Abstract

The nature of dark matter (DM) is one of the hottest topics, and greatest mysteries, of modern physics. One of the most promising candidates for DM is Weakly Interacting Massive Particles which occur naturally in several theories beyond the standard model originally proposed to solve unrelated problems. One such theory is Universal Extra Dimensions (UED)[1], for which the lightest so-called Kaluza-Klein particle (LKP) is stable. In the simplest theory involving UEDs, minimal UED, the LKP is the first Kaluza-Klein (KK) excitation of the photon.

Recently, updated corrections to the masses and couplings of minimal UED have been calculated by Freitas et al. [2]. The main goal of this thesis has been to calculate the relic density of KK dark matter in the mUED framework to test whether the updated corrections have any effect on the phenomenology of mUED. This was done by implementing a UED module in DarkSUSY, a FORTRAN package originally developed to calculate observables of supersymmetric theories. The UED module contains the necessary components to calculate the relic density of KK dark matter including annihilation and the main coannihilation channel.

We have found that the relic density decreases when the updated corrections are included. Based on how the relic density decreases when only the main coannihilation channels are included, we can estimate that the upper bound on the LKP mass increases from approximately 1525 to 1785 GeV. However, more coannihilation channels need to be implemented in order to calculate the relic density precisely and get the correct effect of the updated mass corrections.
In memory of my dad, Bjørn Jacobsen, who always supported me, believed in me and would have loved reading this thesis.
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## Contents

1 Introduction 8

2 Extra dimensions 10
   2.1 Kaluza-Klein Theory 10
   2.2 Universal Extra Dimensions 13
      2.2.1 Kaluza-Klein Parity 14
      2.2.2 The 5D SM Lagrangian 15
      2.2.3 Gauge Fields in 5D 16
      2.2.4 Fermions in 5D 17
      2.2.5 The Higgs mechanism and the Higgs field in 5D 18
      2.2.6 Higher order KK couplings 20
      2.2.7 Radiative Corrections 22

3 Dark Matter 27
   3.1 The Friedmann-Robertsson-Walker Universe 27
   3.2 The Dark Matter particle 30
      3.2.1 Weakly Interacting Massive Particles 32
   3.3 Relic Density of Dark Matter 33
      3.3.1 The Boltzmann Equation 33
      3.3.2 Standard calculation of relic density 34
      3.3.3 Including coannihilations 35
   3.4 Kaluza-Klein Dark Matter 37

4 Annihilation and Coannihilation processes 39
   4.1 Overview of (co-)Annihilation Processes 39
      4.1.1 LKP annihilation 40
      4.1.2 Coannihilation channels 45
   4.2 Calculations and numerical implementation 47
      4.2.1 FORM 47
      4.2.2 Decay Rates 49
   4.3 DarkSUSY 50
      4.3.1 General structure 50
      4.3.2 The (m)UED module 52
      4.3.3 Vertices 54
   4.4 Future of the UED module 55

5 Relic Density of Kaluza-Klein Dark Matter 57
   5.1 Annihilation only 57
   5.2 Including Coannihilations 61
   5.3 Discussion of Parameter Space 66

6 Concluding Remarks 69

A The Standard Model 70
B Masses and vertices

B.1 Useful expressions and parameters ........................................ 73
B.2 KK masses ........................................................................ 74
B.3 KK Feynman Rules ................................................................. 77
   B.3.1 The Gauge sector .......................................................... 78
   B.3.2 The Fermion sector ....................................................... 79
   B.3.3 The Higgs sector .......................................................... 81
   B.3.4 The Yukawa sector ....................................................... 83
B.4 KK number violating couplings .............................................. 85

C Other Coannihilation Channels .................................................. 88

D Tests .................................................................................... 90
1 Introduction

Ever since the beginning of the 20th century, astronomical observations have suggested the existence of Dark Matter (DM): A mysterious component of the Universe that only interacts gravitationally and makes up 75 percent of all matter [3]. Over the course of the 20th century the case for DM has strengthened, and there is now scientific consensus that DM exists, and is most probably composed of one or more new particles.

One of the most promising candidates for DM are Weakly Interacting Massive Particles (WIMPs). These occur naturally in many theories originally proposed to address unrelated problems of the Standard Model, such as supersymmetric theories [4]. WIMPs generically give the correct relic density and have the necessary properties to be DM candidates.

A WIMP can occur in theories involving one or more compactified, extra dimension. The first of these theories was developed in the 1920s by Kaluza and Klein, and is aptly called "Kaluza-Klein" (KK) theory. Kaluza-Klein theory was originally proposed as a way to unify Einstein’s general theory of relativity with electromagnetism. Although it failed, the idea and framework has laid the foundation for modern theories with extra dimensions. One of the most famous of these is String Theory, which requires 7 compactified extra dimensions to describe a consistent theory of quantum gravity.

One of the simplest extra dimensional theories is “minimal Universal Extra Dimensions” which involves one extra dimension that all SM particles can propagate in 1. As the SM particles propagate in the extra dimension they have momenta invisible to a 3D observer, which is interpreted as an extra mass term. Thus, Kaluza-Klein theories gain an infinite number of new states that are identical to the SM particles except for their masses. A new symmetry, KK parity, ensures that the lightest of these states is stable. This gives rise to a DM candidate: the lightest Kaluza-Klein particle.

The goal of this thesis has been to calculate the relic density of the KK photon as the DM candidate. This has been done several times before, but this thesis provides a new twist: Recently new contributions to the masses of the KK particles have been calculated by Freitas et al. [2]. These corrections provide new, finite terms to the radiative corrections of the KK masses and vertices. Since radiative corrections affect the phenomenology of UED, it is important to test whether new contributions to the corrections can affect the relic density.

The relic density has been calculated by implementing a UED module in DarkSUSY [5] including annihilation and the main coannihilation channels.

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1 “Minimal” refers to how radiative corrections at the boundary of the extra dimension are treated. This will be discussed in section 2.2.7.
Thesis Overview
This thesis is divided into four parts. The first part aims to introduce the reader to the basic features of Kaluza-Klein theories, and the specific field content of mUED. It is assumed that the reader has a background in quantum field theory corresponding to an introductory course, and is familiar with the construction of electroweak theory. Note that 5D QCD is not discussed because all coloured states are too heavy to have any phenomenological impact.

Since all calculations are performed at tree-level and no loops involving 3-vector couplings are involved, ghosts in 5D are also not discussed.

The second part aims to motivate the existence of dark matter, give an introduction to the standard way of calculating the relic density and motivate the LKP as the dark matter candidate. First, the standard Model of cosmology is presented and the existence of dark matter is motivated from a purely cosmological point of view. Then, the dark matter particle and its properties are discussed briefly, as well as the main experimental evidence for dark matter. The remaining part of the chapter focuses on the relic density calculation of dark matter and the LKP as the dark matter particle.

The third part of this thesis presents the calculations that have been performed and how the UED module has been implemented in DarkSUSY. The chapter starts with a review of the annihilation channels and the main coannihilation channel, and what diagrams contribute to these processes. Then, some details of the calculations are presented, such as the decay channels of the KK level 2 resonances and the calculation of the amplitude. Finally, this chapter describes the structure of the UED module implemented in DarkSUSY and what improvements might be done in the future.

The last part presents the relic density of KK dark matter including annihilation and the main coannihilation channel, as implemented in DarkSUSY. The first part focuses on the relic density when only the annihilation channels are implemented, and whether the UED module reproduces previous results. The second part presents the relic density including the main coannihilation channels. The last part of the chapter involves a discussion of the parameter space of mUED and whether it is affected by the finite corrections calculated by Freitas et al.

Since many coannihilation channels are missing it is difficult to make any specific conclusions about how the new mass contributions affect the relic density. However, through a naive estimate we have approximated the current upper bound on LKP dark matter to increase by approximately 230 GeV when the new contributions are included. This shifts the previous upper bound on the relic density from $R^{-1} < 1525$ GeV to $R^{-1} < 1785$ GeV within $5\sigma$ of the 2018 results from the Planck collaboration [3].

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2Within the scope of this thesis, this means that coannihilations including KK quarks and KK gluons will be suppressed because their masses are too large compared to the LKP.
2 Extra dimensions

Extra dimensions refers to theories that involve more than the usual three spatial and one time dimensions. These theories were first studied in the beginning of the 20th century by Nordström, Kaluza and Klein, in an attempt to unify gravity and electromagnetism. Today, extra dimensional theories seek to solve a number of different problems, such as the nature of dark matter.

Since no extra dimensions have been discovered yet, the extra dimensions are assumed to be so small that their effects are beyond the energies available in present detectors.

In this thesis, so-called Universal Extra Dimensions [6] have been studied, which is an umbrella term for theories involving one or more extra dimensions that all standard model particles can propagate in (hence, the term "universal"). These theories are interesting because they provide several dark matter candidates. For theories involving one UED, the current bound set by the LHC is at \( R^{-1} > 1.4 \text{TeV} \) [7][8], where \( R \) is the scale of the extra dimension. This bound is roughly at the appropriate scale to provide a dark matter candidate with the correct relic density [9].

This chapter has the following structure:

First, the general features of Kaluza-Klein theory will be introduced, which is the basis of all extra dimensional theories. Then, UED will be discussed in detail: In particular, the particle content in the effective 4-dimensional theory and radiative corrections.

This chapter is primarily based on the works done by Kaluza [10] and Klein [11], and by Appelquist, Cheng and Dobrescu [6].

2.1 Kaluza-Klein Theory

This chapter is based on the original work done by Kaluza and Klein in their 1921 and 1926 articles, "Zum Unitätsproblem in der Physik" [10] and "Quantentheorie und fünfdimensionale Relativitätstheorie" [11].

Kaluza’s theory was a purely classical extension of Einstein’s general theory of relativity to five dimensions. The Einstein equations in 5D get 5 additional components, four associated with the EM vector potential and one associated with an unknown scalar field:

\[
\tilde{g}_{ab} = \begin{bmatrix}
g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\
\phi^2 A_\nu & \phi^2 
\end{bmatrix},
\]

(1)

where \( a, b = 0, 1, 2, 3, 5 \), \( A_\mu \) is the EM field vector and \( \phi \) is the scalar field.

Interestingly, the five-dimensional Einstein equations yield the four-dimensional Einstein equations, the equations associated with Maxwell’s theory of EM and an equation for the scalar field, giving a unified theory of GR and EM. In addition, Kaluza introduced the "cylinder condition": No component of the five-dimensional metric depends on the fifth dimension. This was to ensure that the fifth dimension is not observable, and that the 3+1 dimensions are independent of the fifth dimension.

In 1926, Oscar Klein related Kaluza’s theory to Quantum Mechanics (QM) by proposing that the extra dimension is curled up and microscopic, to explain the cylinder condition. He also suggested that the fifth dimension is closed and periodic, i.e. a circular dimension. This interpretation led to a nice way of explaining the quantization of the electric charge:
By interpreting motion in the fifth dimension as standing waves, with wavelength $\lambda_5$, quantization of the electric charge could be understood in terms of integer multiples of the fifth dimensional momentum. By using de Broglie's relation for momentum $p^5 = \hbar/\lambda_5$, where $\hbar$ is Planck's constant, he found that the radius of the extra dimension was approximately $10^{-30}$ cm.

The original Kaluza-Klein model failed\(^3\), but it is still of importance because it has been the precursor of theories that are still studied today, like string theory and UED. Today, theories with extra dimensions are of interest because they can provide a solution to (at least) two of the major problems of modern physics: the nature of dark matter and a quantum theory of gravity.

**General Features of KK Theories**

As mentioned, Klein proposed that the compact extra dimension has the topology of a circle, $S^1$, with radius $R$. Figure 2.1 illustrates this compactification for a 4+1 spacetime. If $y = x^5$, this compactification corresponds to imposing the condition,

$$y = y + 2\pi R.$$  \hspace{1cm} (2)

Let us consider a scalar field $\phi = \phi (x^\mu, y)$, where $\mu = 0, 1, 2, 3$ denote the regular four dimensions, with mass $m$ that can propagate in 5 dimensions. It is described by the Klein-Gordon equation in 5D:

$$\left( \Box^{(5)} - \partial_y^2 + m^2 \right) \phi (x^\mu, y) = 0$$

$$\left( \partial_i^2 - \partial_y^2 + m^2 \right) \phi (x^\mu, y) = 0,$$  \hspace{1cm} (3)

where $i = 1, 2, 3$ denotes the three ordinary spatial dimensions and $\Box^{(n)} = \partial_i^2 - \partial_M^2, M = 1, 2, ..., n$.

Due to the compactification on a circle, which gives the condition in Eq. (2), the dependence on $y$ can be extracted in the following way:

$$\phi (x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n e^{i \pi y} \phi^{(n)} (x^\mu).$$ \hspace{1cm} (4)

\(^3\)The Kaluza-Klein model failed because it introduces light scalar fields, which are forbidden. In addition, it fails to produce chiral 4D fermions, which we know from experiments must exist. The problems are solved by compactifying the extra dimension on an orbifold, as will be discussed in section 2.2.
By inserting this into the Klein-Gordon equation\(^4\), we find:

\[
\left( \Box^{(4)} - \partial_y^2 + m^2 \right) \phi(x^\mu, y) = \left( \Box^{(4)} - \partial_y^2 + m^2 \right) \frac{1}{\sqrt{2\pi R}} \sum_n e^{i\frac{2\pi}{R} y} \phi^{(n)}(x^\mu) \\
= \left( \Box^{(4)} + \frac{n^2}{R^2} + m^2 \right) \frac{1}{\sqrt{2\pi R}} \sum_n e^{i\frac{2\pi}{R} y} \phi^{(n)}(x^\mu) \\
= 0 .
\]

Thus, the Fourier components \( \phi^{(n)}(x^\mu) \) satisfy the Klein-Gordon equation in 4D:

\[
\left( \Box^{(4)} + m^2 n \right) \phi^{(n)}(x^\mu) = 0, \\
\]

with the new mass,

\[
M^{(n)} = \sqrt{n^2 / R^2 + m^2} .
\]

The same procedure holds for fermions, with the fields described by the Dirac equation,

\[
\bar{\psi}(i\gamma^5 \partial_y + m) \psi = 0 .
\]

Since the Dirac equation is linear, the fermion masses are linearly dependent on the extra dimensional momentum:

\[
M^{(n)} = \frac{n}{R} + m .
\]

The momentum in the fifth dimension is interpreted as an extra mass term, since the extra energy measured in four dimensions cannot be accounted for except as an intrinsic trait of the particle. Thus, by Fourier expansion Kaluza-Klein theories acquire an infinite number of new particles, called a "tower of Kaluza-Klein states", each with a different mass, but with the same quantum numbers as the zero-mode particle. In a multi-particle theory, each Kaluza-Klein (KK) number \( n \) will give a new set of particles with masses given by eqs. (7) and (9). This only applies to the particles that are allowed to propagate in the extra dimension.

The Fourier expansion of the momentum also gives rise to a new conserved quantity, The KK number \( n \), from the conservation of momentum in the fifth dimension. This symmetry is especially important in universal extra dimensional theories, since a subgroup of this symmetry, KK parity, provides a dark matter candidate. This will be further discussed in section 2.2.1.

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\(^4\)Assuming that spacetime is flat
2.2 Universal Extra Dimensions

Universal extra dimensions (UED) denote flat, compact dimensions that all SM particles can propagate in. Instead of compactifying the extra dimension(s) on a circle, it is compacted on an orbifold $S^1/Z_2$, which ensures that the fermions in the 4D effective theory are chiral and that no light scalar fields are present. These features will be discussed further in this section.

In this thesis, we will consider a theory involving one minimal UED (mUED). Minimal UED is defined as the minimum consistent extension of the SM with extra dimensions. It is connected to the way radiative corrections arising from UV-boundary terms are treated. This will be discussed in section 2.2.7.

When referring to "UED" in the remainder of this thesis, it is implied that this refers to "mUED", with one extra dimension, unless these two abbreviations are used together.

Since all SM particles can propagate within the extra dimension, they are all accompanied by an infinite tower of KK states, with masses:

$$M_{\phi}^{(n)} = \sqrt{\frac{n^2}{R^2} + m_{EW}^2}, \quad M_{\psi}^{(n)} = \frac{n}{R} + m_{EW},$$

where $m_{EW}$ is the electroweak mass of the particle. Hence, the SM particles remain with the same mass as in the SM, while the KK particles receive an extra contribution inversely proportional to the scale of the extra dimension.

Effectively, UED models are higher-dimensional versions of the SM. However, since the SM in higher dimensions has couplings that are not dimensionless, UED is not renormalizable and should be interpreted as an effective four-dimensional theory valid up to some cutoff-scale $\Lambda$. Thus, UED adds two new independent variables: $\Lambda$ and $R$.

As mentioned briefly, it turns out that compactifying the extra dimension on a circle, as in Kaluza-Klein theory, give rise to additional massless scalar fields, because the fifth component of the 5D vector fields transform as scalars under 4D Lorentz transformations. However, light scalar fields are heavily constrained by experiments [12]. In addition, one needs to break 5D Lorentz invariance to obtain 4D chiral fermions from a 5D theory. The solution proposed in UED models, is to compactify the extra dimension on an orbifold, for example $S^1/Z_2$, which is used throughout this thesis. The $S^1/Z_2$ orbifold has the usual $S^1$ symmetry, in addition to a mirror symmetry between points that are mapped onto each other under the orbifold projection: $y \rightarrow -y$. This gives the total symmetry:

$$y \approx 2\pi R - y,$$

where $R$ is now the radius of the extra dimension from the circle $S^1$. Effectively, this corresponds to compactifying on the line segment $[0, \pi R]$.

Under this compactification, the Fourier expansion of $\phi(x, y)$ is now a subset of the general transformation since half of the states are projected out. To ensure the symmetry described by Eq. (11), we must differentiate between fields that transform even and odd
under orbifold projections. Even and odd fields transform in the following way under $Z_2$ transformations:

\begin{align*}
\phi(x^\mu, y) &\to \phi(x^\mu, -y) & \text{even fields} , \\
\phi(x^\mu, y) &\to -\phi(x^\mu, -y) & \text{odd fields} .
\end{align*}

Even and odd fields can be Fourier expanded in the following ways:

\begin{align*}
\phi_{\text{even}}(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} \phi_{\text{even}}^{(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \phi_{\text{even}}^{(n)}(x^\mu) \cos \frac{ny}{R} , \\
\phi_{\text{odd}}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \phi_{\text{odd}}^{(n)}(x^\mu) \sin \frac{ny}{R} .
\end{align*}

Hence, zero-mode fields transform even, as we would expect from the symmetry $\phi^{(n)}(x^\mu, y) = \phi^{(n)}(x^\mu, -y)$.

### 2.2.1 Kaluza-Klein Parity

The Fourier expansion of the fields give that the KK number $n$ is a measure of the particle’s momentum in the extra dimension, as shown in Eq. (5). Due to momentum conservation in the fifth dimension, the KK number is a conserved quantity. However, by compacting over the orbifold $S^1/Z_2$, translational symmetry along the extra dimension is broken, and KK number violating interactions are allowed.

A subgroup of this symmetry, called KK parity, remains after compactification. KK parity is a parity flip of the extra dimension. This symmetry comes from the general orbifold symmetry (11), because each term in the Fourier expansion of the fields can be written as:

\begin{align*}
\cos \frac{ny}{R} &= \cos \frac{n(y + \pi R)}{R} = (-1)^n \cos \frac{ny}{R} , \\
\sin \frac{ny}{R} &= \sin \frac{n(y + \pi R)}{R} = (-1)^n \sin \frac{ny}{R} .
\end{align*}

The quantum number corresponding to KK parity is given as $P = (-1)^n$, where $n$ denotes the total KK number of the interacting particles in the initial state. This quantity ensures that the ”evenness” or ”oddness” is conserved in an interaction. KK parity conservation ensures that the lightest KK particle (LKP) is stable, since any decay would violate KK parity, thus providing a dark matter candidate.
2.2.2 The 5D SM Lagrangian

The full, 5D SM Lagrangian is:

\[ \mathcal{L} = -\frac{1}{4} F_{MN} F^{MN} + i \bar{\psi} D \psi + \psi_i \lambda_{ij} \psi_j + h.c. + |D_M \phi|^2 - V(\phi), \]  

(16)

where \( N, M = 0,1,2,3,5 \) are the Lorentz indices, \( x^\mu, \mu = 0,1,2,3 \) are the four ordinary dimensions and \( y = x^5 \) is the extra dimension.

As in the SM this Lagrangian is constructed by adding all possible terms that are both Lorentz invariant and invariant under the SM symmetry group

\[ G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y \rightarrow SU(3) \times U(1)_{EM}, \]  

(17)

where the breaking of the electroweak symmetry group \( SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM} \) is what gives the SM weak gauge bosons their mass.

The covariant derivative for a theory invariant under \( G_{SM} \) is:

\[ D_M = \partial_M - i \hat{g}_3 G^a_M t^a - i \hat{g}_A A^r_M t^r - i Y \hat{g}_Y B_M. \]

There are eight SU(3) Gauge bosons: \( G^a_M \), 4 SU(2) bosons \( A^r_M \) and one U(1) boson \( B_M \).

The 5D couplings constants are related to the 4D ones according to:

\[ g = \frac{\hat{g}}{\sqrt{2\pi R}}. \]  

(18)

To arrive at an effective, 4D Lagrangian one must:

1. Assign odd/even orbifold transformation properties to the 5D SM fields, and make sure to avoid light states that are not in the 4D SM.

2. Integrate out the fifth dimension, by extracting the dependence on the fifth dimension by Fourier expanding as in Eq. (14)

3. Let the \( n = 0 \) fields be independent of the fifth dimension, and identify these as the SM fields.

4. Optional: Apply suitable gauge fixing conditions to remove Goldstone modes
2.2.3 Gauge Fields in 5D

To get an effective four-dimensional theory we first need to compactify the extra dimension on the orbifold $S^1/\mathbb{Z}_2$, with the symmetry described in Eq. (11). The fifth component of a gauge field $A_M$ transforms as a scalar under 4D Lorentz transformations. Thus, the fifth component $A_5$ is odd under the orbifold projection to avoid light scalar fields in the 4D theory, and we have that $A_5^{(0)} = 0$. To reproduce the ordinary 4D fields, we demand that the remaining component $A_\mu$ transform even. The fields then obey the following rules:

\[
A_M(x, y) = A_M(x, 2\pi R - y)
\]
\[
A_\mu(x, y) = A_\mu(x, -y)
\]
\[
A_5(x, y) = -A_5(x, -y).
\]

This also follows from gauge invariance: Since $A_\mu(x^\mu, y) = A_\mu(x^\mu, -y) \approx A_\mu(x^\mu, y) + D_\mu \theta(x^\mu, y)$, $\theta(x^\mu, y)$ must transform even. Thus, $\partial_\mu \theta(x^\mu, y)$ and $A_5$ transform odd.

With these properties, $A_\mu(x, y)$ and $A_5(x, y)$ can be Fourier expanded in the following way:

\[
A_\mu(x, y) = \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)}(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A_\mu^{(n)}(x) \cos \left( \frac{ny}{R} \right),
\]
\[
A_5(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A_5^{(n)}(x) \sin \left( \frac{ny}{R} \right).
\]

The fields $A_\mu^{(0)}$ are the ordinary, four-dimensional fields, while $A_\mu^{(n)}$ and $A_5^{(n)}$ are the n'th KK excitations of the $A_M(x)$ field.

