmetry of space and time derivatives, Dirac found it impossible to include external electromagnetic fields in a relativistically invariant way. So he looked for another equation that can be modified in order to describe the action of electromagnetic forces. In addition, this equation, as it stands, is nonlocal (see also Introduction to nonlocal equations).

Klein and Gordon instead began with the square of the above identity, i.e.

$$\mathbf{p}^2c^2 + m^2c^4 = E^2,$$

which, when quantized, gives

$$\left((-i\hbar\nabla)^2c^2 + m^2c^4\right)\psi = \left(i\hbar\frac{\partial}{\partial t}\right)^2\psi,$$

Conclusion

- Fractional derivative ⇔ Frequency / time power-laws
  - Central role of Caputo and Riemann-Liouville
- Assumption: continuum mechanics/physics:
  - Non-ideal capacitors: Power-law and stretched exp. current / charge ⇔ Cole-Cole++ permittivity
  - Power-law relaxation in lin. viscoelasticity ⇔ power-law attenuation in acoustics ⇔ fractional damper
  - Viscous boundary layer ⇔ 1/2-order fractional derivative ➔ fractional pseudo-differential operator