In a theory with only one gauge field, the (kinetic) gauge part of the effective, 4D Lagrangian$^6$ becomes [13]:

\[
\begin{align*}
\mathcal{L}_{\text{gauge}}^{\text{(kin)}} &= \int_0^{2\pi R} dy \left( -\frac{1}{4} \left( \partial_M A_N - \partial_N A_M \right) \left( \partial^M A^N - \partial^N A^M \right) \\
&\quad - \frac{1}{4} \sum_{\nu=0}^{\infty} \left( \partial_\nu A^{(0)}_\nu - \partial_\nu A^{(0)}_\mu \right) \left( \partial^\mu A^{(0)}^\nu - \partial^\nu A^{(0)}_\mu \right) \\
&\quad - \frac{1}{4} \sum_{n=1}^{\infty} \left( \partial_\nu A^{(n)}_\nu - \partial_\nu A^{(n)}_\mu \right) \left( \partial^\mu A^{(n)}^\nu - \partial^\nu A^{(n)}_\mu \right) \\
&\quad + \frac{1}{2} \frac{2}{n} \sum_{n=1}^{\infty} \left( \partial_\nu A^{(n)}_\nu + \frac{n}{R} A^{(n)}_\nu \right) \left( \partial^\mu A^{(n)}_\mu + \frac{n}{R} A^{(n)}_\mu \right)
\end{align*}
\]

The $A_5$ fields are not physical fields, as they can be removed from the Lagrangian by applying the Gauge fixing condition:

\[
\theta(x^\mu, y) = -\frac{R}{n} A_5^{(n)}.
\]

In fact, they are the Goldstone modes that give KK vector modes the extra mass term related to KK number. The KK weak isospin gauge bosons get masses both from the Goldstone bosons related to the Higgs mechanism, and the scalar fields $Z_5$ and $W_{5}^\pm$ which are

$^6$The result is similar for the electroweak and strong gauge bosons [2].
the corresponding fifth components of the gauge bosons. Since photons do not receive a mass term from the Higgs mechanism, KK photons only receive a mass term from $B_5$.

2.2.4 Fermions in 5D

**Higher-dimensional fermions**

The Dirac representation is the spin-$\frac{1}{2}$ representation of the Lorentz algebra $^8$:

$$[\Sigma^{MN}, \Sigma^{OP}] = i \left( \eta^{NO} \Sigma^{MP} - \eta^{MO} \Sigma^{NP} + \eta^{MP} \Sigma^{NO} - \eta^{NP} \Sigma^{MO} \right).$$  

The generators $\Sigma^{MN}$ of the algebra are constructed of the Dirac $\gamma$-matrices $^9$ in d-dimensions, which represent the d-dimensional Clifford algebra:

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN}. \quad (24)$$

For an even number of dimensions $d = 2k + 2$, the Dirac matrices can be constructed directly from the Pauli matrices (Eq. (102)) in an iterative way $^{[14]}$. For an odd number of dimensions, $d = 2k + 3$, the Dirac matrices are the same as for $d = 2k + 2$ dimensions, but one must add:

$$i\Gamma \equiv i^{1+k}\Gamma^0\Gamma^1 \ldots \Gamma^{2k+1}. \quad (25)$$

The spinors are the eigenvectors of the operators,

$$S_a \equiv \Gamma^a + \Gamma^{-a} - \frac{1}{2}, \quad (26)$$

with eigenvalues $s = \pm 1/2$. The raising and lowering operators $\Gamma^a$, $\Gamma^{-a}$ are defined as

$$\Gamma^{0\pm} \equiv \frac{i}{2} \left( \pm \Gamma^0 + \Gamma^1 \right)$$

$$\Gamma^{a\pm} \equiv \frac{i}{2} \left( \Gamma^{2a} \pm i\Gamma^{2a+1} \right) \quad (\text{for } a = 1, \ldots, k).$$  

The Dirac representation is spanned by the spinors $s$. In an even number of dimensions, this representation can be reduced to 2 inequivalent representations called "Weyl representations". These representations only act on the subspaces $\Gamma S = \pm s$.

The $(2k + 1)$th matrix $\Gamma$ can be used to construct chirality operators:

$$P_{L/R} = \frac{1}{2} \left( 1 \pm \Gamma \right), \quad (28)$$

which project out the chiral parts of a spinor: $P_{L/R}\psi = \psi_{L/R}$. Since the $(2k + 1)$th Dirac matrix is a part of the Lorentz algebra in an odd number of dimensions, it is not possible to construct a chirality operator$^{10}$ and the representation is

---

$^7$As will be discussed later in this chapter, UED introduces four new scalar fields, two Goldstone modes and two $Z_2$ odd scalar fields, which receive mass terms from the fifth components of the weak isospin gauge bosons.

$^8$The Lorentz algebra is briefly presented in appendix A.

$^9$The $\gamma$-matrices in 4 dimensions are defined in Eq. (103).

$^{10}$In 4 + 1 dimensions this means that $\gamma^5$ is part of the Clifford algebra representation, and one cannot construct the usual chirality operators $P_{R/L} = \frac{1}{2} (1 \pm \gamma^5)$. 

17
irreducible. Thus, chiral fermions do not exist in an odd number of dimensions. However, 4D fermions are chiral and even have definite helicity: doublets are left-handed and singlets are right-handed. To ensure that the zero-level states have these properties, we must assign appropriate orbifold transformation properties of the higher-dimensional spinors:

\[
\psi_d = \frac{1}{\sqrt{2\pi R}} \psi_d^{(0)} + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left( \psi_d^{(n)} \cos \frac{ny}{R} + \psi_d^{(n)} \sin \frac{ny}{R} \right),
\]

\[
\psi_s = \frac{1}{\sqrt{2\pi R}} \psi_s^{(0)} + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left( \psi_s^{(n)} \cos \frac{ny}{R} + \psi_s^{(n)} \sin \frac{ny}{R} \right).
\]

Thus, for every SM fermion, there appears two KK fermions:

\[
\psi_s^{(n)} = \psi_s^{(n)} \gamma^5,
\]

\[
\psi_d^{(n)} = \psi_d^{(n)} \gamma^5.
\]

where \( s, d \) stands for "singlet" and "doublet" representations of SU(2). The kinetic part of the fermion Lagrangian is:

\[
L_{\text{kin fermion}} = i \bar{\psi} \gamma^M \partial_M \psi - m_{\text{EW}} \left( \bar{\psi}_s \psi_d + \bar{\psi}_d \psi_s \right).
\]

Thus, the mass eigenstates are related to the fermion doublets and singlets in the following way:

\[
\begin{pmatrix}
\mathcal{f}_d^{(n)} \\
\mathcal{f}_s^{(n)}
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha^{(n)} & \sin \alpha^{(n)} \\
\sin \alpha^{(n)} & -\cos \alpha^{(n)}
\end{pmatrix}
\begin{pmatrix}
\psi_d^{(n)} \\
\psi_s^{(n)}
\end{pmatrix}
\]

where the mixing angle \( \alpha^{(n)} \) is defined as:

\[
\tan 2\alpha^{(n)} = \frac{2m_{\text{EW}}}{m_s^{(n)} + m_d^{(n)}},
\]

including the radiative corrections to the masses (Ch. 2.2.7).

### 2.2.5 The Higgs mechanism and the Higgs field in 5D

The Higgs field is a complex doublet under SU(2) and can be written in the following way:

\[
\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^2 + i\chi^1 \\ H + i\chi^3 \end{pmatrix},
\]

where \( \chi^i \) are the SM Goldstone bosons associated with the breaking of the \( SU(2)_L \times U(1)_Y \) symmetry, while \( H \) is the Higgs boson.

The Higgs Lagrangian in 5D is:

\[
L_{\text{Higgs}} = (D_M \phi)^\dagger (D^M \phi) - V(\phi).
\]

To acquire gauge boson masses, one must choose an appropriate potential so that spontaneous symmetry breaking occurs. A typical choice is the "Mexican hat potential", which is given by:

\[
V(\phi) = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2} (\phi^\dagger \phi)^2.
\]
With this potential, the Higgs field acquires a vacuum expectation value: \( \langle \phi \rangle = \sqrt{\frac{\mu^2}{\lambda}} \).

To find the mass eigenstates of the Higgs Lagrangian we must insert the expression for the Higgs doublet given by Eq. (34), and substitute the weak eigenstates \( A_M, B_M \) with the mass eigenstates in Eq. (113). To get rid of cross-terms that mix scalar and vector states, one must add gauge-fixing terms in the Lagrangian [13]:

\[
\hat{\mathcal{L}}_{\text{gaugefix}} = -\frac{1}{2} \sum_i (G_i^r)^2 - \frac{1}{2} (G_Y^r)^2
\]

\[
G_i^r = \frac{1}{\sqrt{\xi}} \left[ \partial^\mu A_\mu^i - \xi \left( -m_W \chi^i + \partial_5 A_5^i \right) \right]
\]

\[
G_Y^r = \frac{1}{\sqrt{\xi}} \left[ \partial^\mu B_\mu - \xi \left( s_w m_Z \chi^3 + \partial_5 B_5^i \right) \right]
\]

(37)

In addition to the Higgs boson, four new particles appear after symmetry breaking: Two \( \mathbb{Z}_2 \) odd scalar fields and two Goldstone bosons. These are related to the Goldstone bosons \( \chi^3 \) and \( \chi^\pm = \frac{1}{\sqrt{2}} (\chi^1 \pm \chi^2) \) and the fifth components of the gauge bosons:

\[
a_0^{(n)} = \frac{M^{(n)}}{M_Z} \chi^{3(n)} + \frac{m_Z}{M_Z} Z_5^{(n)}
\]

\[
G_0^{(n)} = \frac{m_Z}{M_Z} \chi^{3(n)} - \frac{M^{(n)}}{M_Z} Z_5^{(n)}
\]

\[
a_\pm^{(n)} = \frac{M^{(n)}}{M_W} \chi^{\pm(n)} + \frac{m_W}{M_W} W_5^{\pm(n)}
\]

\[
G_\pm^{(n)} = \frac{m_W}{M_W} \chi^{\pm(n)} - \frac{M^{(n)}}{M_W} W_5^{\pm(n)}
\]

(38)

The masses \( M_Z^{(n)} \) and \( M_W^{(n)} \) are given by Eq. (10), with \( m_Z \) and \( m_W \) as the electroweak masses of the \( Z \) and \( W \) bosons.

The Goldstone bosons \( G_0^{(n)} \), \( G_\pm^{(n)} \) and \( A_5^{(n)} \) \footnote{Fifth component of the photon field} disappear by choosing unitarity gauge: \( \xi \rightarrow \infty \).

The scalars \( a_0^{(n)} \), \( a_\pm^{(n)} \) and the Higgs boson \( H^{(n)} \) appear as physical particles in UED.

The KK Goldstone bosons generate masses for the KK electroweak bosons: \( A_\mu^{(n)}, Z_\mu^{(n)} \) and \( W_\mu^{\pm(n)} \)

At zero-level, one recovers the SM result, and the number of degrees of freedom is preserved before and after SSB:

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 massless vectors ( A_\mu^{(n)}, B_\mu )</td>
<td>3 massive vectors ( Z_\mu, W_\mu^{\pm} )</td>
</tr>
<tr>
<td>1 complex scalar ( \phi )</td>
<td>1 scalar ( H )</td>
</tr>
<tr>
<td>1 massless gauge boson ( \gamma_\mu )</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

At higher KK levels, the number of degrees of freedom is also conserved and the particle content before and after SSB is given below:
Before\[\begin{array}{|c|c|}\hline\text{d.o.f.} & \text{After} \\
\hline 4 \text{ massless vectors } A^{(n)}_{\mu}, B^{(n)}_{\mu} & 4 \times 2 \\
1 \text{ complex scalar } \phi^{(n)} & 1 \times 4 \\
4 \text{ massless scalars } A^{(n)}_5, B^{(n)}_5 & 4 \times 1 \\
& 16 \\
\hline\end{array}\]d.o.f.

2.2.6 Higher order KK couplings

The couplings between KK particles can be divided into two main groups: Those that are in the bulk, and those that are on the boundary of the orbifold. The “bulk”, refers to the entire higher-dimensional spacetime, e.g. the 4+1 spacetime. KK number violating terms in the Lagrangian can appear on the boundaries of the orbifold, as will be discussed in section 2.2.7, while all bulk couplings are KK number conserving.

For now, let’s focus on the bulk couplings.

To find the couplings between higher-order KK particles we start out with the SM Lagrangian in Eq. (116), and apply the scheme in section 2.2.2 to arrive at the effective 4D Lagrangian. When integrating over the fifth dimension, one finds that only certain combinations of the fields remain. The general expressions involving three- and four-point couplings of higher order KK levels are given below.\(^\text{12}\).

Three-point vertices

Let \(X, Y, Z\) denote even fields and \(U, V\) denote odd fields. The only remaining three-point vertices are the combinations: \(XYZ\) and \(XUV\):

\[
\int_0^{2\pi R} dy XYZ = \frac{1}{\sqrt{2\pi R}} \left\{ X^{(0)} Y^{(0)} Z^{(0)} + \sum_{n=1}^{\infty} \left[ X^{(0)} Y^{(n)} Z^{(n)} + X^{(n)} Y^{(0)} Z^{(n)} + X^{(n)} Y^{(n)} Z^{(0)} \right] + \frac{1}{\sqrt{2}} \sum_{n,k,l=1}^{\infty} X^{(n)} Y^{(k)} Z^{(l)} \right\} ,
\]

(39)

\[
\int_0^{2\pi R} dy XUV = \frac{1}{\sqrt{2\pi R}} \left\{ \sum_{n=1}^{\infty} X^{(0)} U^{(n)} V^{(n)} + \frac{1}{\sqrt{2}} \sum_{n,k,l=1}^{\infty} X^{(n)} U^{(k)} V^{(l)} \right\} .
\]

(40)

\(^\text{12}\)These expressions are generalizations of equations (A.131)-(A.133) and (A.137)-(A.138) in [13]
Four-point vertices

Again, let $W, X, Y, Z$ denote even fields and $S, T, U, V$ denote odd fields. The only remaining vertices consists of either four even fields, four odd fields or two even and two odd fields. It is also useful to define the factors [2]:

\[
\Delta_{mn} = \delta_{l,m+n} + \delta_{n,l+m} + \delta_{m,l+n}
\]

\[
\Delta^1_{mnlk} = \delta_{k,l+m+n} + \delta_{l,m+n+k} + \delta_{m,n+k+l} + \delta_{n,k+l+m} + \delta_{k+l,m+n} + \delta_{k+n,l+m}
\]

\[
\Delta^2_{mnlk} = -\delta_{k,l+m+n} - \delta_{l,m+n+k} - \delta_{m,n+k+l} - \delta_{n,k+l+m} + \delta_{k+l,m+n} + \delta_{k+n,l+m}
\]

\[
\Delta^3_{mnlk} = -\delta_{k,l+m+n} - \delta_{l,m+n+k} + \delta_{m,n+k+l} - \delta_{n,k+l+m} - \delta_{k+l,m+n} + \delta_{k+n,l+m}
\]

We get the following four-point couplings in the effective, 4D Lagrangian:

\[
\int_0^{2\pi R} dy W X Y Z = \frac{1}{2\pi R} \sum_{n=1}^{\infty} \left\{ W^{(0)} X^{(0)} Y^{(0)} Z^{(0)} + W^{(n)} X^{(n)} Y^{(n)} Z^{(n)} \right. \\
+ W^{(n)} X^{(0)} Y^{(n)} Z^{(0)} + W^{(n)} X^{(0)} Y^{(0)} Z^{(n)} \\
+ W^{(0)} X^{(n)} Y^{(n)} Z^{(0)} + W^{(0)} X^{(0)} Y^{(n)} Z^{(n)} \\
+ W^{(0)} X^{(0)} Y^{(n)} Z^{(n)} \right\} \\
+ \frac{1}{2\pi R} \sum_{n,l,m=1}^{\infty} \left\{ W^{(n)} X^{(m)} Y^{(l)} Z^{(0)} + W^{(n)} X^{(m)} Y^{(0)} Z^{(l)} \right. \\
+ W^{(n)} X^{(0)} Y^{(m)} Z^{(l)} + W^{(0)} X^{(n)} Y^{(m)} Z^{(l)} \right\} \Delta_{mn}
\]

\[
+ \frac{1}{4\pi R} \sum_{n,l,m,k=1}^{\infty} W^{(n)} X^{(m)} Y^{(l)} Z^{(k)} \Delta^1_{mnlk}
\]

\[
\int_0^{2\pi R} dy S T U V = \frac{1}{4\pi R} \sum_{n,l,m,k=1}^{\infty} S^{(n)} T^{(l)} U^{(m)} V^{(k)} \Delta^2_{mnlk}
\]

\[
\int_0^{2\pi R} dy S T X Y = \frac{1}{2\pi R} \sum_{n,l,m,k=1}^{\infty} \left\{ X^{(0)} Y^{(0)} S^{(n)} S^{(n)} + \frac{1}{2} X^{(n)} Y^{(m)} S^{(l)} S^{(k)} \right\} \Delta^3_{mnlk}
\]
2.2.7 Radiative Corrections

The current lower bound on the inverse scale $1/R$ is approximately 1.4 TeV [8]. From equation (10), this means that all KK masses are nearly degenerate, since the electroweak masses are typically small compared to $1/R$. Note that the top mass is still sufficiently large so that the KK top quarks will be considerably heavier than the rest of the KK particles and the mixing will be stronger.

However, new terms in the Lagrangian arise when the extra dimension is compactified, leading to radiative corrections of the masses and couplings. These corrections are generally much larger than the electroweak masses, e.g. the lepton corrections are $\sim 10^4$ larger than $m_{EW}$, and are thus important for the phenomenology of the UED model.

The corrections come from two main sources: Compactification over a circle $S^1$, bulk corrections, and compactification over the orbifold $S^1/Z_2$.

Bulk Corrections from compactification

The compactification over $S^1$ breaks Lorentz invariance at long distances, which means that the KK masses generally receive radiative corrections. The corrections arise in Feynman diagrams that have an internal loop which winds around the circle of the extra dimension, as in figure 2.2.

These Feynman diagrams are sensitive to the breaking of the Lorentz symmetry. This is a non-local effect since the size of the loop cannot shrink to zero, and thus the corrections are finite and well-defined.

The finite corrections arising from 5D Lorentz violation can be isolated from the divergent Lorentz invariant corrections in the (renormalizable) 4D theory by subtracting every loop diagram in the compactified theory with the corresponding diagram from the uncompactified theory [15]. This removes the divergent terms since both theories are equal at small distances. The KK mass corrections remain unchanged because the subtraction is Lorentz invariant in five dimensions.
For the standard model in 5D, this gives the corrections:\[15\]:

\[
\begin{align*}
\delta (m_{B_n}^2) &= -\frac{39}{2} g_Y^2 \zeta(3) \left(\frac{1}{R}\right)^2 \\
\delta (m_{W_n}^2) &= -\frac{5}{2} g_Y^2 \zeta(3) \left(\frac{1}{R}\right)^2 \\
\delta (m_{g_n}^2) &= -\frac{3}{2} g_Y^2 \zeta(3) \left(\frac{1}{R}\right)^2 \\
\delta (m_{A_0}^2) &= 3 \delta m_{A_0}^2 \\
\delta (m_{f_{n,s}}) &= \delta (m_{f_{n,d}}) = 0 \\
\delta (m_{H_n}^2) &= 0
\end{align*}
\tag{44}
\]

where the factor \(\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202\). To derive the expressions above, Cheng et al. [15] have calculated the mass corrections for a general theory with fields of spin 0, \(\frac{1}{2}\) and 1 and added the appropriate group theory and multiplicity factors in table 1.

### Corrections from orbifold compactification

The compactification on \(S^1/Z_2\) breaks 5D Lorentz and translational invariance at the orbifold boundaries. This gives rise to radiative corrections that generate extra terms in the Lagrangian which contribute to masses and mixings of KK modes. These corrections can be found by following the work done in [16] and [15]. Their solution can be summarized as follows:

Calculate the loop diagrams with modified propagators written in terms of unconstrained fields. This gives rise to log-divergent terms that can be absorbed into the cutoff \(\Lambda\), assuming that \(\Lambda\) is not too large. This is a self-consistent assumption since the boundary terms generated by radiative corrections are loop-suppressed. By assuming that the boundary terms are small, each KK mode receives a mass contribution that is proportional to \(\Lambda\) and no mixing between the KK modes occur.

Note that the extra terms can also include KK number violating interactions, and thus be phenomenologically important. However, in mUED it is assumed that KK number is approximately conserved in the 5D theory (which is non-renormalizable), so that the divergent boundary terms are negligible.

Previously, only the radiative corrections proportional to \(\log(\Lambda)\) have been computed, even though new finite terms also arise due to orbifold compactification. In a recent article by Freitas, Kong and Wiegand [2] from late 2017, these finite corrections have been computed. Since these results are so new, they have been a major motivation for studying Kaluza-Klein dark matter and producing this thesis. According to Freitas et al., the KK modes receive
The following contributions from orbifold compactification:

\[
\bar{\delta}m^2_{V_n} = m_n^2 \frac{g^2}{32\pi^2} \left( C(G) \left( \frac{23}{3} L_n + \frac{154}{9} \right) - \sum_{i \in \text{scalars}} (-1)^{P_i} T(r_i) \left( \frac{1}{3} L_n - \frac{4}{9} \right) \right)
\]

\[
\bar{\delta}m^2_{S_{+n}} = m^2 + m_n^2 \frac{1}{32\pi^2} \left[ C(r) g^2 (6L_n + 16) - \sum_{i \in \text{scalars}} (-1)^{P_i} \lambda_i (L_n + 1) \right]
\]

\[
\bar{\delta}m_{f_n} = m_n \frac{1}{64\pi^2} \left[ C(r) g^2 (9L_n + 16) - \sum_{i \in \text{scalars}} (-1)^{P_i} h_i^2 (L_n + 2) \right]
\]

where \( V_n \) refers to KK vector particles, \( S_{+n} \) to \( \mathbb{Z}_2 \) even KK scalars and \( f_n \) to KK fermions. \( \mathbb{Z}_2 \) odd KK scalars receive the correction [15]:

\[
\bar{\delta}m^2_{S_{-n}} = m_n^2 \frac{1}{32\pi^2} \ln \frac{\Lambda^2}{1/R^2} \left[ 9g^2 C(r) + \sum_{\text{even scalars}} \frac{\lambda_{++}}{2} - \sum_{\text{odd scalars}} \frac{\lambda_{--}}{2} \right]
\]

The group theory factors \( C(G), C(r) \) and the Dynkin index \( T(r) \) depend on the specific gauge group \( G \) and representation \( r \) that the field belongs to. They are defined in table 1, for the \( SU(N) \) and \( U(1) \) groups and fundamental representation.

Further, \( m_n = n/R \), and the sums run over the different scalar fields in the theory with \( h_i = g m_i \) as the Yukawa couplings, \( P_i = (-1)^{n_i} \) as the KK parity and \( \lambda_i \) as a scalar 4-point coupling.

Table 1: Definition of the group theory factors present in the general expressions for the mass corrections due to orbifold compactification. \( r \) is the fundamental representation.
Figure 2.3: Comparison of the masses of level 1 KK particles with and without the finite terms from Ref. [2], when $R^{-1} = 1000\text{GeV}$ and $\Lambda = 20/R$, as implemented in the UED module. The dark blue lines indicate the masses of the level 1 KK particles without the finite terms, using the mass corrections from Ref. [15], whereas the light blue lines indicate the masses with these corrections. The masses are given in GeV. Note that the lower line denotes the KK Higgs and the KK down singlet masses.
Gauge boson masses
The complete expressions for the masses including the old and new corrections from the orbifold can be found in appendix B.

The Z bosons and photons, and their excitations, are the mass eigenstates of the mass matrix that connects the neutral gauge bosons $A_3^{(n)}$ and $B_\mu^{(n)}$:

$$
\begin{pmatrix}
\frac{n^2}{R^2} + \delta m_{B(2)}^2 + \frac{1}{3} g_Y^2 v^2 & -\frac{1}{3} g_Y g v^2 \\
-\frac{1}{3} g_Y g v^2 & \frac{n^2}{R^2} + \delta m_{A_3(2)}^2 + \frac{1}{3} g_Y^2 v^2
\end{pmatrix},
$$

By rotating this matrix, one finds that the eigenstates are:

$$
Z_\mu^{(n)} = \cos \theta_W^{(n)} A_\mu^{(n)} - \sin \theta_W^{(n)} B_\mu^{(n)}
$$

$$
A_\mu^{(n)} = \cos \theta_W^{(n)} B_\mu^{(n)} + \sin \theta_W^{(n)} A_\mu^{(n)}
$$

where $\theta_W^{(n)}$ is the n'th Weinberg angle, and is given by

$$
\theta_W^{(n)} = \frac{1}{2} \arctan \left[ \frac{v^2 g_Y g}{\frac{1}{2} v^2 (g_Y^2 - g_Z^2) + 2 \delta m_{B(2)}^2 - 2 \delta m_{A_3(2)}^2} \right].
$$

The masses of the KK Z bosons and photons are:

$$
m^2_{Z(2)} = \frac{n^2}{R^2} + \frac{1}{2} \left( m_Z^2 + \delta m_{A_3(2)}^2 + \delta m_{B_\mu(2)}^2 + \sqrt{4 m_W^2 (\delta m_{A_\mu(2)}^2 - \delta m_{B_\mu(2)}^2) + \left( m_Z^2 - \delta m_{A_3(2)}^2 + \delta m_{B_\mu(2)}^2 \right)^2} \right)
$$

$$
m^2_{A_\mu(2)} = \frac{n^2}{R^2} + \frac{1}{2} \left( m_Z^2 + \delta m_{A_3(2)}^2 + \delta m_{B_\mu(2)}^2 - \sqrt{4 m_W^2 (\delta m_{A_\mu(2)}^2 - \delta m_{B_\mu(2)}^2) + \left( m_Z^2 - \delta m_{A_3(2)}^2 + \delta m_{B_\mu(2)}^2 \right)^2} \right)
$$

KK number violating couplings
Because the KK number symmetry breaks after orbifold compactification, KK number violating couplings are allowed. These couplings are important for the phenomenology of the theory because they can provide new decay channels. The couplings are induced through loops generated by KK number violating terms at the boundaries of $S^1/Z_2$. These couplings are especially important for level 2n KK particles, that can decay into SM particles, without violating KK parity. In some cases, the KK number violating decay channels dominate, as for the KK particles $\gamma^{(2)}$, $Z^{(2)}$, $a_0^{(2)}$, $a_\pm^{(2)}$ and $H^{(2)}$ [9].

These particles are especially important in the calculation of relic density. Allowing level 2 KK particles in the final state, decreases the relic density significantly\textsuperscript{13}. The KK number violating vertices involving these particles are given in B.4.

\textsuperscript{13}See section 5.2

26
3 Dark Matter

The existence of dark matter was suggested by astronomical observations as early as 1884, when Lord Kelvin discovered that the mass of the Milky Way galaxy was larger than the weight of visible stars, by measuring the velocity dispersion of stars orbiting the center of the galaxy [17]. During the second half of the 20th century several important observations have further motivated the existence of dark matter, such as measurements of galaxy rotation curves [18], the cosmic microwave background (CMB) [3] and gravitational lensing due to galaxies and clusters [19].

Today, there is general consensus among physicists that dark matter must be composed of one, or several, new elementary particle(s).

In this chapter the main motivations for dark matter will be presented, as well as the basic cosmology of the Universe and relic density calculations. For a more detailed, extensive review of the present Universe, chapter 8 of Carroll’s “Spacetime and Geometry” [20] is recommended. A more detailed summary than what is presented here of the current evidence for and status of dark matter can be found in Ref. [21]. A thorough review of the calculation of relic density and the derivation of the Boltzmann equation can be found in Refs. [22] and [23]. The standard calculations of relic density presented in this chapter are based on these works.

The second part of the chapter focuses on the lightest Kaluza-Klein particle as the dark matter candidate, and its experimental constraints. This part is based on Refs. [24] and [9].

3.1 The Friedmann-Robertsson-Walker Universe

The ΛCDM model is referred to as the standard model of cosmology because it is the simplest universe model that describes the Universe. This model includes four spacetime dimensions, and three dominant components that make up the Universe: Ordinary matter, cold dark matter (hence, CDM) and vacuum energy associated with the cosmological constant, Λ. Adding a fifth compactified dimension will not change the cosmology, if we assume the extra dimension is stable.

From astronomical observations, we know that the universe is spatially isotropic at large scales (a few to 10% of the visible universe) and currently has accelerated expansion. The Universe is also homogeneous at large scales. A spatially isotropic and homogeneous universe can be described by the Friedmann-Robertsson-Walker (FRW) metric, given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] ,$$

(50)

where $k$ is a constant that depends on the geometry of the spatial part of the universe.

For a flat universe, $k = 0$, while a universe with positive/negative curvature will have $k = \pm 1$. The scale factor $a(t)$ is a dimensionless quantity, and must be positive to ensure expansion. It is defined as

$$a(t) = \frac{d(t)}{d_0} ,$$

(51)

where $d(t)$ is the proper distance between two objects at time $t$, and $d_0$ is the distance at a reference time $t_0$. The scale factor $a(t)$ describes the relative expansion of the universe.
Table 2: Equation of state for each type of cosmic fluid, assuming they are all perfect fluids.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>e.o.s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dust</td>
<td>( p = 0 )</td>
</tr>
<tr>
<td>Radiation</td>
<td>( p = \frac{1}{3} \rho )</td>
</tr>
<tr>
<td>Vacuum</td>
<td>( p = -\rho )</td>
</tr>
</tbody>
</table>

Note that the FRW metric is written in terms of *comoving* coordinates. These are the coordinates of an observer that observes an isotropic universe, and will not notice the blue- or red-shifts due to expansion.

It is common to assume that all matter (non-relativistic particles) and energy in the universe behaves as a perfect fluid, i.e. a fluid without viscosity and thermal conductivity. Such a fluid can be described by the stress-energy tensor:

\[
T^\mu_\nu = \text{diag}(-\rho, p, p, p),
\]

where \( \rho \) is the isotropic density and \( p \) is the pressure of the fluid.

The Universe consists of either relativistic or non-relativistic matter. One can approximate these limits into three types of cosmic fluids: Dust, radiation and vacuum. Dust refers to non-relativistic fluids, known as "matter", radiation refers to relativistic fluids and vacuum refers to a fluid that is related to Einstein's cosmological constant \( \Lambda \). The equations of state for each type of fluid are given in table 2.

The vacuum energy density can be related to Einstein's cosmological constant:

\[
\rho_v = \frac{\Lambda}{8\pi G},
\]

which implies that the vacuum energy density is constant. This means that any expanding universe will eventually be dominated by vacuum energy, since expansion leads to a decrease of the energy density of matter and radiation. Although adding a term of this kind fits surprisingly well with observations, is it not certain whether we have observed a cosmological constant or not. There is a possibility that what has been observed is a dynamical component of the Universe that mimics the properties of vacuum energy. To take into consideration both of these possibilities, the term "Dark energy" has been introduced to describe what has been detected.

The behaviour of the scale factor \( a(t) \) can be found by inserting eqs. (50) and (52) into Einstein's field equations:

\[
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}.
\]

The resulting cosmological equations of motion, are called the Friedmann equations:

\[
\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda a^2}{3} - k
\]
\[
\ddot{a} = -\frac{8\pi G}{6} (\rho + 3p)a + \frac{\Lambda a}{3}.
\]
Since the Universe is currently undergoing accelerated expansion, both $\dot{a}$ and $\ddot{a}$ are positive. The cosmological constant is necessary to have accelerated expansion when the Universe is dominated by vacuum energy\textsuperscript{14}. The first Friedmann equation can be rewritten in terms of the density parameter. The density parameter is defined as,

$$\Omega = \frac{8\pi G}{3H^2} = \frac{\rho}{\rho_{\text{crit}}},$$

(56)

where $H$ is the Hubble parameter and $\rho_{\text{crit}}$ is the critical density:

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}.$$  

(57)

$H$ is defined as

$$H = \frac{\dot{a}}{a},$$

where the "dot" denotes $\frac{d}{dt}$. The Hubble parameter is a quantity that is used to characterize the rate of expansion, and the present day expansion rate is described by $H_0$.

The geometry of the Universe depends on the density parameter. This comes from the first Friedmann equation, which can be written as,

$$\Omega - 1 = \frac{k}{H^2 a^2},$$

so that $\Omega = 1$ corresponds to a flat universe, while $\Omega > 1$ and $\Omega < 1$ corresponds to a universe with negative and positive curvature. Current measurements of the cosmic microwave background indicates that the Universe is flat [3].

It is useful to express the curvature contribution as an energy density, by defining the curvature density parameter,

$$\Omega_c = -\frac{k}{H^2 a^2}.$$  

(58)

The first Friedmann equation written in terms of the density parameter is:

$$1 = \sum_i \Omega_i = \Omega_m + \Omega_\Lambda + \Omega_c + \Omega_r,$$

(59)

where $\Omega_m, \Omega_\Lambda$ are the matter and vacuum energy density parameters, and $\Omega_r$ is the radiation density parameter.

Since both dark matter and ordinary, baryonic matter are non-relativistic, they cannot be distinguished by using the Friedmann equations alone since they have the same equation of state.

By measurements of the gravitational effects of clustered matter in the Universe, it has been found that the total matter density parameter is currently [3]:

$$\Omega_m h^2 = 0.1430 \pm 0.0011 \quad (68\%, \ \text{Planck } TT, \text{TE, EE} + \text{lowE} + \text{lensing}),$$

\textsuperscript{14}This can be worked out directly from the second Friedmann equation by inserting the proper e.o.s. and the definition $\rho_v$. 

29
where $h$ is the present day Hubble parameter in units of $100\text{km s}^{-1}\text{Mpc}^{-1}$. This implies that the Universe is made up of approximately 30% matter. It is possible to measure the density parameter of baryonic matter separately by various methods, such as direct counting of baryonic matter or comparison with the CMB power spectrum. These measurements have shown that the current baryonic density parameter is [3]:

$$\Omega_b h^2 = 0.02237 \pm 0.00015 \,(68\%, \text{ Planck TT, TE, EE} \,, \text{lowE} \,, \text{lensing})$$

Thus, the Universe must be made up of a large component of dark matter today. The results from Planck [3] imply that there is approximately 5 times as much dark matter as ordinary matter in the Universe today.

### 3.2 The Dark Matter particle

The last section demonstrated that a non-relativistic, non-baryonic component of the Universe, coined Dark Matter, is highly necessary to explain the observations of the CMB. Several candidates have been proposed and refuted, such as primordial black holes, neutrinos and MACHOs (Massive Compact Halo Objects). Today most scientists agree that the discrepancy between the total and baryonic matter density must be due to undiscovered particles that only interact gravitationally, and possibly weakly with other particles. A few percent of the dark matter consists of baryonic objects that emit little to no radiation, such as black holes and brown dwarfs, but the density of these objects are far from ruling out DM as a new particle [25].

Based on the most recent observations by the Planck Collaboration [3], the DM density is approximately $\Omega_{CDM} h^2 = 0.120 \pm 0.001$. In total, dark matter makes up approximately 25% of the total energy density of the Universe and 85% of the total matter density.

The current observational evidence of dark matter suggests that it has the following properties [21]:

- **Electrically neutral:** DM does not interact electromagnetically. This means that it does not emit/absorb/reflect any kind of radiation. Hence it is, astronomically speaking, "dark" [15].

- **Non-relativistic:** From simulations of the structure formation in the early universe we know that DM must be cold (non-relativistic) if our simulations are to match with what we can observe.

- **Non-baryonic:** From observations we know that only $4 - 5\%$ [3] of the total energy density of the universe is made up of baryonic matter. However, the total matter energy density is approximately 30%. Hence, dark matter must be non-baryonic to account for the missing density.

- **Long-lived:** DM must have a lifetime at least comparable to cosmological time scales, so that it has been around from freeze-out until today.

---

[15] A better name would be “transparent” since DM does not absorb light.
Experimental Evidence for Dark Matter

There are several astronomical observations that suggest the existence of dark matter. In section 3.1, it has already been demonstrated that the observed matter density in the universe is much larger than what can be accounted for by baryonic matter, meaning that dark matter, or some unknown form of matter, must exist.

This is one of the most compelling arguments for dark matter, because it gives the most precise value of the dark matter relic density. Other important astronomical evidence are [26][27]:

- **Galaxy rotation curves:** The density of luminous matter in spiral galaxies scale as $\rho(r) \propto r^{-3.5}$ as a function of the distance $r$ from the center. According to Kepler's third law, the velocity of a star at $r$ should be $v(r) = \sqrt{GM(r)/r}$, where $M(r)$ is the total mass of the galaxy within the star's orbit. This implies that the velocity decreases as $r$ increases. However, observations by e.g. Vera Rubin [18] of spiral galaxies, like the Andromeda galaxy, have shown that the rotation velocity becomes constant for large $r$. The observed velocities indicate that the galaxy density goes as $\rho(r) \propto r^{-2}$. This suggests that some other matter is present, which is not as concentrated in the center of the galaxy as the luminous matter. The disparity between expected and observed galaxy rotation curves is widely used as an argument for dark matter because it involves simple, Newtonian mechanics. However, it is important to note that this argument is one of the weaker for dark matter because theories of modified gravity can reproduce galaxy rotation curves similar to those observed of spiral galaxies.

- **Structure formation:** If only ordinary matter existed, the large-scale structures we see today would not have had time to form. This is because ordinary matter is affected by radiation. In the early universe, the density perturbations would be diluted by radiation, and thus not be able to grow fast enough. Since dark matter is not affected by radiation, its density perturbations grew faster creating gravitational potential "wells" that attracted the ordinary matter and thus hastened the structure formation process.

- **Gravitational lensing:** One of the consequences of general relativity is that a massive object positioned between some source and an observer will bend the light coming from the source. The more massive the object, the more the light will bend. Observations of gravitational lensing due to clusters and galaxies indicate that dark matter must exist to account for the observed lensing effect.

As discussed above, the gravitational effects of dark matter are relatively well-known. However, dark matter has yet to be detected. There are three main detection methods of dark matter: Production in a collider, direct detection and indirect detection. DM production in a collider is the process where two SM particles collide and form a DM pair. DM would be characterized in a detector by missing transverse energy because the DM particle does not interact with the detector. DM production in colliders are especially interesting for WIMP dark matter because the electroweak scale is probed by the LHC. In addition, a collider would provide a controlled environment where both the DM candidate and the mediator, the force-carrying particle that mediates between DM and SM particles, can be studied.

Direct detection experiments aim to measure the nuclear recoil of dark matter particles.
scattering off nucleons [28][29]. After the recoil, the dark matter particle will emit energy in the form of scintillation light (a flash of light produced in a transparent material by the passing of a particle) or phonons (collective excitation of the atoms in the detector), which can be detected. Direct detection experiments are sensitive to background processes, and thus often situated underground to reduce the background from cosmic rays. Indirect detection measures the products of dark matter annihilation, such as gamma rays or particle-antiparticle pairs. The flux of the radiation produced in DM annihilation is proportional to the square of the DM density, \( \Gamma \propto \rho_{DM}^2 \). Thus, DM annihilation would lead to an excess of gamma rays, positrons and antiprotons in places where DM accumulates, such as the galactic center [30].

### 3.2.1 Weakly Interacting Massive Particles

Weakly interacting massive particles (WIMP’s) are one of the most promising candidates for DM. As evident from their name, they are particles that only interact weakly and have a mass around 100 GeV - \( \mathcal{O}(\text{TeV}) \). There are several reasons why WIMP’s are so attractive, the main one being what is referred to as the "WIMP miracle": In many theories, the annihilation cross section of a particle is determined by its mass. The annihilation cross section can then be approximated, on dimensional grounds, by,

\[
\sigma v = k \frac{g^4}{16\pi^2 m^2},
\]

where \( g \) is the \( SU(2) \) coupling constant and \( k \) is some deviation from \( g \). The relic density today can be approximated as,

\[
\Omega h^2 \sim \frac{10^{-26} \text{cm}^3/s}{\langle \sigma v \rangle} \approx 0.12.
\]

When we let \( k \) vary from 0.5 to 2, we get that the mass of the DM particle must lie between 100 GeV and \( \mathcal{O}(\text{TeV}) \) to get the observed relic density. Since this is already the relevant mass scale for WIMP’s, they naturally give the correct relic density.

The WIMP miracle refers to the fact that several theories addressed to solve unrelated problems of the SM independently predict stable WIMP candidates that give the observed relic density.
Figure 3.2: From [31]: Illustration of the evolution of the number density as a function of $x = m/T$, where $m$ is the mass of the particle and $T$ is the temperature of the surrounding universe. The dashed lines show the comoving number density, while the solid line shows the equilibrium number density. The freeze-out is indicated where the comoving depart from the equilibrium number density.

3.3 Relic Density of Dark Matter

3.3.1 The Boltzmann Equation

The early universe was so dense and hot that DM particles existed in thermal equilibrium with SM particles. As the rate of interaction became smaller than the rate of expansion, the comoving number density of the DM particles became approximately constant. This is what is referred to as freeze-out. An illustration of this process is presented in figure 3.2. Note that the number density according to a non-comoving observer will continue to decrease due to expansion. The number density of a particle without any interaction will only be affected by the expansion of the universe. The number density is $n = N/V = NR^{-3}$, where $R(t) = a(t)R_0$ (from Eq. 51). The time-evolution of the number density is related to the Hubble parameter:

$$\frac{dn}{dt} = \frac{dn}{dR} \frac{dR}{dt} = -3n \frac{\dot{R}}{R} = -3Hn .$$

The number density will be affected by annihilation and production processes. This gives two new terms, one that accounts for the annihilation processes: $\langle \sigma_{\text{ann}}v \rangle n^2$ and an arbitrary function that accounts for the production: $\psi$ (the annihilation term must be proportional to $n^2$, while the production term has no such restriction). $v$ is the invariant relative velocity \footnote{The velocity of one one particle in the rest frame of the other.}, and $\langle \rangle$ denotes the thermal average.

The number density is then given by:

$$\frac{dn}{dt} = -3Hn - \langle \sigma_{\text{ann}}v \rangle n^2 + \psi .$$

(62)
Consider a static universe: \( \frac{dn}{dt} = 0 \) defines the equilibrium distribution \( n_{eq} \). Detailed balance requires that the number of created particles must be equal to the number of destroyed particles. The reaction partners of the DM are assumed to be in equilibrium, so we can replace: \( \psi = \psi_{eq} = (\sigma_{\text{ann}}v)n_{eq}^2 \). This gives the Boltzmann equation:

\[
\frac{dn}{dt} + 3Hn = -\langle \sigma v \rangle (n^2 - n_{eq}^2).
\]

(63)

A rigorous derivation of the Boltzmann equation has been performed by Gondolo and Gelmini in [23].

### 3.3.2 Standard calculation of relic density

As described in the previous section, the evolution of the number density of a particle in an expanding universe can be described by the Boltzmann equation:

\[
\frac{dn}{dt} + 3Hn = -\langle \sigma v \rangle (n^2 - n_{eq}^2),
\]

(64)

where \( n_{eq} \) is the number density at thermal equilibrium and \( \langle \sigma v \rangle \) is the thermal average of the total annihilation cross section times the relative velocity of the particle. Since WIMPs are both massive and non-relativistic, we are interested in the non-relativistic limit of \( n_{eq} \). This is given by

\[
n_{eq} = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-m/T},
\]

(65)

where \( m \) is the mass of the particle, \( T \) is the temperature and \( g \) is the number of d.o.f. of the relic particle.

We introduce a new parameter: \( Y = \frac{n}{s} \), where \( s \) is the entropy density. The entropy is conserved in a comoving volume, and is given by \( sa^3 = \text{const.} \) Thus, \( Y \) takes into account the dilution of the number density due to the expansion of space, since \( s \propto a^{-3} \). The entropy density is given by \( s = 2\pi^2 g_s T^3/45 \), where \( g_s \) is the number of relativistic d.o.f., and is defined as:

\[
g_s = \sum_{i=\text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left( \frac{T_i}{T} \right)^4
\]

The Boltzmann equation (64) can be rewritten as:

\[
\frac{dn}{dt} = s \frac{dY}{dt} - 3Hn
\]

(66)

\[
s \frac{dY}{dt} = -\langle \sigma v \rangle s^2 (Y^2 - Y_{eq}^2)
\]

We now introduce the parameters

\[
x \equiv \frac{m}{T}
\]

\(^{17}\)Assume equilibrium
and \( \Delta = Y - Y_{eq} \). The Boltzmann equation can then be further rewritten:

\[
\Delta' = -Y'_{eq} - f(x)\Delta(2Y_{eq} + \Delta),
\]

where the "prime" denotes \( \frac{d}{dx} \) and \( f(x) = \sqrt{\frac{\pi g_*}{45}} M_{Pl} m \langle \sigma v \rangle x^{-2} \), where \( M_{Pl} \) is the Planck mass.

This expression is quite useful, because it can be solved analytically in two distinct regions: Very long before, and very long after the freeze-out, which occurs at temperature \( T_F = x_F m/T_F \). Very long after freeze-out, we have \( x \gg x_F \), and \( \Delta' = -f(x)\Delta^2 \), which can be solved analytically. To find the relic density today a couple of simplifications are needed. We assume that the DM particle is heavy (consistent with the LKP based on the current bound on \( R^{-1} \)), and can then approximate the thermal average of the cross section:

\[
\langle \sigma v \rangle = a + b \langle v^2 \rangle + O(\langle v^4 \rangle) \approx a + 6b/x.
\]

Further, we assume that \( \Delta x_F \gg \Delta_0 \), where the subscript "0" denotes the present value. This is a fair assumption because the Universe has expanded after freeze-out, and no new DM particles have been produced.\(^{18}\) The Boltzmann equation reduces to,

\[
\Delta' = -f(x)\Delta^2,
\]

which gives:

\[
Y_0^{-1} = \sqrt{\frac{\pi g_*}{45}} M_{Pl} m x_F^{-1}.
\]

The present dark matter density for any particle is simply given as the number density times the mass:

\[
\rho_X = m_X n_X = m_X s_0 Y_0,
\]

where \( s_0 \) is the total entropy density. The dark matter density in terms of the critical density is:

\[
\Omega = \frac{8\pi G}{3H_0^2} m_X s_0 Y_0
\]

### 3.3.3 Including coannihilations

According to Griest and Seckel [22], the standard calculation of the relic abundance fails if one or more particles have a mass that is similar to the relic particle and if they share a quantum number. The following calculations are based on their works. If the mass difference \( \delta m = m - m_\chi \) is much larger than the freeze-out temperature, the \( \chi \)-annihilations will freeze out and the extra particles that are present won’t have an effect on the relic density. However, if \( \delta m \approx T_F \), then the extra particles are thermally accessible and can be nearly as abundant as the relic particle species.

In the latter case, one needs to take into account coannihilations: Coannihilations are processes where a pair of particles mutually annihilate as they collide.

Consider a system with \( N \) particles \( \chi_1, \chi_2, \ldots, \chi_N \) with masses \( m_\chi \), so that

\[
m_{\chi_1} < m_{\chi_2} < \ldots < m_{\chi_N}.
\]

\(^{18}\) Assume that all higher-mass KK particles are sufficiently unstable so that all have decayed to the LKP.
There are three types of reactions that change the number density of these particles, and thus determine their abundance in the early universe:

\begin{align}
\chi_i \chi_j &\leftrightarrow XX' \quad \text{(73a)} \\
\chi_i X &\leftrightarrow \chi_j X' \quad \text{(73b)} \\
\chi_i &\leftrightarrow \chi_j XX' \quad \text{(73c)}
\end{align}

where $i$ and $j$ are different in (73b) and (73c) and $X, X'$ are any SM particles. If the third process can take place at a reasonable rate, it is assumed that all $\chi_i$ particles have now decayed into the lightest particle, the relic. Thus, the number density of the relic particle today depends on the rates of these processes.

The Boltzmann equation that describes the evolution of the number density in a system as described above, is:

\[
\frac{dn}{dt} = -3Hn - \sum_{i,j=1}^{N} (\langle \sigma_{ij} v_{ij} \rangle n_in_j - n_{eq_i} n_{eq_j}) ,
\]

where $n$ is the number density of the relic particle. and $n = \sum_{i=1}^{N} n_i$ because the age of the universe is much larger than the decay rate of $\chi_i$ particles. $\sigma_{ij}$ is the total annihilation rate of a process of the type $\chi_i \chi_j \rightarrow X_{SM}$ and $v_{ij}$ is the relative velocity of the annihilating particles:

\[
v_{ij} = \frac{\sqrt{s - (m_i + m_j)^2} \sqrt{s - (m_i - m_j)^2}}{s - m_i^2 - m_j^2} .
\]

The scattering rate with SM particles is much larger than the annihilation rate. This is because the cross sections for these two processes are roughly the same, but the number of background SM particles are much larger than the number of KK particles. The light SM particles are relativistic, while $\chi_i$ particles are non-relativistic. The $\chi_i$ particles are thus suppressed by a Boltzmann factor $e^{-m_i/T_i}$ compared to the light SM particles.

Hence, the $\chi_i$ distributions remain in thermal equilibrium and their ratios are the same as the equilibrium ratios:

\[
\frac{n_i}{n} \approx \frac{n_{eq_i}}{n_{eq}}.
\]

Then, the Boltzmann equation (74) can be written in terms of an effective annihilation rate:

\[
\frac{dn}{dt} = -3Hn - \sum_{i,j=1}^{N} (\langle \sigma_{eff} v \rangle n_in_j - n_{eq_i} n_{eq_j}) .
\]

According to Edsjö and Gondolo [32], $\langle \sigma_{eff} v \rangle$ can be written as:

\[
\langle \sigma_{eff} v \rangle = \sum_{ij} \langle \sigma_{ij} v_{ij} \rangle \frac{n_{eq_i}}{n_{eq}} \frac{n_{eq_j}}{n_{eq}} = \int_0^{\infty} dp_{eff} p_{eff}^2 W_{eff} K_1 \left( \frac{\sqrt{s}}{T} \right) m_i^4 T^2 \left[ \sum_{i} \frac{g_i}{g_1} \frac{m_i^2}{m_1^2} K_2 \left( \frac{m_i}{T} \right) \right]^2 ,
\]

where $K_n$ is the modified Bessel function of the second kind, $T_\gamma$ is the heat-bath temperature, $g_i$ are the internal d.o.f and the effective invariant rate, $W_{eff}$ is:

\[
W_{eff} = \sum_{ij} \frac{p_{ij} g_i g_j}{p_{11} g_1^2} W_{ij} .
\]
The relative momenta $p_{ij}$ are:

$$p_{ij} = \frac{\sqrt{s} - (m_i + m_j)^2}{2\sqrt{s}} \sqrt{s - (m_i - m_j)^2}$$

(79)

with $p_{\text{eff}} = p_{11}$. $W_{ij}$ is the invariant rate of annihilation between $\chi_i$ and $\chi_j$, and defined as:

$$W_{ij} = \frac{4m_im_jv_{ij}}{(1 - v_{ij}^2)^{1/2}}\sigma_{ij}$$

(80)

where $v_{ij} = \frac{p_{ij}\sqrt{s}}{p_i\cdot p_j}$.

The Boltzmann equation can also be solved analytically when coannihilations are included, however, $\langle \sigma v \rangle$ can have a complicated structure. Hence, the relic density plots that are presented in section 5 have been created numerically by using the DarkSUSY package [5].

### 3.4 Kaluza-Klein Dark Matter

KK parity conservation ensures that the first-level KK states cannot decay to standard model particles. Thus, all states with $n = 1$ decay through the process:

$$\chi^{(1)} \rightarrow \chi'^{(1)} + \text{SM},$$

(81)

where $m_{\chi} > m_{\chi'}$. After a long time, this means that all higher-level KK states will eventually decay into the lightest Kaluza-Klein particle, the LKP, except those that dominantly decay into SM particles (Note that these must have an even KK number). Ignoring radiative corrections, it would be obvious that the LKP is the Kaluza-Klein photon, by considering the KK masses for bosons and fermions in Eq. (10). In mUED the radiative corrections ensure that the KK photon is the LKP.

The KK photon as the LKP makes a good WIMP candidate because it is stable, electrically neutral, colourless, and quite heavy according to present collider bounds: $R^{-1} > 1.4 \text{ TeV}$ [8].

To test whether the LKP is a suitable candidate for dark matter, we must check whether it gives the observed relic density of the universe. As mentioned previously, the current relic density is $\Omega_{\text{CDM}}h^2 = 0.120 \pm 0.001$. The results from Ref. [9], presented in figure 3.3, show that the relic density depends strongly on the coannihilation processes. When all coannihilation channels are included, the LKP must have a mass that is similar to $R^{-1} \approx 1300 - 1500$ GeV for $\Lambda R = 20$, if the LKP makes up all the DM in the Universe. Since the LHC runs I and II has set the lower limit on $R^{-1}$ at approximately 1400 GeV [8][33], this is in conflict with the present collider limits. However, these limits might change when the finite corrections obtained by Ref. [2] are included. Also, increasing $\Lambda R$ to 50 pushes the upper limit up to almost 1600 GeV [9].

Thus, $R^{-1}$ is bound from above by $R^{-1} < 1600$ GeV by the observed relic density. The relic density only gives an upper bound because the total dark matter density might exist of several different types of dark matter particles. In addition, we already know that a minority of the total dark matter in the Universe exists of black holes, neutron stars and faint old white or brown dwarfs [25].
Prospects of detection

Since the LKP can interact weakly, it is possible to detect it both by direct and indirect detection.

As mentioned previously, the relic density calculations performed by Belangér et al. gives the correct relic density when $1300 < R^{-1} < 1600$ GeV. The LHC is currently not able to probe all of this region, but the LHC14 run might be able to probe energies up to 1700 GeV, according to a preliminary result by Belyaev et al. [34].

Indirect detection of the LKP is also promising, as there are several signatures that can be studied. For instance, annihilation in the Milky Way can lead to production of an electron pair and thus lead to a positron signal from the LKP [35].

To conclude, the LKP makes a good WIMP dark matter candidate as it is stable, electrically neutral, massive and can give the correct relic density. The LHC14-run will be very exciting, as it might exclude mUED as a viable theory beyond the SM. However, UED theories are not necessarily excluded because the boundary terms that are ignored in mUED might have a significant effect on the phenomenology of the model.
4 Annihilation and Coannihilation processes

The main goal of this thesis was to calculate the relic density of Kaluza-Klein dark matter, the first KK excitation of the photon $\gamma^{(1)}$. The relic density depends on the rate of all coannihilation and annihilation processes that change the number of DM particles. Since all higher level KK particles eventually decay to the LKP, there are several processes that change the number density. The most important ones, the annihilation and main coannihilation channels, are presented in this chapter along with their analytical expressions. However, for realistic scenarios, including mass corrections and without simplifications, the cross section should be calculated numerically as the final expressions become too large to be calculated by hand without losing important information. Hence, the relic density can be very complicated, if not impossible, to calculate analytically. The solution to this, as well as one of the main goals of this thesis, has been to implement a UED module in the DarkSUSY package. The DarkSUSY package contains the necessary routines to calculate the relic density fast, provided that the (co-)annihilation channels are implemented.

The first part of this chapter presents the (co-)annihilation channels that have been included in the relic density calculations, as well as their analytic expressions. The second part focuses on the numerical calculation of the diagrams, while the third part describes DarkSUSY and the UED module.

4.1 Overview of (co-)Annihilation Processes

There are numerous processes that change the number density of the dark matter particle, since coannihilation channels contribute. However, it is sufficient to consider those that involve KK level 1 particles, because we can assume all higher-level KK particles have already decayed. By expanding the Bessel functions in Eq. (77), one can rewrite the expression of $\langle \sigma v \rangle$ in terms of exponentials factors dependent on the mass differences $\Delta m_i$ of the coannihilating particles [9]:

$$\langle \sigma v \rangle = \sum_{i,j} \langle \sigma v \rangle_{ij} \frac{g_i g_j \exp\left(-\frac{\Delta m_i + \Delta m_j}{T_f}\right)}{\left(\sum_i g_i \exp\left(-\frac{\Delta m_i}{T_f}\right)\right)^2}.$$ (82)

Since higher-order KK states have greater masses compared to lower order KK states, their contribution to the effective annihilation rate is exponentially suppressed. Hence, we do not need to take into account their contribution to the relic density. This reduces the number of processes significantly, but there are still many left. The most recent calculation of the relic density including all coannihilation channels was performed by Belangèr et al. in 2010 [9].

According to figure 4.1, taken from Ref. [9], the main contributions come from the coannihilations of: $l^{(1)}\gamma^{(1)} \rightarrow$ level 2 particles, $l^{(1)}l^{(1)}$, $l^{(1)}H^{(1)}$ and $\gamma^{(1)}H^{(1)}$.

Only the main channel, $l^{(1)}\gamma^{(1)} \rightarrow$ level 2 particles, and the annihilation channels have been implemented in the UED and contribute to the relic density plots presented in the next chapter. 19 The remaining diagrams have been calculated analytically and are presented in appendix C. Due to lack of time, these will be implemented in DarkSUSY later.

19Note that due to lack of time and complications, only the channels $l^{(1)}\gamma^{(1)} \rightarrow \gamma^{(2)}l$ are included in the calculation of relic density.
Figure 4.1: From [9]. Relative contributions to the relic density as a function of $R^{-1}$ of the LKP for different pairs of coannihilating particles. The relic density has been computed for $m_H = 120 \text{GeV}$ and $\Lambda R = 20$. The contributions of each coannihilating pair is a sum over the whole class of initial states and all final states. However, $\gamma^l l^*$ is divided into two groups: Final states involving a level 2 KK particle, and final states only involving SM particles. All remaining channels contribute less than 1%.

4.1.1 LKP annihilation

There are four main processes\textsuperscript{20} that contribute to the annihilation cross section of $\gamma^{(1)}$:

$$\gamma^{(1)} \gamma^{(1)} \rightarrow f \bar{f}, \ H H, \ ZZ, \ W^+ W^-,$$  \hspace{1cm} (83)

with the corresponding Feynman diagrams presented in figures 4.3, 4.4 and 4.5. The fermion final states have the greatest contribution to the relic density. The contributions of each type of fermion final state are given in figure 4.2. Clearly, the leptonic final states dominate, when $R^{-1}$ is large. In the limit where only the $t$- and $u$-channels are present, presented in the top plot, the cross section depends strongly on the hypercharges of the final state fermions, as in Eq. (84). Thus, the leptonic states will dominate since the cross section is proportional to $Y^4$. However, adding the s-channel diagrams with $H$ and $H^{(2)}$ increases the quark contribution, as shown in figure 4.2. This is almost entirely due to the $H^{(2)}$ channel, which has a strong Yukawa coupling to $t \bar{t}$ due to the large top mass. The $H$-channel is not as strong because $H$ primarily decays into a $b \bar{b}$-pair which has a smaller Yukawa coupling and is not resonant. An interesting aspect of the $H^{(2)}$ resonance is that its contribution decreases as $R^{-1}$ increases, since the decay rate is proportional to the mass of $H^{(2)}$. Since $m_{H^{(2)}} \sim R^{-1}$, the decay width increases with $R^{-1}$ and the resonance becomes weaker. Thus, the $H^{(2)}$ resonance will dominate only for small $R^{-1}$ (Based on figure 4.2 this happens when $R^{-1} < 821 \text{ GeV}$).

\textsuperscript{20}Annihilation to a $\gamma \gamma$, $\gamma Z$ and $\gamma H$ are also allowed, and have been calculated in ref. [36]. These are loop-suppressed, and have a small contribution to the relic density at the typical freeze-out temperature (see figure 4.10)
The annihilation fractions to gauge and Higgs bosons are only of approximately one percent, and have a small contribution to the relic density.

**Fermion final states**

This is the main annihilation channel and includes the four diagrams presented in figure 4.3. The total cross section can be calculated analytically by ignoring radiative corrections and assuming that the SM fermions have negligible masses. Then, the cross section only depends on the \( t \)- and \( u \)-channel diagrams, because the \( s \)-channel diagrams are proportional to the Yukawa couplings with Higgs, which depend on the SM masses. The remaining diagrams are proportional to the hypercharges of the fermions:

\[
\sigma \left( \gamma_1 \gamma_1 \rightarrow ff \right) = \frac{N_c g^4_Y \left( Y^4_{f_L} + Y^4_{f_R} \right)}{72 \pi s^2 \beta^2} \left( -5 s \left( 2m^2 + s \right) L - 7s\beta \right) , \tag{84}
\]

where \( \beta = \sqrt{1 - \frac{4m^2}{s}} \) and \( L = -2 \tan^{-1} \beta \). This calculation has been performed by Kong and Matchev in \cite{38}. In the limit where all KK masses are equal, and the SM masses are negligible, the complete cross section implemented in DarkSUSY reduces to Eq. (84).

**Boson final states**

The analytical expression for the processes \( \gamma^{(1)} \gamma^{(1)} \rightarrow HH, ZZ, W^+W^- \) are found by neglecting electroweak symmetry breaking (EWSB) so that the gauge boson final states are not allowed. The final state is then a complex doublet, \( \phi \phi^* \). Note that this implicitly includes decay to longitudinal \( Z \) and \( W \) bosons, and the Higgs boson, after EWSB effects have been included.

In addition, one must assume that all SM masses are negligible and that radiative corrections can be ignored to arrive at the analytical result. Using these simplifications, the only remaining diagrams are those without couplings proportional to SM masses: The \( t \)- and \( u \)-channel with \( a_{0}^{(1)} \) and SM Higgs in the final state, and the four-point coupling between \( \gamma^{(1)} \gamma^{(1)} \) and SM Higgs bosons. Using unitarity gauge, the final state only consists of Higgs bosons, since the Goldstone bosons are removed.

Similar to the fermion final state, the cross section depends on the hypercharge of the Higgs boson and the LKP mass \cite{38}:

\[
\sigma \left( \gamma_1 \gamma_1 \rightarrow \phi^* \right) = \frac{g^4_Y Y^4_{\phi}}{12 \pi s \beta} . \tag{85}
\]

Thus, in the analytic limit the total annihilation cross section, including fermions and bosons in the final state, only depends on the mass of the LKP. This is similar to the general case of WIMP’s, where the annihilation cross section can be approximated as a function only of the coupling to the WIMP and its mass (Eq. (60)). The relic density is expected to increase when \( R^{-1} \) increases because the annihilation cross section is inversely proportional to the LKP mass squared.

Again, the complete results that are implemented in DarkSUSY reduce to Eq. (85) when the above simplifications are applied.
Figure 4.2: Relative annihilation fractions for the process $\gamma^1\gamma^1 \rightarrow f\bar{f}$, as a function of the inverse scale of the extra dimension. The dashed lines correspond to $\Lambda R = 50$, and the solid lines correspond to $\Lambda R = 20$. The annihilation fractions without the s-channels reproduce those from table 2 in Ref. [37], when the KK masses are completely degenerate. The annihilation fraction to up-type quarks is dominated by the annihilation to the top quark, via a $H^{(2)}$ s-channel. The contribution from the s-channel with $H$ is less than 1%. When $R^{-1} = 200\text{GeV}$, the $\gamma^{(1)}\gamma^{(1)}$ pair annihilate almost entirely to a top-pair. As $R^{-1}$ increases, the annihilation fraction to $t\bar{t}$ decreases because the width of the resonance is proportional to the mass of $H^{(2)}$ (Eq. (91)). Since the Breit-Wigner propagator (Eq. (87)) is inversely proportional to the decay width, the process is suppressed when $R^{-1}$ increases because $\Gamma_{H^{(2)}} \propto m_{H^{(2)}}$, which increases with $R^{-1}$ according to Eq. (7).
Figure 4.3: Feynman diagrams that contribute to the cross section for the process $\gamma^1 \gamma^{(1)} \rightarrow f \bar{f}$.

Figure 4.4: Feynman diagrams that contribute to the cross section for the process $\gamma^{(1)} \gamma^{(1)} \rightarrow HH$. 
Figure 4.5: Feynman diagrams that contribute to the cross section for the process $\gamma^{(1)}\gamma^{(1)} \rightarrow VV$. 

\[ \begin{array}{c}
\gamma_{\mu}^{(1)} & Z_{\rho} & \gamma_{\mu}^{(1)} & W_{\rho}^+ & \gamma_{\mu}^{(1)} & Z_{\rho} \\
H^{(1)} & Z_{\sigma} & q_{\perp}^{(1)} & W_{\sigma}^- & \gamma_{\nu}^{(1)} & Z_{\sigma} \\
\gamma_{\nu}^{(1)} & Z_{\sigma} & \gamma_{\nu}^{(1)} & W_{\sigma}^+ & \gamma_{\nu}^{(1)} & Z_{\sigma} \\
W_{\rho} & \gamma_{\mu}^{(1)} & Z_{\rho}, W_{\rho}^+ & \gamma_{\mu}^{(1)} & Z_{\sigma}, W_{\sigma}^- & \gamma_{\nu}^{(1)} & W_{\sigma}^- \\
& \gamma_{\nu}^{(1)} & H & Z_{\rho}, W_{\rho}^+ & \gamma_{\nu}^{(1)} & W_{\sigma}^- \\
& W_{\sigma} & \gamma_{\mu}^{(1)} & H & Z_{\sigma}, W_{\sigma}^- & \gamma_{\nu}^{(1)} & W_{\sigma}^-
\end{array} \]
4.1.2 Coannihilation channels

The main coannihilation channels, and the only ones implemented in DarkSUSY (so far), are:

\[ \gamma^{(1)} l^{(1)} \rightarrow V^{(2)} l , \quad S^{(2)} l \quad (86) \]

where \( V^{(2)} \) and \( S^{(2)} \) are the KK level 2 vectors and scalars that decay dominantly into SM particles. These are: \( \gamma^{(2)}, a^{(2)}_0 \) and \( H^{(2)} \). The corresponding channels involving quarks are completely negligible \[9\].

The KK level 2 particles above decay dominantly to SM particles, so all processes of this kind reduce the number of relic particles. The Feynman diagrams that contribute to the matrix element for these channels are presented in figure 4.6. The couplings present in the diagrams are given in tables 7 and 10, and appendix B.4.

The channels involving \( \gamma^{(2)} \) in the final state make up approximately 50% of the (co-)annihilation channels, according to Ref. \[9\]. The channels involving \( H^{(2)}, a^{(2)}_0 \) in the final state make up at most a few percent of the coannihilation channels, because the Yukawa coupling to leptons is suppressed. Thus, all remaining channels make up close to 50%.

These have not been implemented in the UED module, but the most important remaining channels, \( l^{1\bar{l}^1}, l^1 H^1 \) and \( \gamma^1 H^1 \) have been calculated, but not checked due to lack of time. These channels are expected to make up at least 30% \[9\] of the coannihilation channels, and should be included to give a precise calculation of the relic density. The total cross sections for these channels have been calculated analytically in Ref. \[37\].

The remaining channels make up between 10% and 15% of the coannihilation channels, with \( H^{(1)} H^{(1)} \) being the most important of these as it makes up a few percent \[9\].

In general, the most important coannihilation channels are those that have a mass maximally 10% larger than the LKP mass. A large mass difference between the coannihilating particles lead to a stronger suppression according to Eq.(82).

There are other factors besides the mass difference that determine whether a coannihilation channel is important or not. Table 3 presents all coannihilation channels with a mass difference less than 10 % when the improved mass corrections from Ref. \[2\] are included. One might argue that all these channels should be implemented in the UED module for maximal precision of the relic density calculation, but the added computation time might be too large compared to the increased precision if the channels are suppressed. On quite general grounds, coannihilation channels can be suppressed due to e.g. the following reasons:

a) Suppressed Yukawa coupling
b) Mass difference close to 10%
c) Annihilation is suppressed by the small degree of freedom
Figure 4.6: Feynman diagrams that contribute to the cross section for the process $\gamma^{(1)} l^{(1)} \rightarrow V^{(2)} l$, where $V^{(2)}$ is a level 2 KK vector particle. Similar diagrams contribute to the process $\gamma^{(1)} l^{(1)} \rightarrow S^{(2)} l$, with $S^{(2)}$ as a KK level 2 scalar. Note that the s-channel with a SM lepton involves a KK number violating vertex, and is thus loop-suppressed.

<table>
<thead>
<tr>
<th></th>
<th>$\gamma^{(1)}$</th>
<th>$l^{(1)}$</th>
<th>$H^{(1)}$</th>
<th>$a_0^{(1)}$</th>
<th>$Z^{(1)}$</th>
<th>$W^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma^{(1)}$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l^{(1)}$</td>
<td></td>
<td>$\checkmark$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H^{(1)}$</td>
<td></td>
<td></td>
<td>$\checkmark$</td>
<td></td>
<td>$\checkmark$</td>
<td>$c$</td>
</tr>
<tr>
<td>$a_0^{(1)}$</td>
<td></td>
<td>$c$</td>
<td>$a, c$</td>
<td>$c$</td>
<td>$c$</td>
<td></td>
</tr>
<tr>
<td>$Z^{(1)}$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b, c$</td>
<td>$b, c$</td>
<td>$b$</td>
<td></td>
</tr>
<tr>
<td>$W^{(1)}$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b, c$</td>
<td>$b, c$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
</tbody>
</table>

Table 3: Table specifying the most important coannihilation channels when $\gamma^{(1)}$ is the LKP, using the mass corrections obtained by Ref. [2]. The table includes all the coannihilations between particles with a mass difference less than 10% compared to the LKP. The checkmark indicates that the coannihilation channels has been implemented in the UED module, while the checkmark in parentheses indicates that it should be implemented. *Note that only the channels that include KK level 2 fermions in the final state have been included. The channels with only SM particles in the final state contribute to the relic density with less than 1% [9]
4.2 Calculations and numerical implementation

By using the simplifications described in the previous section, it is possible to find analytic expressions for the cross section of all these processes. However, the full calculations including the correct masses are too complicated to be performed analytically and often involve several hundred terms. Hence, the amplitude of each process has been calculated numerically by using FORM [39] and later simplified using Mathematica [40].

4.2.1 FORM

FORM is a symbolic manipulation program. It is especially popular among theoretical physicists due to its fast evaluation of Feynman diagrams. Since it was developed by a particle physicist, Jos Vermaseren, it is especially good at evaluating the trace of Dirac matrices and vectors.

To describe a simple FORM program, let us evaluate the Feynman diagram:

\[
\begin{array}{c}
p_2 \\
\gamma \\
p_1 \\
k_1 \\
k_2
\end{array}
\]

The matrix element of this diagram is:

\[
i\mathcal{M} = e^2 \bar{v}^r (p_2) \gamma^\mu g_{\gamma ff} u^s (p_1) \frac{i g_{\mu\nu}}{s} \bar{u}^{s'} (k_1) g_{\gamma ff}^\dagger \gamma^\nu \bar{v}^{r'} (k_2).
\]

FORM cannot evaluate spinors, so we must calculate the unpolarized, squared matrix element analytically:

\[
\frac{1}{4} \sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{e^2}{4s^2} \text{Tr} \left( (p_2 - m) \gamma^\mu g_{\gamma ff} (p_1 + m) g_{\gamma ff}^\dagger \gamma^\nu \right) \text{Tr} \left( (k_1 + m) g_{\gamma ff}^\dagger \gamma^\mu (k_2 - m) \gamma^\nu g_{\gamma ff} \right).
\]

This is the expression that can be implemented in FORM and evaluated. The basic structure of a FORM program is quite intuitive. It consists of three parts, in the following order:

1. Declaration of variables
2. Expression(s) to be evaluated
3. Simplifications and definitions

FORM has three types of variables: Vectors, symbols and indices. For the diagram above, with four vectors and two indices, the variables are declared as:

Vectors p1, p2, k1, k2;
Indices u, v;
Symbols s, t, r, m, g, gd;

where s, t, r are the Mandelstam variables and g, gd are \( g_{ff\gamma}, g_{ff\gamma}^\dagger \).

The expression one wants to calculate can be written almost identically to how it is written analytically. When calculating the trace, one must make sure to assign different values to
the “spin line”: The vectors and Dirac matrices one wishes to take the trace of are written as \( p(1), p(2), q(1), q(2) \) where equal number denotes that they are part of the same trace.

The 4D Dirac matrices are given as: \( g_a \) where \( a \) can either be an index to be summed over or a number from 1 to 4 denoting one of the Dirac matrices. The fifth Dirac matrix is \( g_5(a) \) where \( a \) is now a number denoting which trace the matrix is a part of. Note that FORM uses a different convention for \( \gamma^5 \) than what is presented in Appendix A. FORM defines \( \gamma^5 \) as,

\[
\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 .
\]

This is important to take into account when calculating the matrix element, because \( \gamma^5 \) is hermitian using this convention.

Slashed momenta are written as: \( \not{p} = \gamma^\mu p_\mu = g_\mu(1, p) \).

The main program will look like:

```plaintext
Local Msquared =
    (g_(1, p2) - m)*g_(1, u)*g*(g_(1, p1) + m)*gd*g_(1, v))*
    (g_(2, k1) + m)*gd*g_(1, u)*(g_(2, k2) - m)*g_(1, v)*g/s^2/4 ;
Trace4,1;
Trace4,2;
```

The `Trace4` command takes the trace of 4D vectors and Dirac matrices.

Last, one can define certain relations between the vectors or symbols to simplify the expressions. For example, replacing the vector products by Mandelstram variables and masses:

```plaintext
Id p1.p1 = m^2;
Id p1.p2 = (s - 2*m^2)/2;
Id p1.k1 = (2*m^2 - t)/2;
```

where the “dot” indicates the dot product.

After replacing all vector products by masses and Mandelstam variables, the resulting output will only yield three terms. However, calculations of matrix elements involving several Feynman diagrams and different masses can yield hundreds of terms. To avoid unnecessarily long DarkSUSY routines, and shorten computation time, the output should be simplified. This is most easily done by Mathematica [40], which is a more powerful tool than FORM. Since the expressions are too large to use `Simplify` directly, they have been divided into smaller pieces by applying the `Coefficient` function and simplifying each part.
4.2.2 Decay Rates

Whenever an s-channel involving an unstable particle is present, one must replace the regular propagator in Eq. (117) with the Breit-Wigner propagator:

$$\frac{i}{s - m^2} \rightarrow \frac{i}{s - m^2 - i\Gamma},$$

(87)

where $m$ is the mass and $\Gamma$ is the decay rate of the particle. The implemented channels have resonances at the masses of: $H^{(2)}$ and $\ell^{(2)}_{d/s}$. The decay rates of these particles have been found using FORM, and include only the main decay channels. The decay rate in the center of mass frame for a two-particle decay is:

$$d\Gamma = \frac{1}{32\pi^2 M^2} |k_1|^2 |\mathcal{M}|^2 d\Omega,$$

(88)

where $\mathcal{M}$ is the invariant matrix element, $M$ is the mass of the decaying particle, $p_1$ is the momenta of one of the final state particles, and $d\Omega = d\cos\theta d\phi$. Note that one has to divide by two for identical final state particles.

The KK level 2 Higgs decay dominantly to a $t\bar{t}$ pair. Decays of the type $2 \rightarrow (1) + (1)$ and $2 \rightarrow (2) + SM$ are kinematically suppressed, so the loop-induced coupling to SM particles dominate.

The KK level 2 leptons have two main decay channels:

$$l^{(2)} \rightarrow \gamma^{(1)} + l^{(1)}$$
$$l^{(2)} \rightarrow \gamma^{(2)} + f l^{(0)}.$$

The KK level 2 leptons can also decay to SM particles via KK number violating vertices calculated in Ref. [2]. These decay channels have decay fractions of maximally 2% because they are loop-suppressed.

The decay rates are, in the limit where $\alpha^{(n)} = \theta_W^{(n)} = 0$:

$$\Gamma \left( l^{(2)} \rightarrow \gamma^{(2)} l \right) = \frac{g_Y^2 Y^2}{32\pi m^3_{l^{(2)}} m^2_{\gamma^{(2)}}} \sqrt{\left( m^2_{l^{(2)}} - (m_l - m_{\gamma^{(2)}})^2 \right) \left( m^2_{l^{(2)}} - (m_l + m_{\gamma^{(2)}})^2 \right) \left( m^2_{l^{(2)}} - m^2_l - m^2_{\gamma^{(2)}} \right)}$$

$$\times \left( m^2_{l^{(2)}} m^2_{\gamma^{(2)}} + m^2_l m^2_{\gamma^{(2)}} - m^4_{\gamma^{(2)}} + \left( m^2_{l^{(2)}} - m^2_l - m^2_{\gamma^{(2)}} \right) \left( m^2_{l^{(2)}} - m^2_l + m^2_{\gamma^{(2)}} \right) \right),$$

(89)

$$\Gamma \left( l^{(2)} \rightarrow \gamma^{(1)} l^{(1)} \right) = \frac{g_Y^2 Y^2}{32\pi m^3_{l^{(2)}} m^2_{\gamma^{(2)}}} \sqrt{\left( m^2_{l^{(2)}} - (m_l - m_{\gamma^{(1)}})^2 \right) \left( m^2_{l^{(2)}} - (m_l + m_{\gamma^{(1)}})^2 \right) \left( m^2_{l^{(2)}} + m^4_l - 6m_{l^{(2)}} m_{l^{(1)}} m^2_{\gamma^{(1)}} + m^2_{l^{(2)}} m^2_{\gamma^{(1)}} - 2m^4_{\gamma^{(1)}} + m^4_{l^{(1)}} \right)}$$

$$\times \left( m^4_{l^{(2)}} + m^4_{l^{(1)}} - 6m^2_{l^{(2)}} m_{l^{(1)}} m^2_{\gamma^{(1)}} + m^2_{l^{(2)}} m^2_{\gamma^{(1)}} - 2m^4_{\gamma^{(1)}} + m^4_{l^{(1)}} \left( 2m^2_{\gamma^{(1)}} \right) \right),$$

(90)

where $Y = Y_d, Y_s$ for doublet and singlet leptons respectively. Note that the decay rates implemented in the UED module are not in the limit above. The branching ratios of the lepton singlet and doublet decay to the two main decay channels are presented in figure 4.7, both with and without the finite corrections to the masses. The branching ratios have been compared with Ref. [2], both with the new and old mass corrections in appendix D.
Figure 4.7: The branching ratios of the two main decay channels of $l_s^{(2)}$ and $L_d^{(2)}$: $l_{s,d}^{(1)}\gamma^{(1)}$ and $l_{R,L}^{(0)}\gamma^{(2)}$. The solid lines are the branching ratios when the finite corrections to the masses are included, and the dashed lines are the BRs when they are not.

The decay width of $H^{(2)} \rightarrow t\bar{t}$ is,

$$\Gamma_{H^{(2)}} = \frac{3}{8\pi} m_{H^{(2)}} s_{\text{eff}}^{2} \sqrt{\left(1 - 4 \frac{m_t^2}{m_{H^{(2)}}^2}\right)^3},$$

(91)

where the effective coupling between $H^{(2)}t\bar{t}$ is given in Eq. 131.

4.3 DarkSUSY

DarkSUSY [5] is an advanced FORTRAN package that calculates the properties of dark matter particles. DarkSUSY allows for precise calculations of e.g. the relic density, direct and indirect detection rates and the self-interaction rate of a specific DM candidate. Until recently, the only particle physics model implemented was the minimal supersymmetric model (hence the name, DarkSUSY), but now DarkSUSY has implemented the Silveira-Zee model, a generic WIMP model, a generic decaying-DM model and a dark sector model with velocity-dependent self-interactions.

A large part of this thesis consist of implementing a UED module in DarkSUSY. The specific structure of a typical particle physics module and the current structure of the UED module will be presented in this section.

4.3.1 General structure

DarkSUSY is roughly divided into two main structures: The core library, which contains all general routines, and the particle physics modules which contain routines specific for each model.

The main library, `ds_core`, contains a set of general routines independent of the particle physics modules. The main program (a separate program that calls DarkSUSY) must specify the chosen particle physics module, as well as the quantity one wants to calculate. The main program links to a specific routine in the core library, which then calls the particle physics module specified by the user in order to perform the calculation. This calling sequence is illustrated in figure 4.8.
Figure 4.8: From ref. [5]: Conceptual illustration of the DarkSUSY linking sequence. The main program must specify a particle physics module and which quantity it wants to calculate. The main program typically calls the core library first, which then links to the specified particle physics module.

**Particle physics modules**

The particle physics modules contain all necessary information about the specific model. They are divided into two main parts: One that sets up the model and one that facilitates the calculation of observables. The actual calculation of the observables is done in `ds_core`, but with the help of interface functions provided by the particle physics module. At a minimum, the model setup includes the number of particles in the model, their quantum numbers and the DM candidate. This is implemented in the ini folder. The ge folder contains more detailed information about the model, such as the masses of the particles included, their decay widths and Feynman rules for vertices between these particles, and SM particles. Note that the vertices must be implemented in a specific way, described in section 4.3.3.

All other folders contain information that either calculates or are relevant for specific routines. In this thesis, only the relic density calculation has been implemented which includes the folders an and rd which contain the necessary routines and information to calculate $W_{\text{eff}}$ and the relic density.

**Interface functions**

Each particle physics module contains a number of functions that the core library routines might need to perform calculations. These functions provide some parameter or result that is unique to the model. Examples of such functions are `dsanwx`, which returns the effective invariant annihilation rate $W_{\text{eff}}$ and `dsrdparticles` which returns information about the
particles and resonances included in the coannihilation channels. All interface functions are specifically labeled as an interface function in the file header.

User replaceables
DarkSUSY provides the possibility of replacing a specific routine with one of your own. This is done by creating a new file with the same name as the one you want to replace and let DarkSUSY link to this one instead. DarkSUSY provides tools to help create or delete new functions, and update the makefiles in the proper way.

4.3.2 The (m)UED module
The UED module currently includes all information necessary to set up the model and calculate the relic density. The UED module contains five variables: $R_{\text{inv}}, \Lambda, \text{masshowto}, \text{vrtxhowto}$ and $\text{widthhowto}$. $R_{\text{inv}}$ and $\Lambda$ are the free parameters of the mUED model, while $\text{masshowto}$, $\text{vrtxhowto}$ and $\text{widthhowto}$ lets you decide whether you want to implement the old, $\text{masshowto}, \text{vrtxhowto}, \text{widthhowto} = 0$, or new, $\text{masshowto}, \text{vrtxhowto}, \text{widthhowto} = 1$, corrections of the KK particles for the masses, vertices and decay widths respectively. The “new” corrections refers to the radiative corrections including the finite corrections calculated by Freitas et al. [2], while the “old” corrections refer to the radiative corrections calculated by Cheng et al. [15].

Note that the KK number violating vertices in appendix B.4 between SM fermions and a KK 2 gauge boson are only included if $\text{vrtxhowto}=1$.

Model setup
ini/: The initialization folder contains two subroutines that set up the model, $\text{dsinit\_module}$ and $\text{dsmodelsetup}$. $\text{dsinit\_module}$ contains the total number of particles in the model, and all their quantum numbers and degrees of freedom. Only KK particles up to level 2 are included in the UED module. KK parity is not included as a new quantum number, because this primarily affects whether a vertex is allowed or not. Only vertices allowed by conservation of KK parity are included in the UED module. $\text{dsmodelsetup}$ calls the subroutines where the masses, decay widths and vertices of the KK particles are implemented.

gel/: This folder contains general information about the UED model. That includes the masses, decay widths and allowed vertices of the KK particles, as well as the DM candidate. The masses presented in B.2 are implemented in the subroutine $\text{dssetmuedmasses}$, while the vertices presented in B.3 and B.4 are implemented in $\text{dssetmuedvertices}$. Note that only the constant term of the coupling, $g_{\phi_1\phi_2\phi_3}$, and only tree-point couplings are implemented in $\text{dssetmuedvertices}$. The sign of the coupling depends on both the particles involved and whether they are in the initial or final state. The signs of the couplings are implemented in $\text{dssetmuedcouplings}$. How the signs are determined will be discussed in section 4.3.3.

---

21In general, $\text{dsrdparticles}$ returns a list of all coannihilating particles, and information about resonances and tresholds which are areas where the relic density routines in $\text{ds\_core}$ needs to take special care.
Figure 4.9: The calling structure in \texttt{an/} for the calculation of the effective invariant rate. Note that the last two steps, that calculate the amplitude of a process, has slightly different names for the coannihilation channels: \texttt{dsanmsqkgamma} and \texttt{dsfindamp_k<initial state>_<final state>}. 

**Relic density routine**

The routine that calculates the relic density is a part of the core library. It needs the effective invariant rate returned by \texttt{dsanwx} and information about the coannihilating particles and possible resonances, which is implemented in the folders \texttt{an/} and \texttt{rd/}.

\texttt{an/}: The (co-)annihilation rate folder contains all the subroutines necessary to find the invariant rate. The subroutine \texttt{dsanwx} returns the invariant rate, for an initial center-of-mass momentum $p$. This is the only routine required by \texttt{ds_core} in order to calculate the relic density. The structure that has been implemented to calculate the invariant rate follows closely the setup in the mssm module.

The calling structure from the amplitude to the invariant rate is presented in figure 4.9. The effective invariant rate returned by \texttt{dsanwx} is,

$$ W_{\text{eff}} = \int_{-1}^{1} \frac{dW_{\text{eff}}}{d\cos \theta} d\cos \theta , $$

where $\frac{dW_{\text{eff}}}{d\cos \theta}$ is returned by \texttt{dsandwdcos}.

\texttt{dsandwdcos} sums over all the invariant rate of all coannihilation and annihilation channels:

$$ \frac{dW_{\text{eff}}}{d\cos \theta} = \sum_{i,j} g_i g_j \frac{\sqrt{p_{ij}}}{g_{\text{eff}}^2} \frac{dW_{ij}}{d\cos \theta} , $$

where $g_i, g_j$ are the number of d.o.f. of the coannihilating particles and $g_{\text{eff}}$ is the number of d.o.f. of the DM particle.
The invariant rate \( \frac{dW_{ij}}{d\cos\theta} \) for each channel is returned by \( \text{dsandwcosij} \):

\[
\frac{dW_{ij}}{d\cos\theta} = \frac{\sqrt{s}}{8\pi s} \sum_f \frac{d}{d\cos\theta} |\vec{p}_f| |M_f|^2,
\]

where \( \vec{p}_f \) is the three-momentum of one of the final state particles and \( |M|^2 \) is the spin averaged amplitude of the process. The sum of all amplitudes times final state momenta is returned by \( \text{dsanmsqkgamma} \). The amplitude and final state momenta for a specific final state is calculated by the \( \text{dsfindamp_kgamma_<final state>} \) subroutines. The amplitude is first calculated by form, and the output is calculated in DarkSUSY by substituting the couplings and masses with those implemented by DarkSUSY.

**rd/**: The folder **rd/** contains information about which particles are included in the coannihilation channels. Specifically, the subroutine \( \text{dsrdparticles} \) decides which particles are included by imposing a condition on the masses of the coannihilating particles. If,

\[
\frac{m_{\text{coann}}}{m_{\text{DM}}} > m_{\text{cofr}},
\]

where \( m_{\text{cofr}} \) is some number larger than 1, then the particle is not included in the coannihilation channels. Although these channels may be kinematically allowed, they are not included because their effect is so small compared to the added computation time. \( \text{dsrdparticles} \) also contains the resonances of the coannihilation channels, that is, the decay widths and masses of the particles that appear in the propagator of \( s \)-channel diagrams.

### 4.3.3 Vertices

The vertices should be implemented in DarkSUSY so that they can be generalized and used in different types of diagrams. The convention that is applied to achieve this was invented by Edsjö in Ref. [32]. The couplings are implemented in DarkSUSY as:

\[
g_{\phi_1\phi_2\phi_3} = g(\phi_1, \phi_2, \phi_3),
\]

with the following convention:

- Differentiate between left-handed and right-handed couplings: \( g_L P_L + g_R P_R \)
- Divide all couplings by \( i \)
- The first fermion has a bar and is hence outgoing
- The order of the indices is equal to the order of appearance in the Lagrangian

Following this convention, the sign of the coupling changes as the order of the particles change. This is done by the subroutine \( \text{dssetmuedcouplings} \). The sign change according to the position of the particles is presented in table 4 for different types of couplings.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$\phi_1\phi_2\phi_3$</th>
<th>$\phi_1\phi_3\phi_2$</th>
<th>$\phi_2\phi_3\phi_1$</th>
<th>$\phi_2\phi_1\phi_3$</th>
<th>$\phi_3\phi_1\phi_2$</th>
<th>$\phi_3\phi_2\phi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>$g$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Pseudo-scalar</td>
<td>$g\gamma^5$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Pseudo-vector</td>
<td>$g\gamma^\mu\gamma^5$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Vector</td>
<td>$g\gamma^\mu$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Complex Scalar</td>
<td>$ig$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Vector-Scalar-Scalar</td>
<td>$g(q_1 - q_2)^\mu$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Vector-Vector-Vector</td>
<td>Eq. (119)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4: This table shows how the sign of the coupling $g_{\phi_1\phi_2\phi_3}$ changes according to the order of the particles, as implemented in DarkSUSY. The left column presents all the different types of couplings, with $g$ as the numerical value of the coupling. The row corresponding to each type of coupling gives the sign of the coupling according to the permutation of the fields involved. Note that the pseudo-scalar, pseudo-vector and vector couplings only appear in vertices involving two fermions. In that case $\phi_1$ is the boson. Also, note that for a vector-scalar-scalar coupling, $\phi_1$ is the vector and both momenta are ingoing.

### 4.4 Future of the UED module

Although the necessary structures to calculate the relic density are implemented, the UED module is far from finished. There are still plenty of interface functions that need to be implemented, but that might be the goal of another Master`s project.

However, there are a few changes that can be implemented in the existing module without having to implement new interface functions. These are described below.

**Improvement of the relic density calculation**

LKP annihilation can also result in a gamma ray pair, a $\gamma Z$ or a $\gamma H$ pair in the final state. Their cross section times relative velocity is presented in figure 4.10. Since $\sigma v$ is typically of the order $10^{-27}$, the resonances with $H^{(2)}$ have a significant contribution, and should be included in the UED module for an even more precise calculation of the relic density when the $v/c$ is smaller than 0.3. However, since freeze out typically happens when $v/c = 0.3$, these contributions are small.

So far, approximately 50% of the coannihilation channels are not included in DarkSUSY. Hence, the next step should be to implement these channels in order to increase the precision of the relic density. Since all the couplings present in the Feynman diagrams in appendix C are implemented in the UED module, no extra work beside the implementation of new coannihilation channels is required.

**Implementing a Beyond mUED module**

Implementing a beyond mUED module is straightforward since the only difference is how KK number violating couplings are treated at the boundaries. Since this only appears in dssetmuedvertices, one only needs to add a separate program dssetuedvertices with the correct couplings in the case of UED. The subroutine dssetmodelsetup is already modified so that choosing modeltype=2 will call the UED vertices.

By adding the remaining coannihilation channels $l^{(1)}l^{(1)}$, $l^{(1)}H^{(1)}$ and $H^{(1)}\gamma^{(1)}$ it will be easy to implement a UED module with the NLKP (Next to lightest Kaluza-Klein particle), the KK neutrino, as the DM candidate, since the annihilation channels are already included. This would make it easy to compare $\gamma^{(1)}$ and $\nu^{(1)}$ as the DM particle.
Figure 4.10: From ref. [36]: Cross sections of the annihilation of $\gamma^{(1)}\gamma^{(1)}$ into different final states, as a function of the speed of the dark matter particle. Since the $B^{(2)}$ and $A^{(2)}_3$ resonances are so weak, they will not be discussed.
5 Relic Density of Kaluza-Klein Dark Matter

According to the Planck 2018 results, the current density of dark matter in the Universe is: \( \Omega_{\text{CDM}} h^2 = 0.120 \pm 0.001 \) within 68 % CL [3]. Any viable DM candidate should be able to return the same relic density within the allowed parameter space.\(^{22}\)

The main goal of this thesis was to implement a UED module in the DarkSUSY package in order to calculate the relic density of KK dark matter. The UED module contains two sets of masses and couplings: One where the finite corrections recently calculated by Freitas et al. have been implemented, and one where only the corrections calculated by Cheng et al. have been implemented. These two sets are referred to as ”new” and ”old” corrections in this chapter.

This chapter is devoted to the results obtained from the relic density calculations in DarkSUSY based on the coannihilation channels implemented in the UED module. The relic density contains the main coannihilation channels, as well as the dominating annihilation channels. Consequently, the chapter is divided into two parts: The first part presents the results when only the annihilation channels are included in the calculations, while the second part presents the results when coannihilations are included.

5.1 Annihilation only

Although the coannihilation channels make up a larger contribution to the relic density than the annihilation channels, annihilation alone can give us a decent approximation of the relic density.

Comparison with earlier results

One of the earliest full calculations of the relic density of LKP dark matter was performed by Servant et al. in Ref. [24]. They performed the calculation including the two main coannihilation channels:

\[ \gamma(1) \gamma(1) \rightarrow f \bar{f}, \phi \phi^* , \]  

(92)

where \( \phi \phi^* \) implicitly includes the SM Higgs bosons and the weak gauge bosons.

The results obtained by Servant et al. are in what is referred to in this thesis as the ”analytic limit”, where all SM masses are negligible and the KK masses are fully degenerate. In addition, there is no weak or fermionic mixing, so the KK photon is the KK weak hypercharge gauge boson, \( B^{(1)}_\mu \). In this limit, we find that the results obtained from DarkSUSY are in good correspondence with the results obtained by Servant et al., as shown in figure 5.1. The relative difference between the results obtained by Servant et al. and DarkSUSY are maximally 4% and go towards zero as the inverse scale \( R^{-1} \) increases. Figure D.4 shows explicitly this difference for \( R^{-1} \) ranging from 200 to 2000 GeV in appendix D. The plot of the relic density obtained by Servant et al. has been obtained by using WebPlotDigitizer [41] which is not a precise method. Thus, the 4% difference is most likely due to this method. Nevertheless, figure 5.1 suggests that the results obtained by Servant et al. and the UED module in the analytic limit are in good correspondence. In addition, the cross sections of the processes above have been compared in the analytic limit to those obtained by

\(^{22}\)That is, the parameter space that is kinematically allowed and not within the current energies of the LHC if the signal cannot be hidden by the SM background (as DM has not yet been detected).
Figure 5.1: Comparison of relic density obtained by the native DS relic density solver and Ref. [24] in the limit where all KK masses are degenerate and the SM masses are negligible.

Servant et al. prior to implementation in the UED module, and gave perfect correspondence.

**Including the $H^{(2)}$ resonance**

One of the main results obtained by Belangér et al. [9] is that the addition of the $H^{(2)}$ resonance in the $s$-channel reduces the relic density by approximately 10%. Using the effective vertex between $H^{(2)}t\bar{t}$ given by Eq. (131) we find that the relative effect of adding $H^{(2)}$ is a decrease of the relic density of maximally 50% when the finite corrections from Ref. [2] are included and 70% when they are not. The $H^{(2)}$ resonance has a maximal effect when $R^{-1} \approx 250 \text{GeV}$.

When the relic density is large, this effect goes toward 5% and 7% with and without the finite corrections. Generally, the effect is larger than what was found by Belangér et al. when $400 < R^{-1} < 1600 \text{ GeV}$, that is, within the parameter space discussed in [9]. Adding $a_0^{(2)}$ in the $s$-channel might decrease the relic density further by a few percent.

The difference between our results and those obtained by Belangér et al. is most likely due to the difference in the effective coupling between $H^{(2)}$ and the top pair. Since $\mathcal{M} \propto g_{\text{eff}}/\Gamma^2$ at the resonance, and $\Gamma \propto g_{\text{eff}}$, an increase in the effective coupling leads to a suppressed resonance and thus a smaller contribution to the relic density.

The effective coupling implemented by Belangér et al. is $g_{\text{eff}} \approx 0.1$, while the coupling calculated in Ref. [36] (and implemented in DarkSUSY for the old corrections) is $g_{\text{eff}} \approx 4.73 \times 10^{-2}$ when $\Delta R = 20$.

This effect is slightly weaker when the new masses and a new effective coupling between $H^{(2)}t\bar{t}$ is implemented. The effective coupling derived by Ref. [2] given in Eq. (131) is $g_{\text{eff}} \approx 6.1 \times 10^{-2}$ when $\Delta R = 20$. Since the decay width is approximately proportional to the mass of $H^{(2)}$ the increased mass due to the new corrections further suppress the decay channel.
Figure 5.2: The effect of adding the $H^{(2)}$ resonance in the annihilation channel $\gamma^{(1)}\gamma^{(1)} \rightarrow t\bar{t}$. The plots show the difference between the relic density calculated with and without the resonance, for two different cases:

a) Using the new mass corrections obtained by Ref. [2] and the effective coupling between $H^{(2)}t\bar{t}$ in Eq. (131), and

b) Using the old corrections from Ref. [15] and the effective coupling obtained in Ref. [36].

Using b), the $H^{(2)}$ resonance reduces the relic density by maximally 70\% and has a larger contribution than what is expected by Belanger et al in Ref. [9] when $R^{-1}$ increases because the effective coupling between $H^{(2)}t\bar{t}$ is smaller than the one used in Ref. [9].

Using a) the relic density is reduced by maximally 50\%, because the $H^{(2)}$ mass and decay width are larger compared to b).

In both cases, adding $H^{(2)}$ reduces the relic density with $\sim 10\%$ percent when $R^{-1}$ is large. Since the correction term from Ref. [2] is logarithmic, the effect of using the new corrections should decrease when the cut-off $\Lambda$ increases.
Figure 5.3: When only annihilation is included, the relic density increases by $6 - 24\%$ when the new corrections are implemented. The difference decreases as $R^{-1}$ increases because the decay width of $H^{(2)}$ increases which weakens the resonance.

**Effect of adding the new mass corrections**

The effect of adding the finite terms in the orbifold corrections calculated by Ref. [2] is a $6 - 24\%$ increase of the relic density when only annihilation is included. The difference is larger when $R$ is small because the $H^{(2)}$ resonance is stronger. Since the coupling between $H^{(2)}\bar{t}t$ is larger with the new corrections, the resonance is weaker and thus the relic density decreases less than when the old corrections are implemented. As $R^{-1}$ increases, the effect of the finite corrections decrease because the bulk corrections are proportional to $R^{-1}$. 

![Graph showing relic density and difference with and without new corrections](image-url)
5.2 Including Coannihilations

Expected contribution

It has already been shown in Ref. [9] that adding KK level 2 particles that decay dominantly to SM particles in the final state leads to a significant decrease of the relic density. The strongest contribution is from the channels:

\[ l_s^{(1)} \gamma^{(1)} \rightarrow l^{(0)} \gamma^{(2)} \]  

(93)

The coannihilation channels with KK level 2 particles in the final state make up approximately 50% of the coannihilation channels, and are enhanced by the \( l_s^{(2)} \) resonance. Thus, the channels presented above make up the largest contribution to the relic density although the exact contribution is unknown.

Due to lack of time, not all of the main coannihilation channels in Eq. (86) have been implemented. Due to complications with the initial state doublet leptons and a lack of time implementing the channels with KK level 2 scalar final states, only the channels in Eq. (93) have been included in the relic density calculations.

Adding the \( l_s^{(1)} \gamma^{(1)} \rightarrow \gamma^{(2)} l \) is the natural next step of the UED module, since the channels \( l^{(1)} \gamma^{(1)} \rightarrow \gamma^{(2)} l \) make up 50% of the coannihilation channels [9].

We can roughly estimate the change in \( \Omega h^2 \) when the doublet leptons are added. First, we consider how the integrand in Eq. (77) changes:

The singlet and lepton doublet have the same number of d.o.f (4), while the neutrinos have half (2) since they are not Dirac fermions. In addition, since the relative difference in mass between the singlet and doublet is small, we can assume the invariant momentum is approximately the same for all particles. In addition, the decay widths only differ by \( \mathcal{O}(10)\% \), so they will have approximately the same contribution to the relic density (see fig. 5.7).

The invariant rate is then:

\[ W_{\text{eff}} \approx \frac{4 \sqrt{p_l \gamma}}{3} \left( W_{l_s \gamma} + W_{l_d \gamma} + \frac{1}{2} W_{l \nu \gamma} \right) \ . \]

To consider the invariant rate of each channel, it’s sufficient to consider the diagram: \( l^{(1)} \gamma^{(1)} \rightarrow l^{(2)} \rightarrow l^{(0)} \gamma^{(2)} \) because the \( l^{(2)} \) resonance is generally so strong that this diagram has a much greater contribution than the other diagrams. The invariant rate is proportional to the couplings of the diagram:

\[ W \propto g_{l^{(1)} \gamma^{(1)} l^{(2)}}^2 \times g_{l^{(2)} \gamma^{(2)} l}^2 \ . \]

The product of the couplings for the doublet leptons and neutrinos are approximately 1/16 and 1/9 of the product for the singlet leptons. The effective invariant rate can thus be approximated as,

\[ W_{\text{eff}} = \frac{4 \sqrt{p_l \gamma}}{3} \frac{161}{144} W_{l_s \gamma} \ . \]

Next, one needs to take into account the added degrees of freedom. When only the lepton singlets are included, the denominator in Eq. (77) is \( 16m_l^4 T_5 \frac{m_l}{m_1} K_2 \left( \frac{m_{\nu}}{m_{\tau}} \right) \), where all the KK level 1 singlet masses have been approximated as identical because the electroweak masses
are small compared to $R^{-1}$.

Adding the doublet leptons and neutrinos gives the denominator:

$$16T_\gamma \left[ m^2_{\epsilon_e^{(1)}} K_2 \left( \frac{m_{\epsilon_e^{(1)}}}{T} \right) + m^2_{\epsilon_d^{(1)}} K_2 \left( \frac{m_{\epsilon_d^{(1)}}}{T} \right) + \frac{1}{2} m^2_\nu K_2 \left( \frac{m_\nu}{T} \right) \right]^2$$

By replacing the effective invariant rate and the denominator, we get that adding the doublet leptons results in a 42% decrease of $\langle \sigma v \rangle$. The relic density is inversely proportional to $\langle \sigma v \rangle$, which means that adding all the doublet leptons increases the relic density by approximately 2.4 times.

The relic density including the channels $l_s^{(1)} \gamma^{(1)} \rightarrow \gamma^{(2)} l$ is presented in the top plot of figure 5.4. The bottom plot shows the relic density as it should be when the lepton doublets in addition to the singlets are included in the initial state, e.g. the relic density multiplied by 2.4. The relic density is larger than what is obtained by Belangér et al. when the doublet leptons are “included” because 50% of the coannihilation channels are still missing. The missing coannihilations with the largest contributions to the relic density will only add the Higgs and $a_0$ d.o.f., which will not increase the denominator of Eq. (77) by much. Adding the coannihilation channels with $Z^{(1)}$ and $W^{(1)}$ will not affect the relic density much because the added d.o.f are suppressed by the mass difference between $Z^{(1)}, W^{(1)}$ and the LKP.

**Effect of adding the new mass corrections**

The new mass corrections lead to a decrease between 6 and 7%, as shown in figure 5.6. One would naively expect the relic density to increase since both the $l_s^{(1)}$ mass and decay width increases with the new corrections. This results in a smaller resonance which is centered around a larger center-of-mass momentum since the resonance particle is heavier. However, $\langle \sigma v \rangle$ receives a contribution from $p^2_{\text{eff}} K_1(\sqrt{s}/T)$ with a peak around $p_{\text{eff}} = 146$ GeV when $R^{-1} = 800$GeV and $\Lambda R = 20$, as shown in figure 5.5. Since the resonances are generally on the left side of the peak, a shift towards larger CM-momenta leads to a larger $\langle \sigma v \rangle$ thus decreasing the relic density. For the case presented in figure 5.5, the old and new mass corrections have peaks at approximately $W_{\text{eff}} = 200$ and 168, respectively. Multiplying by $p^2_{\text{eff}} K_1(\sqrt{s}/T)$ shifts the peaks from 200 to $6 \cdot 10^{-21}$ and from 168 to $7.1 \cdot 10^{-21}$.
Figure 5.4: The top plot shows the relic density as generated by the UED module when the main coannihilation channels $l^{(1)}_s \gamma^{(1)} \rightarrow \gamma^{(2)} l$ are implemented. The bottom plot shows the relic density as it when the channels $l^{(1)}_d \gamma^{(1)} \rightarrow \gamma^{(2)} l$ have been “added” by multiplying the relic density by 2.4, which is the estimated contribution of these channels. In both cases adding the new mass corrections decreases the relic density because the $l^{(2)}_s$ resonance, which is the main contribution, becomes easier to access thermally (see fig. 5.5). The solid lines represent the relic density when $\Lambda R = 20$, while the dashed lines represent the relic density when $\Lambda R = 50$. The upper bound on the relic density within $5\sigma$ increases by approximately 200 and 230 GeV when the new mass corrections are implemented and $\Lambda R = 20$ and 50, respectively.
Figure 5.5: The computation of the effective invariant rate $W_{\text{eff}}$ leads to a smaller resonance when the new radiative corrections are implemented because both the mass and decay width of $l^{(2)}$ increases. However, the relic density decreases when the new corrections are implemented because the resonance is shifted towards energies that are easier to access thermally. The dashed, red line shows $p_{\text{eff}}^2 K_1(\sqrt{s}/T)$ rescaled by an arbitrary factor as a function of $p = p_{\text{eff}}$.

Figure 5.6: Difference between the relic density with the old and new mass corrections, when the main coannihilation channels are included.
Decay width of the KK level 2 singlet lepton

The relic density is very sensitive to error in the decay width of $l_{s}^{(2)}$. Adding a small term $\delta \Gamma = \epsilon \times m_{l^{(2)}}$, so that the total decay width is:

$$\Gamma'_{l^{(2)}} = \Gamma_{l^{(2)}} + \delta \Gamma,$$

changes the relic density drastically when the term is of the same order as the decay width. Figure 5.7 shows how the relic density changes as a function of $\Gamma' / \Gamma$. If $\delta \Gamma$ is of the same order as $\Gamma$, the relic density increases significantly.

The decay widths implemented in the UED module have been checked in the limit where there is no weak or fermionic mixing, and gives the same analytic expressions as found in Ref. [42]. In addition, the branching fractions of $l_{s}^{(2)}$ to the main decay channels generated by routines in the UED module are almost exactly equal to those found in Ref. [2] when $R^{-1}$ is large. Thus, the decay channels implemented in the UED module are most likely correct when $R^{-1}$ is large, although more channels should be implemented because the resonance is very sensitive to small changes.

Figure 5.7: The relic density when $R^{-1} = 1500\text{GeV}$ and $\Lambda R = 20$ as a function of $\Gamma' / \Gamma$. 
5.3 Discussion of Parameter Space

The cut-off, $\Lambda$

Since theories with an odd number of dimensions are not renormalizable it is necessary to define a cut-off, $\Lambda$. Beyond the cut-off, the theory is no longer valid. $\Lambda$ is one of the two free parameters of mUED and is included in the radiative corrections of the KK masses and vertices. Hence, the phenomenology of mUED depends on $\Lambda$.

Since $\Lambda$ defines the cut-off after which the theory is no longer valid, we want it to be as large as possible. However, when $\Lambda$ becomes very large, the radiative corrections can make the KK number violating couplings too large for perturbation theory to still be valid. The upper bound has been set at $\Lambda R = 50$ based on the calculations of Ref. [43] which show that the $U(1)$ coupling blows up when $\Lambda R = 50$ and $R^{-1} = 1$ TeV. In addition, one can estimate that no new physics should appear before $\Lambda = 20R^{-1}$ [1], which sets the lower bound at $\Lambda R = 20$.

The relic density with and without coannihilations has been calculated as a function of $R^{-1}$ for $\Lambda R = 20, 30, 40, 50$ and $60$ in figure 5.8. As one can tell, when $\Lambda$ becomes larger, the relic density becomes less sensitive to changes in $\Lambda$. This is because the radiative corrections only depend on the logarithm of $\Lambda$.

The masses of the KK particles generally increase when $\Lambda$ increases. This results in a larger relic density when only DM annihilation is included because the cross section is inversely proportional to the DM mass. The relic density including coannihilations decreases when $\Lambda$ increases because the increased mass of $l_s^{(2)}$ shifts the resonance to a greater CM momentum, and thus towards energies that are easier to access thermally. The increased contribution from $p_{\text{eff}}^2 K_1(\sqrt{s_T})$ is sufficiently large so that the relic density decreases even though the resonance becomes weaker due to a larger $\Gamma_{l_s^{(2)}}$ decay width.
Figure 5.8: Relic density with and without coannihilation as a function of $R^{-1}$ for different $\Lambda R$. In both cases, there is a logarithmic dependence on $\Lambda$ that goes towards a constant as $\Lambda R$ increases. The relic density decreases as a function of $\Lambda R$ when coannihilation is included because the $l_s^{(2)}$ mass increases. This pushes the resonance towards the peak of $p_{\text{eff}}^2 K_1(\sqrt{s}/T)$ so that $\langle \sigma v \rangle$ increases. The increased contribution from $p_{\text{eff}}^2 K_1(\sqrt{s}/T)$ is sufficiently large so that $\langle \sigma v \rangle$ increases even though the decay width also increases due to the larger mass difference between the decay products and $l_s^{(2)}$. 

67
The radiative corrections to the KK photon mass are generally small, so that the LKP mass can be approximated as $R^{-1}$. Thus, the relic density depends strongly on the LKP mass, as one would expect for WIMP dark matter (see Eq. (60)). As discussed in the previous section, adding coannihilation channels with KK level 2 particles in the final state decreases the relic density significantly. The bounds set by Belangér et al. put an upper limit at $R^{-1} \approx 1300 - 1600$ GeV when $\Lambda R = 20 - 50$.

Adding the finite contributions to the masses increases the relic density by a few percent when only the annihilation channels are implemented. We naively expect the relic density including the main coannihilation channel to increase as well, since both the mass difference between the LKP and the NLKP increases and the decay widths of the KK level 2 leptons increase. However, the relic density decreases because the resonance becomes easier to access thermally. Implementing the new corrections increases the upper bound by approximately 200 GeV when $\Lambda R = 20$ and 230 GeV when $\Lambda R = 50$ compared to the relic density calculated with the old corrections.

The bounds set by Belangér et al. are outdated since the relic density has been measured much more precisely since they performed their calculations in 2010. Based on the 2018 results from Planck [3] the calculations by Belangér et al. including all coannihilations (corresponds to line c1 in fig. 3.3) give the correct relic density when $1400 < R^{-1} < 1460$ GeV when $\Lambda R = 20$ and $1460 < R^{-1} < 1525$ GeV when $\Lambda R = 50$, within 5$\sigma$.

Based on the decrease of the relic density when only the $l_s(1)\gamma(1) \rightarrow \gamma(2)l$ coannihilation channels are included, we naively expect the bounds set by Belangér et al. to increase to $1600 < R^{-1} < 1660$ GeV ($\Lambda R = 20$) and $1630 < R^{-1} < 1785$ GeV ($\Lambda R = 50$) when the new mass corrections are implemented.

However, the effect of including the new mass corrections will most probably change when all coannihilation channels are implemented in the UED module, because the decrease of the relic density is due to the shift of the $l_s^{(2)}$ resonance when only the $l_s(1)\gamma(1) \rightarrow \gamma(2)l$ channels are implemented. The new corrections lead to an increase in the masses of the KK particles, which leads to a stronger suppression according to Eq. (82).

Nevertheless, we can naively approximate the current bounds on $R^{-1}$ within 5$\sigma$, including the finite corrections from Freitas et al., to be,

$$1400 < R^{-1} < 1660 \text{ GeV}, \quad \Lambda R = 20$$
$$1400 < R^{-1} < 1785 \text{ GeV}, \quad \Lambda R = 50 . \quad (95)$$

The upper bounds obtained by Belangér et al. are still outside of current collider bounds, but a LHC run with CM energy of 14 TeV might be able to probe these energies according to Ref. [34]. Including the new corrections to the masses increases the upper limit when $\Lambda R = 50$ to energies that barely cannot be reached by the LHC at a CM energy of 14 TeV. Thus, mUED might still be discovered at the LHC, but based on the naive estimates above it cannot be refuted either.
6 Concluding Remarks

The main goal of this thesis was to calculate the relic density of Kaluza-Klein dark matter by implementing a UED module in DarkSUSY. A secondary goal was to implement the finite terms of the radiative corrections arising from orbifold corrections, in order to check whether they have a significant effect on the LKP relic density. These have recently been calculated by Freitas et al. [2].

The UED relic density routine contains the main annihilation and coannihilation channels. The relic density reproduces previous results well when only the annihilation channels are included in the calculations, indicating that the relic density routines in the UED module work well.

There are two main results obtained by the work performed in this thesis. The first is that the effect of adding the $H^{(2)}$ resonance is generally larger than what was obtained in Ref. [9]. This is because the coupling implemented in the UED module, which is calculated in Ref. [36], is smaller resulting in a stronger resonance. This effect is weaker when the finite corrections are included, because these include a correction term to the $H^{(2)}t\bar{t}$ effective coupling. This suggests that either the coupling calculated by Ref. [36] or implemented by Ref. [9] is wrong.

The second main result is that the new mass corrections lead to a change in the relic density. When only the annihilation channels are included in the relic density calculations, the new mass corrections lead to an increase ranging from 5 to 24 percent.

The opposite happens when the main coannihilation channels are included. Implementing the finite corrections decreases the relic density by approximately 6%. Since the experimental accuracy of the DM density obtained by the Planck collaboration is $\sim 1\%$, this is a significant result.

Based on the decrease of the relic density with the new mass corrections as calculated by the currently implemented routines in the UED module, one can naively estimate new bounds on the scale of the extra dimension:

$$1400 < R^{-1} < 1785\text{GeV} ,$$

when $\Lambda R = 50$ and within $5\sigma$.

However, more coannihilation channels should be added to the UED module in order to calculate the relic density precisely, as discussed in sections 5.2 and 5.3.

To conclude, the results presented in this thesis suggest that the updated mass corrections have a significant effect on the relic density, but more coannihilation channels need to be implemented in order to ascertain this.
A The Standard Model

Cross section and Decay width

The cross section of a $2 \to 2$ process is the likelihood that two incoming particles produce a specific pair of final state particles. It is related to the Feynman diagrams that describe the process, by the invariant matrix element. The cross section in the center-of-mass frame for a process $A, B \to 1, 2$ is defined as:

$$\left( \frac{d\sigma}{d\Omega} \right)_{CM} = \frac{1}{2E_A2E_B|v_A-v_B|} \frac{|p_1|}{(2\pi)^24E_{cm}} |M(p_A, p_B \to p_1, p_2)|^2,$$  \hspace{1cm} (96)

where $p_i, v_i$ are the three-momenta and velocities of the particles, $E_i = \sqrt{p_i^2 + m_i^2}$ are their energies and $d\Omega = d\cos\theta d\phi$ is the differential solid angle.

The decay width of a particle $A$ is defined as,

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of A particles present}}.$$

For a decay process with two final state particles, the decay width in the rest frame of the decaying particle, is defined as:

$$\frac{d\Gamma}{d\Omega} = \frac{1}{32\pi^2} \frac{|p_1|}{M^2} |M|^2,$$  \hspace{1cm} (97)

where $M$ is the mass of $A$ and $p_1$ is the three-momentum of one of the final state particles.

A short introduction to group theory

A group is defined as a set of mathematical objects with some operation that takes two elements of the group to form a third element. We are mainly interested in two types of groups: Abelian and non-Abelian. These groups are part of what is called Lie groups. Lie groups are continuous groups that contain elements arbitrarily close to the identity. This means that any group element within the group can be written as

$$g(\alpha) = 1 + i\alpha^a T^a + O(\alpha),$$  \hspace{1cm} (98)

where $\alpha^a$ are the coefficients of the infinitesimal group parameters and $T^a$ are the (hermitian) generators of the symmetry group. The generators have the commutation relations

$$[T^a, T^b] = i f^{abc} T^c,$$  \hspace{1cm} (99)

where $f^{abc}$ are the structure constants of the group.

All the structure constants are zero for an Abelian group, meaning that all generators commute, whereas the structure constants for a non-Abelian group do not all have to be zero.

Groups have been important for the development of the standard model to classify the transformations a theory is invariant under, and the relationship between different types of particles. The standard model contains two types of groups: $SU(N)$ and $U(1)$. $U(1)$
is an Abelian group which has the same structure as the group of phase rotations. Invariance under $U(1)$ means that the equations of motion (e.o.m) do not change under the transformation,

$$\psi \rightarrow e^{i\alpha} \psi,$$

where $\alpha$ is some phase. The family of non-Abelian groups, $SU(N)$, consists of all unitary $N \times N$ transformations $U$, with Det($U$) = 1. Invariance under some general non-Abelian group, means that the e.o.m. are invariant under the transformation,

$$\psi \rightarrow e^{i\alpha^a t^a} \psi,$$

where $t^a$ are the generators of the group.

For the purpose of using groups as a way to describe elementary particles and their interactions, it is necessary to use group representations. These can be used to describe group elements as matrices, such that the group operations can be reduced to matrix multiplications.

**SU(2)**

The generators of SU(2) are: $T^a = \sigma^a/2$, where $\sigma^a$ are the Pauli matrices:

$$\sigma^1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

The Pauli matrices have commutation relations $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, where $\epsilon^{abc}$ is the Levi-Civita symbol.

Fundamental representation: The fundamental representation of SU(2) is a complex doublet.

**U(1):**

U(1) is the group of all 1-dimensional, unitary matrices. The generator $T = 1$, and thus the structure constant is zero.

**Dirac Matrices**

The Dirac matrices are a set of matrices that are essential in the construction of spinors and the Dirac equation. In 3+1 dimensions, they are 4-dim. matrices defined as:

$$\gamma^0 = \left( \begin{array}{cc} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{array} \right), \quad \gamma^i = \left( \begin{array}{cc} 0 & \sigma^i \\ -\sigma^i & 0 \end{array} \right),$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix and $\sigma^i$ are the Pauli matrices.

The Dirac matrices have the following anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

The Dirac matrices can be used to construct the fifth Dirac matrix: $\gamma^5$:

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3,$$

with the properties:

$$\{\gamma^\mu, \gamma^5\} = 0,$$

$$(\gamma^5)^\dagger = -\gamma^5,$$

$$(\gamma^5)^2 = -1.$$
Chirality Projection Operators

The fifth Dirac matrix is used to define the chirality projection operators $P_{L/R}$ which project out the LH or RH parts of a spinor in an even number of dimensions:

$$P_L \psi = \psi_L, \quad P_R \psi = \psi_R.$$  \hfill (107)

They are defined as:

$$P_{L/R} = \frac{1}{2} (1 \pm \gamma^5),$$  \hfill (108)

with properties:

$$(P_{L/R})^2 = P_{L/R}$$

$$P_L P_R = 1$$  \hfill (109)

$$P_{L/R} \gamma^\mu = \gamma^\mu P_R/L$$

The Lorentz Group

The Lorentz group is the group of symmetries of the SR spacetime. This includes rotations and linear transformations of the coordinates that do not change the line element. I.e. A Lorentz transformation $\Lambda_{\mu}^{\nu}$ is an element of the Lorentz group and must satisfy:

$$x_\mu x^\mu = x'_\mu x'^\mu = \eta_{\mu \nu} \Lambda_\mu^\alpha x_\alpha \Lambda_\nu^\beta x^\beta.$$  \hfill (110)

The Dirac matrices can be used to construct the Lorentz algebra:

$$[\Sigma^{\mu \nu}, \Sigma^{\sigma \rho}] = i (\eta^{\nu \sigma} \Sigma^{\mu \rho} - \eta^{\mu \sigma} \Sigma^{\nu \rho} + \eta^{\mu \rho} \Sigma^{\nu \sigma} - \eta^{\nu \rho} \Sigma^{\mu \sigma}),$$  \hfill (111)

where the generators of the algebra are $\Sigma^{\mu \nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$.

Electroweak theory

Electroweak theory is a part of the SM together with the strong interaction and $SU(3)$ group. It describes the interactions between fermions, the Higgs boson, the weak gauge bosons and the photon. The fields are invariant under the symmetry group:

$$G_{EW} = SU(2)_L \times U(1)_Y.$$  \hfill (112)

where the "L" means that the $SU(2)$ group only couples to left-handed fermions and the "Y" stands for hypercharge, which is the (eigenvalue of the) Noether charge associated with this group. The $Y$ is included to emphasize that this $U(1)$ group is different from the electromagnetic group $U(1)_{EM}$ which is associated with the electromagnetic charge. After SSB, the $U(1)_{EM}$ group remains, ensuring that the photon is massless.

After SSB, the mass eigenstates of the gauge group are:

$$W^\pm_\mu = \frac{1}{\sqrt{2}} (A^1_\mu + i A^2_\mu)$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g_Y^2}} (g A^3_\mu - g_Y B_\mu)$$  \hfill (113)

$$A_\mu = \frac{1}{\sqrt{g^2 + g_Y^2}} (g_Y A^3_\mu + g B_\mu)$$
with masses $m_W = g^2_{W_2}, m_Z = \sqrt{g^2 + g^2_{Y,\gamma}}$, $m_A = 0$. These eigenstates are what we call the $W^\pm, Z$ bosons and the photon. $Z_\mu^0$ and $A_\mu$ are related by the Weinberg angle:

$$
\theta_W = \frac{1}{2} \arctan \left[ \frac{g^2 g^2 v^2}{v^2 (g^2 - g^2)} \right].
$$

(114)

The Standard Model

The Standard Model Lagrangian is invariant under the symmetry group:

$$
G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y,
$$

(115)

The SM Lagrangian can be compactified into the following form:

$$
L_{SM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} D\psi + \psi_i \lambda_{ij} \bar{\psi}_j \phi + h.c. + |D_\mu \phi|^2 - V(\phi),
$$

(116)

where $D_\mu = \partial_\mu + ig_3 G_{\mu}^A + ig A_\mu^a + ie B_\mu$ is the covariant derivative, where the gauge indices $A$ and $a$ of the SU(3) and SU(2) group run from 1 to 8 and from 1 to 3. $\lambda_{ij}$ is the Yukawa-couplings between the Higgs field and fermions and $V(\phi)$ is the Higgs potential.

B Masses and vertices

B.1 Useful expressions and parameters

The following notation, constants and parameters are present in the masses of KK particles and the Feynman rules for vertices involving Kaluza-Klein particles. Some of these parameters have already been defined earlier in the thesis, but for the ease of the reader, they are all presented here as well.

**Standard Model Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine-structure constant</td>
<td>$\alpha_{EW} = 1/128$</td>
</tr>
<tr>
<td>Strong fine-structure constant ($m_Z$-scale)</td>
<td>$\alpha_S = 0.1172$</td>
</tr>
<tr>
<td>Elementary charge</td>
<td>$e = \sqrt{4\pi \alpha_{EW}}$</td>
</tr>
<tr>
<td>Weinberg angle</td>
<td>$\theta_W = \arccos(m_W/m_Z)$</td>
</tr>
<tr>
<td>$U(1)$ coupling constant</td>
<td>$g_Y = \frac{e}{\cos(\theta_W)}$</td>
</tr>
<tr>
<td>$SU(2)$ coupling constant</td>
<td>$g = \frac{e}{\sin(\theta_W)}$</td>
</tr>
<tr>
<td>$SU(3)$ coupling constant</td>
<td>$g_3 = \sqrt{4\pi \alpha_s}$</td>
</tr>
<tr>
<td>Higgs VEV</td>
<td>$v = \frac{2\alpha_W}{g}$</td>
</tr>
<tr>
<td>Yukawa coupling with Higgs</td>
<td>$2\pi \frac{\alpha_s}{v}$</td>
</tr>
</tbody>
</table>
mUED Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weinberg angle</td>
<td>$\theta_W^{(n)} = \frac{1}{2} \arctan \left[ \frac{g_2^2 g_\nu^2 v^2}{\sqrt{(g_2^4 - g_\nu^4) + 2\delta m^2_{B(n)} - 2\delta m^2_{A(n)}}} \right]$</td>
</tr>
<tr>
<td>Fermion mixing angle</td>
<td>$\tan 2\alpha^{(n)} = \frac{2m_{EW}}{2m_n + \delta m_d^{(n)} + \delta m_s^{(n)}}$</td>
</tr>
<tr>
<td>Log-term in radiative corrections</td>
<td>$[2]: L_n = \ln \left( \frac{\Lambda^2}{m_n^2} \right)$</td>
</tr>
<tr>
<td>Geometric mass relation</td>
<td>$m_n = \frac{n}{R}$</td>
</tr>
<tr>
<td>Cut-off</td>
<td>$\Lambda = x/R$, where $x$ is a free parameter</td>
</tr>
</tbody>
</table>

SM charges

<table>
<thead>
<tr>
<th>Particle type</th>
<th>$Y_d$</th>
<th>$Y_s$</th>
<th>$T_3$</th>
<th>$N_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutrinos</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>Charged leptons</td>
<td>-1/2</td>
<td>-1</td>
<td>-1/2</td>
<td>1</td>
</tr>
<tr>
<td>Up-type quarks</td>
<td>1/6</td>
<td>2/3</td>
<td>1/2</td>
<td>3</td>
</tr>
<tr>
<td>Down-type quarks</td>
<td>1/6</td>
<td>-1/3</td>
<td>-1/2</td>
<td>3</td>
</tr>
<tr>
<td>Higgs boson</td>
<td>1/2</td>
<td>1/2</td>
<td>-1/2</td>
<td>1</td>
</tr>
</tbody>
</table>

B.2 KK masses

Table 5 shows the KK masses implemented in the calculation of the various cross sections. The first element for each particle type is the implemented mass, while the second and third elements are the total mass correction: bulk correction + corrections from the boundary terms. The second element is the currently most updated mass correction, from [2]. This correction involved a second contribution to the correction from the boundary terms. The third element is the previous standard for the mass correction, from [15].
<table>
<thead>
<tr>
<th>Particle type</th>
<th>KK mass</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photon $\gamma^{(n)}$</td>
<td>$m_{A^{(n)}}^2 = m_n^2 + \frac{1}{2} m_Z^2 + \frac{1}{2} \delta m_{B^{(n)}}^2 + \frac{1}{2} \delta m_{A^{(n)}}^2$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$(\delta m_{B^{(n)}}^2 - \delta m_{A^{(n)}}^2 + m_Z^2 - 2 m_W^2) \sqrt{1 + \tan^2 2\theta_W^{(n)}}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\delta m_{B^{(n)}}^2 = -\frac{39}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 - m_n^2 \frac{g_s^2}{16\pi^2} (\frac{1}{R} \frac{L_n - 2}{5})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{A^{(n)}}^2 = -\frac{39}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 - m_n^2 \frac{g_s^2}{64\pi^2} \frac{1}{5} L_n$</td>
<td>[15]</td>
</tr>
<tr>
<td>W-boson $W^{(n)}$</td>
<td>$m_{W^{(n)}}^2 = m_W^2 + m_n^2 - \frac{5}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} (15L_n + \frac{104}{3})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{W^{(n)}}^2 = -\frac{5}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} (15L_n + \frac{104}{3})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{W^{(n)}} = -\frac{5}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} 15L_n$</td>
<td>[15]</td>
</tr>
<tr>
<td>Z-boson $Z^{(n)}$</td>
<td>$m_{Z^{(n)}}^2 = m_n^2 + \frac{1}{2} m_Z^2 + \frac{1}{2} \delta m_{B^{(n)}}^2 + \frac{1}{2} \delta m_{A^{(n)}}^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\delta m_{B^{(n)}}^2 - \delta m_{A^{(n)}}^2 + m_Z^2 - 2 m_W^2) \sqrt{1 + \tan^2 2\theta_W^{(n)}}$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{A^{(n)}} = \frac{5}{2} g_s^2 \xi(3) (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} (15L_n + \frac{104}{3})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{A^{(n)}} = m_n^{2a} - \frac{5}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} 15L_n$</td>
<td>[15]</td>
</tr>
<tr>
<td>Gluon $G^{(n)}$</td>
<td>$m_{G^{(n)}}^2 = m_n^2 - \frac{3}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} (23L_n + \frac{154}{3})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{G^{(n)}} = -\frac{3}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} (23L_n + \frac{154}{3})$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{G^{(n)}} = -\frac{3}{2} \frac{g_s^2 \xi(3)}{16\pi^2} (\frac{1}{R})^2 + m_n^2 \frac{g_s^2}{32\pi^2} 23L_n$</td>
<td>[15]</td>
</tr>
<tr>
<td>Higgs boson $H^{(n)}$</td>
<td>$m_{H^{(n)}}^2 = m_n^2 + m_H^2 + m_n^2 \frac{1}{16\pi^2} (g^2 (\frac{2}{3} L_n + 6) + g_Y^2 (\frac{3}{4} L_n + 2) - \lambda_H (L_n + 1))$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{H^{(n)}} = \frac{m_H^2}{2} + m_n^2 \frac{1}{16\pi^2} ((\frac{3}{2} g^2 + \frac{1}{8} y_Y^2) - \lambda_H (L_n + 1))$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\delta m_{H^{(n)}} = \frac{m_H^2}{2} + m_n^2 \frac{1}{16\pi^2} (\frac{3}{4} g^2 + \frac{3}{4} g_Y^2 - \lambda_H) L_n + \frac{m_H^2}{4}$</td>
<td>[15]</td>
</tr>
<tr>
<td>Scalar states $a_0^{(n)}, a_\pm^{(n)}$</td>
<td>$m_{a_0}^{2(n)} = m_n^2 + \delta m_{a_0}^{2(n)}$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$m_{a_0}^{2(n)} = m_n^2 + \delta m_{a_0}^{2(n)}$</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4} \frac{g_s^2 m_n^2}{(M_W^{(n)})^2} \left[ \left( \frac{1}{2} g^2 + 4 g_Y^2 \right) \frac{9 L_n + 16}{} \right]$</td>
<td></td>
</tr>
</tbody>
</table>
\[ \delta m^2_{a_\pm} = m_n^2 \frac{1}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} \left[ \left( \frac{1}{2} g^2 + 4g_Y^2 \right) - \frac{\pi R}{4} \frac{g^2 m_n^2}{(M_n^2)^2} \left( \frac{m_n^2}{c_W} + \frac{\mu^2}{4m_W^2} \right) \right] \]  

\[ \delta m^2_{a_\pm} = m_n^2 \frac{1}{32\pi^2} \left[ \left( \frac{1}{2} g^2 + 4g_Y^2 \right) (9L_n + 16) - \frac{\pi R}{4} \frac{g^2 m_n^2}{m_W} \left( \frac{m_n}{M_n^2} \right)^2 (L_n + 1) \right] \]

Fermion doublet \( f_d^{(n)} \)

\[ \delta m_d^{(n)} = m_n \frac{1}{64\pi^2} \left( d_3 C_2(3) g_3^2 + d_2 C_2(2) g_2^2 + Y_d^2 g_Y^2 \right) (9L_n + 16) - \frac{m_n}{32\pi^2} h^2 (\frac{1}{2} L_n + 1) \]

\[ \delta m_s^{(n)} = m_n \frac{1}{64\pi^2} \left( d_3 C_2(3) g_3^2 + d_2 C_2(2) g_2^2 + Y_s^2 g_Y^2 \right) 9L_n - \frac{3 m_n}{32\pi^2} h^2 L_n \]

Fermion singlet \( f_s^{(n)} \)

\[ m_{f_s^{(n)}} = -\frac{1}{2} (\delta m_d^{(n)} - \delta m_s^{(n)}) + m_{EW} \left( 1 + \tan^{-1}(2\alpha^{(n)}) \right) \]

\[ m_{f_s^{(n)}} = -\frac{1}{2} (\delta m_d^{(n)} - \delta m_s^{(n)}) + m_{EW} \left( 1 + \tan^{-1}(2\alpha^{(n)}) \right) \]

Table 5: The complete expressions, including radiative corrections, of the Kaluza-Klein field masses. The first element of each particle type is the complete expression of the mass, while the second and third elements are the new [2] and old [15] mass corrections. \( d_3 \) and \( d_2 \) are added to distinguish between the couplings to quarks/leptons and doublets and singlets: \( d_3 = 1 \) for quarks and 0 for leptons, while \( d_2 = 1 \) for doublets and 0 for singlets.
B.3 KK Feynman Rules

The KK number conserving vertices can be found by using the general expressions in eqs. (39), (40), (41), (42) and (43) and inserting the mass eigenstates, if necessary.

The Feynman rules can be read off from the effective 4D Lagrangian in Ref. [2] by inserting the mass eigenstates. The structure of this appendix is as follows: The general structure of a vertex is presented in each subsection (for 3- and 4-point vertices, if implemented). The couplings between specific particles are presented in the tables.

No couplings involving Goldstone bosons have been implemented, since unitarity gauge has been used in all calculations. The propagators for the different fields, with 4-momentum $q$, are presented below:

- **Scalars:** \[ \frac{i}{q^2 - m^2 + i\epsilon} \]
- **Fermions:** \[ \frac{i(q + m)}{q^2 - m^2 + i\epsilon} \]
- **Massive vectors:** \[ \frac{i}{q^2 - m^2 + i\epsilon} \left( \eta_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{m^2} \right) \]
- **Massless vectors:** \[ \frac{-i}{q^2 + i\epsilon} \eta_{\mu\nu} \]

The corresponding part of the full 5D Lagrangian is presented together with the Feynman rules. The inverse expressions of the mass eigenstates are useful when deriving the Feynman rules:

\[ Z_5^{(n)} = \frac{m_Z}{M_Z^{(n)}} a_0^{(n)} - \frac{M^{(n)}}{M_Z^{(n)}} G_0^{(n)} \]
\[ \chi^{3(n)} = \frac{M^{(n)}}{M_Z^{(n)}} a_0^{(n)} + \frac{m_Z}{M_Z^{(n)}} G_0^{(n)} \]
\[ W_5^{\pm(n)} = \frac{m_W}{M_W^{(n)}} a_\pm^{(n)} - \frac{M^{(n)}}{M_W^{(n)}} G_\pm^{(n)} \]
\[ \chi^{\pm(n)} = \frac{M^{(n)}}{M_W^{(n)}} a_\pm^{(n)} + \frac{m_W}{M_W^{(n)}} G_\pm^{(n)} \]
\[ \psi_s^{(n)} = \sin \alpha^{(n)} \xi_s^{(n)} - \cos \alpha^{(n)} \gamma_5 \xi_s^{(n)} \]
\[ \psi_d^{(n)} = \cos \alpha^{(n)} \xi_d^{(n)} + \sin \alpha^{(n)} \gamma_5 \xi_s^{(n)} \]

The term “cross check” means that the Feynman rules have been compared to Ref. [13], which has already been checked with Ref. [44].
B.3.1 The Gauge sector

The 4D effective Lagrangian for all couplings between gauge bosons have been calculated in [2], for a general gauge field $V^a$. The full Lagrangian can be found in A.1 in [2].

$$\mathcal{L}_{\text{Gauge}} \supset 2gC^{abc} \sum_{n=1}^{\infty} F_{\mu \nu}^{n, a} V^{n, a \mu} V^{n, c \nu} + 2gC^{abc} \sum_{n=1}^{\infty} (\partial_{\mu} V_{\nu}^{n, a} - \partial_{\nu} V_{\mu}^{n, a}) \left( V^{0, a \mu} V^{n, c \nu} + V^{0, c \mu} V^{n, a \mu} \right)$$

(118)

The Feynman rules of a three-point vector coupling have the general form:

$$V_{1}^{\rho} \rightarrow \sum_{q_{2}} V_{2}^{\mu} \Rightarrow q_{2} \quad V_{1}^{\rho} \rightarrow \sum_{q_{3}} V_{3}^{\nu} \Rightarrow q_{3}$$

$$i g V_{1} V_{2} [(q_{1} - q_{2})^{\nu} \eta^{\rho \mu} + (q_{2} - q_{3})^{\rho} \eta^{\nu \mu} + (q_{3} - q_{1})^{\mu} \eta^{\rho \nu}] \quad (119)$$

<table>
<thead>
<tr>
<th>$g V_{1} V_{2} V_{3}$</th>
<th>Feynman rule</th>
<th>Ref.</th>
<th>Cross check</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{\mu}^{+}(n) W_{\mu}^{-}(n) A_{\mu}^{(0)}$</td>
<td>$-e$</td>
<td>[2]</td>
<td>[13] (A.10) ✓</td>
</tr>
<tr>
<td>$W_{\mu}^{+}(n) W_{\mu}^{-}(n) Z_{\mu}^{(0)}$</td>
<td>$-c_{w} g$</td>
<td>[2]</td>
<td>[13] (A.11) ✓</td>
</tr>
<tr>
<td>$W_{\mu}^{+}(n, 0) W_{\mu}^{-}(0, n) A_{\mu}^{(n)}$</td>
<td>$-\sin \theta_{w}^{(n)} e$</td>
<td>[2]</td>
<td>[13] (A.12) ✓</td>
</tr>
<tr>
<td>$W_{\mu}^{+}(n, 0) W_{\mu}^{-}(0, n) Z_{\mu}^{(n)}$</td>
<td>$-\cos \theta_{w}^{(n)} e g$</td>
<td>[2]</td>
<td>[13] (A.12) ✓</td>
</tr>
</tbody>
</table>

Table 6: Vector-Vector-Vector All couplings above should be multiplied with $i [(q_{1} - q_{2})^{\nu} \eta^{\rho \mu} + (q_{2} - q_{3})^{\rho} \eta^{\nu \mu} + (q_{3} - q_{1})^{\mu} \eta^{\rho \nu}]$, according to Eq. (119).
B.3.2 The Fermion sector

The 4D effective Lagrangian for all couplings between fermions and gauge bosons have been calculated in [2], for a general gauge field $V^a$ and a doublet fermion $\Psi(x, x^5)$. The Lagrangian for a singlet fermion can be constructed in complete analogy. The Feynman rules are found by inserting the mass eigenstates of the fermions (32) and gauge bosons (47).

$$\mathcal{L}_\Psi \supset \sum_{n=1}^{\infty} \left[ i \bar{\Psi}_R \gamma^\mu (\partial_\mu + ig V_\mu^0) \Psi_R^n + i \bar{\Psi}_L \gamma^\mu (\partial_\mu + ig V_\mu^0) \Psi_L^n - g \bar{\Psi}_L \gamma^\mu V_\mu^n \Psi_L^n + g \bar{q}_L i \gamma^5 V_5^n \Psi_R^n \right]$$

(120)

The Feynman rules for a three-point coupling between two fermions and a vector particle has the general form:

$$V^\mu \rightarrow \bar{\xi}_1 \xi_2 = i \gamma^\mu g_V \bar{\xi}_1 \xi_2$$

(121)
Table 7: **Fermion-Fermion-Vector couplings.** All couplings above should be multiplied with $i\gamma^\mu$, according to Eq. (121). Note that these expressions are given in the $A^{(n)}_\mu$, $B^{(n)}_\mu$ basis for simplicity (except when $n = 0$). The Feynman rules in the $Z^{(n)}_\mu$, $A^{(n)}_\mu$ basis are easily derived by using the inverse expressions of the mass eigenstates stated in Eq. (47). The generic vector field $V^{(n)}_\mu$ generally has a different eigenvalue associated with singlet and doublet fermion fields, denoted by $C_s$ and $C_d$. The eigenvalues corresponding to the weak gauge fields $B_\mu$ and $A^3_\mu$ are given in Table 11.

<table>
<thead>
<tr>
<th>$g V \xi \xi$</th>
<th>Feynman rule</th>
<th>Ref.</th>
<th>Cross check</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(0)}_\mu \xi^{(n)}_d \xi^{(n)}_s$</td>
<td>$cQ$</td>
<td>[2]</td>
<td>[13] (A.97) ✓</td>
</tr>
<tr>
<td>$Z^{(0)}_\mu \xi^{(n)}_d \xi^{(n)}_d$</td>
<td>$\frac{g}{c_w} \left(-Y_d s^2_w + T_3 c^2_w \cos^2 \alpha(n) - T_3 s^2_w \sin^2 \alpha(n)\right)$</td>
<td>[2]</td>
<td>[13] (A.99) ✓</td>
</tr>
<tr>
<td>$Z^{(0)}_\mu \xi^{(n)}_d \xi^{(n)}_s$</td>
<td>$\frac{g}{c_w} \left(-Y_d s^2_w + T_3 c^2_w \sin^2 \alpha(n) - T_3 s^2_w \cos^2 \alpha(n)\right)$</td>
<td>[2]</td>
<td>[13] (A.100) ✓</td>
</tr>
<tr>
<td>$Z^{(0)}_\mu \xi^{(n)}_s \xi^{(n)}_d$</td>
<td>$2 \frac{g}{c_w} T_3 \sin \alpha(n) \cos \alpha(n)$</td>
<td>[2]</td>
<td>[13] (A.101)</td>
</tr>
<tr>
<td>$A^{(3)}_\mu \xi^{(0)}_d \xi^{(n)}_d$</td>
<td>$-\frac{1}{2} g T^3 \cos \alpha(n) P_L$</td>
<td>[2]</td>
<td>[13] (A.102) ✓</td>
</tr>
<tr>
<td>$A^{(3)}_\mu \xi^{(0)}_s \xi^{(n)}_s$</td>
<td>$\frac{1}{2} g T^3 \sin \alpha(n) P_L$</td>
<td>[2]</td>
<td>[13] (A.103) ✓</td>
</tr>
<tr>
<td>$B^{(n)}_\mu \xi^{(n)}_d \xi^{(n)}_d$</td>
<td>$-g Y (Y_d \cos \alpha(n) P_L + Y_s \sin \alpha(n) P_R)$</td>
<td>[2]</td>
<td>[13] (A.104) ✓</td>
</tr>
<tr>
<td>$B^{(n)}_\mu \xi^{(n)}_s \xi^{(n)}_s$</td>
<td>$g Y (Y_d \sin \alpha(n) P_L + Y_s \cos \alpha(n) P_R)$</td>
<td>[2]</td>
<td>[13] (A.105) ✓</td>
</tr>
<tr>
<td>$V^{(m+n)}_\mu \xi^{(m)}_d \xi^{(n)}_d$</td>
<td>$\frac{g}{\sqrt{2}} \left( \cos \alpha(n) \cos \alpha(m) C_d + \sin \alpha(n) \sin \alpha(m) C_s \right) \gamma^5$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(m+n)}_\mu \xi^{(m)}_s \xi^{(n)}_s$</td>
<td>$\frac{g}{\sqrt{2}} \left( \sin \alpha(n) \sin \alpha(m) C_d + \cos \alpha(n) \cos \alpha(m) C_s \right) \gamma^5$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(m+n)}_\mu \xi^{(m)}_d \xi^{(n)}_s$</td>
<td>$\frac{g}{\sqrt{2}} \left( \cos \alpha(m) \sin \alpha(n) C_d - \sin \alpha(m) \cos \alpha(n) C_s \right)$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(m+n)}_\mu \xi^{(m)}_s \xi^{(n)}_d$</td>
<td>$\frac{g}{\sqrt{2}} \left( \sin \alpha(m) \cos \alpha(n) C_d - \cos \alpha(m) \sin \alpha(n) C_s \right)$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(n-m)}_\mu \xi^{(n)}_d \xi^{(m)}_d$</td>
<td>$\sqrt{2} g \left( \sin \alpha(m) \sin \alpha(n) C_d + \cos \alpha(m) \cos \alpha(n) C_s \right)$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(n-m)}_\mu \xi^{(n)}_s \xi^{(m)}_s$</td>
<td>$\sqrt{2} g \left( \cos \alpha(m) \sin \alpha(n) C_d - \sin \alpha(m) \cos \alpha(n) C_s \right) \gamma^5$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$V^{(n-m)}_\mu \xi^{(n)}_d \xi^{(m)}_s$</td>
<td>$\sqrt{2} g \left( \cos \alpha(n) \sin \alpha(m) C_d + \sin \alpha(n) \cos \alpha(m) C_s \right) \gamma^5$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
</tbody>
</table>
B.3.3 The Higgs sector

Both three- and four-point couplings have been implemented in DarkSUSY and are relevant for the annihilation cross section. This sector includes couplings between the scalar particles present in mUED and gauge bosons. The vertex rules have been derived from the effective 4D Lagrangian in Ref. [2].

**Scalar-Vector-Vector**
The Feynman rule between two vector particles and a scalar has the following, general form:

\[
S \rightarrow V_2^\nu \rightarrow V_1^\mu = ig_s V_1^\nu V_2^\mu \eta^{\mu\nu} \tag{122}
\]

**Scalar-Scalar-Vector**
The Feynman rule between two scalar particles and a vector has the following, general form:

\[
V \rightarrow V_1^\mu \rightarrow S_2 = ig_{V_1 S_2} (q_1 - q_2)^\mu \tag{123}
\]

**Scalar-Scalar-Scalar**
The Feynman rule between two vector particles and a scalar has the following, general form:

\[
S_1 \rightarrow S_2 \rightarrow S_3 = ig_{S_1 S_2 S_3} \tag{124}
\]

**Vector-Vector-Scalar-Scalar**
The Feynman rule between two vector particles and two scalars has the following, general form:

\[
V_2^\nu \rightarrow V_1^\mu \rightarrow S_1 = ig_{V_1 V_2 S_1} \eta^{\mu\nu} \tag{125}
\]
<table>
<thead>
<tr>
<th>$g_{SV_1V_2}$</th>
<th>Feynman rule</th>
<th>Ref.</th>
<th>Cross check</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\mu}^{(n)} Z_{\nu}^{(0)} H^{(n)}$</td>
<td>$g m_Z$</td>
<td>[2]</td>
<td>[13] (A.21) ✓</td>
</tr>
<tr>
<td>$B_{\mu}^{(n)} Z_{\nu}^{(0)} H^{(n)}$</td>
<td>$g_Y m_Z$</td>
<td>[2]</td>
<td>[13] (A.22) ✓</td>
</tr>
<tr>
<td>$A_{\mu}^{(n)} A_{\nu}^{(n)} H^{(0)}$</td>
<td>$g m_W$</td>
<td>[2]</td>
<td>[13] (A.32) ✓</td>
</tr>
<tr>
<td>$B_{\mu}^{(n)} A_{\nu}^{(n)} H^{(0)}$</td>
<td>$g_Y m_W$</td>
<td>[2]</td>
<td>[13] (A.33) ✓</td>
</tr>
<tr>
<td>$B_{\mu}^{(n)} B_{\nu}^{(n)} H^{(0)}$</td>
<td>$g m_W \frac{m^2_w}{c m^2}$</td>
<td>[2]</td>
<td>[13] (A.34) ✓</td>
</tr>
<tr>
<td>$A_{\mu}^{(n)} A_{\nu}^{(n)} H^{(0)}$</td>
<td>$i \frac{m_w}{g} (g_Y \cos \theta_W^{(n)} - g \sin \theta_W^{(n)})^2$</td>
<td>[2]</td>
<td>[13] Not implemented</td>
</tr>
<tr>
<td>$Z_{\mu}^{(n)} Z_{\nu}^{(n)} H^{(0)}$</td>
<td>$i \frac{m_w}{g} (g \cos \theta_W^{(n)} + g_Y \sin \theta_W^{(n)})^2$</td>
<td>[2]</td>
<td>[13] Not implemented</td>
</tr>
<tr>
<td>$Z_{\mu}^{(n)} A_{\nu}^{(n)} H^{(0)}$</td>
<td>$2 i \frac{m_w}{g} (g \cos \theta_W^{(n)} + g_Y \sin \theta_W^{(n)}) (g_Y \cos \theta_W^{(n)} - g \sin \theta_W^{(n)})$</td>
<td>[2]</td>
<td>[13] Not implemented</td>
</tr>
<tr>
<td>$A_{\mu}^{(n)} a_0^{(n)} H^{(0)}$</td>
<td>$(g \sin \theta_w^{(n)} - g_Y \cos \theta_W^{(n)}) \frac{m_n}{M_Z^{(n)}}$</td>
<td>[2]</td>
<td>[13] (A.61), (A.66) ✓</td>
</tr>
<tr>
<td>$Z_{\mu}^{(n)} a_0^{(n)} H^{(0)}$</td>
<td>$(g \cos \theta_w^{(n)} + g_Y \sin \theta_W^{(n)}) \frac{m_n}{M_Z^{(n)}}$</td>
<td>[2]</td>
<td>[13] (A.61), (A.66) ✓</td>
</tr>
<tr>
<td>$H^{(n)} a_0^{(n)} Z^{(0)}$</td>
<td>$- \frac{g}{2 c w} \frac{m_n}{M_Z^{(n)}}$</td>
<td>[2]</td>
<td>[13] (A.48) ✓</td>
</tr>
<tr>
<td>$A_{\mu}^{(2n)} A_{\nu}^{(n)} a_0^{(n)}$</td>
<td>$\frac{m_n}{M_Z^{(n)}} (g^2 \sin \theta_w^{(n)} \sin \theta_w^{(2n)} - g_Y^2 \cos \theta_w^{(n)} \cos \theta_w^{(2n)}) / 4$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$A_{\mu}^{(2n)} A_{\nu}^{(n)} H^{(n)}$</td>
<td>$\frac{3}{4} \frac{c}{w} (g \sin \theta_w^{(n)} + g_Y \cos \theta_w^{(n)}) (g \sin \theta_w^{(2n)} + g_Y \cos \theta_w^{(2n)})$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$A_{\mu}^{(n)} a_{\pm}^{(n)} W_{\mp}^{(0)}$</td>
<td>$\pm m_W \frac{m_n}{M_W^{(n)}} (g_Y \cos \theta_W^{(n)} + g \sin \theta_W^{(n)})$</td>
<td>[2]</td>
<td>[13] (A.23), (A.26) ✓</td>
</tr>
<tr>
<td>$W_{\mu}^{+(n)} W_{\nu}^{-}(n) H^{(0)}$</td>
<td>$g m_W$</td>
<td>[2]</td>
<td>[13] (A.35) ✓</td>
</tr>
<tr>
<td>$H^{(n,0)} H^{(n,0)} H^{(0)}$</td>
<td>$- \frac{3}{2} \frac{m_n^2}{g m_W}$</td>
<td>[2]</td>
<td>[13] (A.77) ✓</td>
</tr>
</tbody>
</table>

Table 8: **Vector-Vector-Scalar, Vector-Vector-Scalar** and **Scalar-Scalar-Scalar**. All
couplings between two vectors and a scalar should be multiplied with $i \eta_{\mu \nu}$ according to Eq.
(122) , whereas the coupling between three scalars should be multiplied with $i$, Eq. (124).
Table 9: Vector-Vector-Scalar-Scalar
All couplings should be multiplied by $i \eta_{\mu\nu}$, according to Eq. (125). Since these are not implemented in neither [13] nor [44], they have not been cross-checked.

B.3.4 The Yukawa sector

The effective Lagrangian for the Yukawa part for a down-type fermion is [2]:

$$L_{\text{Yukawa}} = h_i \overline{\Psi}_L \psi R \Phi_0 + h_i \sum_{n=1}^{\infty} \left[ \overline{\Psi}_L^n \psi_R^n \Phi_0 + \overline{\Psi}_R^n \psi_L^n \Phi_0 \right] + h_i \sum_{n=1}^{\infty} \left[ \overline{\Psi}_L \psi_R^n \Phi_n + \overline{\Psi}_R \psi_L^n \Phi_n \right]$$  \hspace{1cm} (126)

where $\Psi$ are the doublet spinors defined in Eq. (29) and $\psi$ are the singlet spinors. The Lagrangian for an up-type fermion can be constructed in the same way.

The Feynman rules for a general Yukawa coupling has the following form:

\[
\begin{align*}
\bar{\xi}_1 & \\
\xi_2 & \\
S \quad & \quad \quad \quad \quad \quad = \quad ig S \bar{\xi}_1 \xi_2 \\
\end{align*}
\]  \hspace{1cm} (127)
<table>
<thead>
<tr>
<th>$g_S \bar{\xi} \xi$</th>
<th>Feynman rule</th>
<th>Ref.</th>
<th>Cross check</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^{(0)} \bar{\xi}^{(n)} \xi^{(n)}<em>{d,s} S</em>{d,s}$</td>
<td>$-g \frac{m_e}{m_W} \sin \alpha^{(1)} \cos \alpha^{(1)}$</td>
<td>[2]</td>
<td><a href="A.117">13</a>, (A.118) ✓</td>
</tr>
<tr>
<td>$H^{(0)} \bar{\xi}^{(n)} \xi^{(n)}<em>{s,d} S</em>{d,s}$</td>
<td>$-g \frac{m_e}{m_W} \left(1 - 2 \cos^2 \alpha^{(1)}\right) \gamma^5$</td>
<td>[2]</td>
<td><a href="A.119">13</a> ✓</td>
</tr>
<tr>
<td>$H^{(n)} \bar{\xi}^{(0)} \xi^{(n)}_d$</td>
<td>$-g \frac{m_e}{m_W} \left(\sin \alpha^{(1)} P_R + \cos \alpha^{(1)} P_L\right)$</td>
<td>[2]</td>
<td><a href="A.120">13</a> ✓</td>
</tr>
<tr>
<td>$H^{(n)} \bar{\xi}^{(0)} \xi^{(n)}_s$</td>
<td>$g \frac{m_e}{m_W} \left(\cos \alpha^{(1)} P_R + \sin \alpha^{(1)} P_L\right)$</td>
<td>[2]</td>
<td><a href="A.121">13</a> ✓</td>
</tr>
<tr>
<td>$a^{(n)}_0 \bar{\xi}^{(0)} \xi^{(n)}_d$</td>
<td>$i \frac{m_Z}{M^2} \left{ \left(T_3 g_c w - Y_d g_Y s_w\right) \cos \alpha^{(1)} P_R + Y_d g_Y s_w \sin \alpha^{(1)} P_L\right}$</td>
<td>[2]</td>
<td>[13] (A.122) ✓</td>
</tr>
<tr>
<td></td>
<td>$-4igT_3 \frac{m_e}{m_W} \frac{M^{(1)}}{M^2} \left(\sin \alpha^{(1)} P_R - \cos \alpha^{(1)} P_L\right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a^{(n)}_0 \bar{\xi}^{(0)} \xi^{(n)}_s$</td>
<td>$i \frac{m_Z}{M^2} \left{ \left(T_3 g_c w - Y_d g_Y s_w\right) \sin \alpha^{(1)} P_R + Y_d g_Y s_w \cos \alpha^{(1)} P_L\right}$</td>
<td>[2]</td>
<td>[13] (A.123) ✓</td>
</tr>
<tr>
<td></td>
<td>$+4igT_3 \frac{m_e}{m_W} \frac{M^{(1)}}{M^2} \left(\cos \alpha^{(1)} P_R - \sin \alpha^{(1)} P_L\right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H^{(n)} \bar{\xi}^{(2n)} \xi^{(n)}_d$</td>
<td>$\frac{h}{\sqrt{2}} \left(\cos \alpha^{(n)} \sin \alpha^{(2n)} + \sin \alpha^{(n)} \cos \alpha^{(2n)}\right)$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$H^{(n)} \bar{\xi}^{(2n)} \xi^{(n)}_s$</td>
<td>$-\frac{h}{\sqrt{2}} \left(\cos \alpha^{(n)} \sin \alpha^{(2n)} + \sin \alpha^{(n)} \cos \alpha^{(2n)}\right)$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
<tr>
<td>$H^{(n)} \bar{\xi}^{(2n)} \xi^{(n)}_d$</td>
<td>$-\frac{h}{\sqrt{2}} \left(\sin \alpha^{(n)} \sin \alpha^{(2n)} + \cos \alpha^{(n)} \cos \alpha^{(2n)}\right) \gamma^5$</td>
<td>[2]</td>
<td>Not implemented</td>
</tr>
</tbody>
</table>

Table 10: **Fermion-Fermion-Scalar** All couplings above should be multiplied with $i\gamma^\mu$, according to Eq. (127). Note that these expressions are given in the $A_\mu^{(n)}, B_\mu^{(n)}$ basis for simplicity (except when $n = 0$). The Feynman rules in the $Z_\mu^{(n)}, A_\mu^{(n)}$ basis are easily derived by using the inverse expressions of the mass eigenstates (47).
Table 11: Eigenvalue corresponding to $C^a$ depending on the fermion type and gauge boson present in the interaction.

### B.4 KK number violating couplings

As discussed in section 2.2.7, level 2 KK particles have a significant contribution to the relic density of the dark matter particle. In particular, the decays of level 2 KK bosons are of importance, because they decay dominantly to SM fermions. The vertices in this section have been implemented in DarkSUSY and are calculated by [2].

**SM fermions + KK 2 gauge boson**

These couplings have the general structure

$$V_{2}^{\mu} = -i(C_{\psi_0\psi_0}V_2 + D_{\psi_0\psi_0}V_2)\gamma^\mu C^a P_{R/L}$$

(128)

where $C^a$ are the eigenvalues of the gauge group that the boson belongs to. The corrections $C_{\psi_0\psi_0}V_2$ come from the KK number violating terms that arise due to orbifold compactification, while $D_{\psi_0\psi_0}V_2$ comes from the terms due to bulk corrections. Note that only the $C_{\psi_0\psi_0}V_2$-terms are implemented in mUED.

The terms arising from bulk corrections from compactification all have the same structure [15]:

$$D_{\psi_0\psi_0} = (-i\gamma^\mu g C^a P_{R/L}) \frac{\sqrt{2}}{2} \left[ \frac{\delta (m_A^2)}{m_2^2} - 2 \frac{\delta (m_f^2)}{m_2^2} \right]$$

(129)

The chirality operator $P_{L/R}$ is always $P_L$ for doublet SM fermions and $P_R$ for singlet SM fermions, because the SM fermions have definite helicity. The eigenvalues $C^a$ can be replaced by the following eigenvalues of the gauge groups, depending on the fermions and gauge bosons considered (see table 11).

The specific eigenvalues for each fermion type can be found in B.1. The terms that arise from radiative corrections due to orbifold compactification that are implemented in DarkSUSY are presented in table 12.
<table>
<thead>
<tr>
<th>Vertex</th>
<th>Feynman rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{Q_0Q_0Z_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{tL_0tL_0Z_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{bL_0bL_0Z_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{tL_0tL_0Z_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{Q_0Q_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{bL_0hL_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{u_0u_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{tR_0tR_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{d_0d_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{tL_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
<tr>
<td>$C_{c_0c_0B_2}$</td>
<td>$\frac{\sqrt{2}g_2}{64\pi^2} \left[ g_3^2 \left( -12L_1 - \frac{52}{3} + \frac{7\pi^2}{3} \right) + g_2^2 \left( \frac{33}{4}L_1 + \frac{299}{12} - \frac{43\pi^2}{16} \right) + g_1^2 \left( -\frac{1}{4}L_1 - \frac{13}{36} + \frac{7\pi^2}{144} \right) \right] T^3 P_L$</td>
</tr>
</tbody>
</table>

Table 12: KK number violating couplings for the vertex $V^{(2)} \bar{f}^{(0)} f^{(0)}$ due to radiative corrections from orbifold compactification. All couplings are from [2].
SM fermions + KK 2 scalar

Both the $\mathbb{Z}_2$ odd and even scalar decay dominantly into a $t\bar{t}$-pair. The couplings have the general structure:

$$S^{(2)} \quad \xrightarrow{\downarrow i} \quad \bar{t} = -i \left( \frac{m_t}{v} C^{(0)} \bar{t} S^{(2)} + g'_\text{eff} \right) ,$$

where the specific couplings for the decay of the KK level 2 Higgs and $a_0$ have been implemented:

$$C^{(0)}_{t\bar{t}h^{(2)}} = \frac{\sqrt{2} g_3^2}{64\pi^2} C_F \left[ -4L_1 - 4 + \frac{\pi^2}{2} \right] [2] \quad (131)$$

$$C^{(0)}_{t\bar{t}a^{(2)}_0} = \gamma^5 \frac{m_n}{M^{(0)}_Z} \frac{\sqrt{2} g_3^2}{64\pi^2} C_F \left[ -4L_1 - 4 + \frac{\pi^2}{2} \right] [2] \quad (132)$$

and $g'_\text{eff} = 0$ when $S^{(2)} \neq H^{(2)}$ and otherwise,

$$g'_\text{eff} = \frac{y_t}{12} \left[ \frac{33}{4} g^2 + \frac{23}{6} g'^2 - 9 y_t^2 + 3 \lambda_h \right] \log \frac{\Lambda^2}{\mu^2} \quad [36] \quad (133)$$

when the finite corrections from Ref. [2] are implemented and

$$g_{\text{eff}} = \frac{y_t}{12} \left[ 16 g_s^2 + \frac{33}{4} g^2 + \frac{23}{6} g'^2 - 9 y_t^2 + 3 \lambda_h \right] \log \frac{\Lambda^2}{\mu^2} \quad (134)$$

when they are not.
Figure C.1: Possible Feynman diagrams for $\gamma^{(1)}H^{(1)}$ coannihilation. Note that only a $W^+W^-$ pair is allowed in the boson-boson final state.

Figure C.2: Possible Feynman diagrams for $l^{(1)}H^{(1)}$ coannihilation.

C Other Coannihilation Channels

Neglecting KK level 2 final states, there are in total 9 different processes that contribute to the total cross sections of the coannihilation channels $l^{(1)}l^{(1)}, \gamma^{(1)}H^{(1)}$ and $l^{(1)}H^{(1)}$:

\[
\begin{align*}
    l^{(1)}\bar{l}^{(1)} &\rightarrow l\bar{l} & \gamma^{(1)}H^{(1)} &\rightarrow f\bar{f} & l^{(1)}H^{(1)} &\rightarrow lV \\
    l^{(1)}\bar{l}^{(1)} &\rightarrow VV & \gamma^{(1)}H^{(1)} &\rightarrow VH & l^{(1)}H^{(1)} &\rightarrow lH \\
    l^{(1)}\bar{l}^{(1)} &\rightarrow HH & \gamma^{(1)}H^{(1)} &\rightarrow WW \\
    l^{(1)}\bar{l}^{(1)} &\rightarrow HV
\end{align*}
\]

The cross sections of most of these processes have been calculated analytically in Ref. [38]. They are calculated in the limit where all SM masses are negligible, all KK particles are degenerate in mass and EWSB effects can be ignored.
Figure C.3: Possible Feynman diagrams for the coannihilation channel $\ell^{(1)}\overline{\ell}^{(1)}$. 
Figure D.1: $\sin^2 \theta_W^{(n)}$ as a function of $R^{-1}$ for KK levels 1 and 2, using the new and old radiative corrections obtained by Refs. [2] and [15], respectively. $\sin^2 \theta_W^{(n)}$ implemented in DarkSUSY is indicated by dots, while $\sin^2 \theta_W^{(n)}$ obtained by Ref. [2] is indicated by stars.

D Tests

The Weinberg Angle

The Weinberg angle in mUED is dependent on the radiative corrections of $B_{\mu}^{(n)}$ and $A_{\mu}^{(n)}$, as well as the cut-off scale $\Lambda$ and $R$. The Weinberg angle implemented in DarkSUSY is presented in figure D.1, for the new and old mass corrections. These are in good correspondence with Ref. [2].

Branching ratios

The branching ratios of the KK level 2 leptons have been calculated in Ref. [2]. We have found that the branching ratios of the KK level 2 singlet lepton are in good correspondence when $R^{-1}$ is large, and a few percent off when $R^{-1} < 1000$ GeV. The branching ratios of the doublet leptons are in good correspondence when $R^{-1} \geq 1000$ GeV, but behave quite differently when $R^{-1} < 1000$ GeV. However, since the current lower bound on $R^{-1}$ is at 1400 GeV, this won’t affect the results in the relevant parameter space.
Figure D.2: Branching ratios of the two main decay channels of $l^{(2)}$ using the old and new mass corrections obtained by Refs. [15] and [2], respectively. The solid lines show the BRs obtained by the implemented decay widths in the UED module, while the dashed lines are the BRs obtained by Freitas et al. [2]. The ratio between the results obtained by the UED module and Freitas et al. are presented in the bottom figures. The BRs are almost identical when $R^{-1}$ is larger than 1000 GeV, which is approximately the lower bound on $R^{-1}$.
Figure D.3: Branching ratios of the two main decay channels of $L^{(2)}$ using the old and new mass corrections obtained by Refs. [15] and [2], respectively. The solid lines show the BRs obtained by the implemented decay widths in the UED module, while the dashed lines are the results obtained by Freitas et al. [2]. The ratio between the results obtained by DarkSUSY and Freitas et al. are presented in the bottom figures. As for the KK level 2 singlet, the BRs of the decay channels implemented in the UED module are in good correspondence with those obtained by Freitas et al. when $R^{-1} > 1000\text{GeV}$.
Comparison of annihilation relic density

The relic density in the limit where all KK masses are degenerate, all SM masses are negligible and the mixing angles are zero are in good correspondence with the relic density found by Servant et al. [24]. The plot of the relic density is presented in figure 5.1. The difference between the results generated by DarkSUSY and Servant et al. are presented in figure D.4.

Figure D.4: The difference in percentage between the relic density found by Servant et al. and the one generated by the relic density solver in DarkSUSY in the limit where all KK masses are degenerate, all SM masses are negligible and all mixing angles are zero. The annihilation relic density is generally very close to the result obtained by Servant et al. indicating that the relic density routine works well.
References


Alexander Belyaev, “Exploring universal extra-dimensions at the lhc.”


