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reading:
- any standard cosmology textbook
  [here, I (mostly) follow the conventions of Weinberg]
- K. A. Olive, G. Steigman, T. P. Walker, "Primordial nucleosynthesis: Theory and observations," Phys. Rept. **333**, 389-407 (2000). [astro-ph/9905320].

1 Recap – the big bang in a nutshell

The three main building blocks of modern cosmology are general relativity, the cosmological principle and a description of matter as a perfect fluid. According to the *cosmological principle*, the universe is homogeneous and isotropic on large scales, i.e. it looks basically the same at all places and in all directions. This means that spacetime can be described by the Friedmann-Robertson-Walker (FRW) metric:

$$-\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right], \quad (1)$$

where $k = 0, 1, -1$ for a flat, positively and negatively curved universe, respectively. For the approximately flat geometry that we observe, the scale factor $a(t)$ thus relates physical distances $\lambda_{\rm phys}$ to coordinate (or comoving) distances r via $\lambda_{\rm phys} = ar$. Note that a change of coordinates would re-scale a, so its normalization does not correspond to a physical quantity; conventionally one often sets its value today to $a_0 \equiv a(t_0) = 1$.

A *perfect fluid* can be completely characterized by its rest-frame energy density ρ and and isotropic pressure p (i.e. shear stresses, viscosity and heat conduction are assumed to be absent). In the rest frame of the fluid, its energy-momentum tensor takes the form $T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$; in a frame that is moving with 4-velocity v, it then takes the form

$$T^{\mu\nu} = (\rho + p)v^\mu v^\nu + pg^{\mu\nu}. \quad (2)$$

Inserting (1) and (2) into the field equations of general relativity,

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\, g_{\mu\nu} - \Lambda\, g_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad (3)$$

one arrives at the Friedmann[1] equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \quad (4)$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \quad (5)$$

[1]The second equation is often also called Raychaudhuri equation.

which describe the time evolution of the scale factor a and are thus the basic cosmological equations of motion. The cosmological constant Λ in these equations appears in the form of a perfect fluid with constant energy density and pressure $\rho_\Lambda = -p_\Lambda = \Lambda/(8\pi G)$. By expressing all energy densities in terms of a *critical density* $\rho_c \equiv 3\dot{a}^2/(8\pi G a^2)$, one can bring the first Friedmann equation into the form

$$\Omega + \Omega_\Lambda = 1 + k/\dot{a}^2\,, \tag{6}$$

where $\Omega \equiv \rho/\rho_c$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$. The various contributions to the total energy density in the universe today are rather well-determined and add up to a value consistent with the critical density – which means that the universe is close to or exactly flat ($k = 0$); for early times (when $\dot{a}$ was much bigger than its present value) we can thus safely neglect the effect of k and always set it to zero.

Observations of remote galaxies reveal spectra that are more redshifted the further they are away – which is usually taken as a strong indication that we, indeed, live in an expanding universe that is described by the FRW metric: In this case, the redshift z is obtained as

$$1 + z \equiv \frac{\nu_{\rm emit}}{\nu_0} = \frac{a_0}{a_{\rm emit}}\,, \tag{7}$$

where $\nu_{\rm emit}$ is the signal frequency at the time of emission and ν_0 the frequency observed today.[2] To first order this recovers Hubble's law, $z \simeq H_0\, d$, because $a_0/a_{\rm emit} = 1 + (t_0 - t_{\rm emit})\, \frac{\dot{a}}{a}\big|_{t_0} - \frac{1}{2}(t_0 - t_{\rm emit})^2\, \frac{\ddot{a}}{a}\big|_{t_0} + \ldots$ and the distance to some remote galaxy is given by $d = (t_0 - t_{\rm emit})$.

The strong energy condition requires on the other hand that

$$\rho + 3p > 0\,. \tag{8}$$

From the second Friedmann equation it therefore follows that the growth of the scale factor a has always been decelerating (neglecting for the moment the possible existence of a cosmological constant, which does not satisfy the above condition). Since we observe $\dot{a} > 0$ today, this indicates that the universe started off with a "*Big Bang*" at which $a \approx 0$.[3] An important

[2] The second step follows directly for light-like geodesics in the FRW metric (1). Concretely, $\mathrm{d}s^2 = 0 \rightsquigarrow \int_{t_{\rm emit}}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} = \text{const.} \rightsquigarrow \frac{\delta t_0}{a(t_0)} - \frac{\delta t_{\rm emit}}{a(t_{\rm emit})} = 0$.

[3] In this picture, the Big Bang is sometimes referred to as the actual singularity ($a = 0$), indicating the beginning of both space and time. Instead, one should rather think of it as an "initial" state of very high energy density and temperature beyond which it would not make any sense to extrapolate the above arguments because the underlying theory of general relativity can no longer be assumed valid in these extreme conditions.

observation is the fact that in such a spacetime there appears a horizon which characterizes the maximal length that any particle or piece of information can have propagated since the Big Bang:

$$d_H(t) = a(t) \int_0^{r(t)} \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} \,. \tag{9}$$

This *particle horizon* d_H is usually well approximated by the *Hubble radius* $H^{-1} \equiv a/\dot{a}$.[4]

From the two Friedmann equations, or directly from energy conservation, $T^{\mu\nu}_{\ ;\mu} \equiv \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\nu\lambda} T^{\lambda\nu} = 0$, one obtains

$$(\rho a^3)^{\cdot} + p(a^3)^{\cdot} = 0 \,. \tag{10}$$

To relate energy density ρ and pressure p, one usually also specifies an equation of state,

$$p = w\rho \,. \tag{11}$$

In that case, and assuming w to be constant, Eq. (10) can be integrated to

$$\rho \propto a^{-3(w+1)}. \tag{12}$$

Inserting this scaling behavior into the first Friedmann equation, one finds the following time-dependence of the scale factor:

$$a(t) \propto t^{\frac{2}{3(w+1)}} \,. \tag{13}$$

For a cosmological fluid composed of several components, $\rho = \sum \rho_i$ and $p = \sum p_i$, Eq. (10) applies to each of the fluids separately *if* they do not interact (other than gravitationally). Radiation (highly relativistic matter), for example, has $w = 1/3$ and thus evolves as $\rho_r \propto a^{-4}$. At early times, it will therefore dominate the total energy budget over any contribution from (non-relativistic) matter with $w = 0$ and $\rho_m \propto a^{-3}$ – or, of course, a constant contribution ρ_Λ.

More generally, we can use the fact that the entropy in a co-moving volume stays constant *in thermal equilibrium*. During radiation domination, the entropy density reads

$$s(T) = \frac{\rho(T) + p(T)}{T} = \frac{2\pi^2}{45} g^s_{\mathrm{eff}} T^3 \,. \tag{14}$$

[4] Whenever $a \propto t^n$, with $n < 1$, as is the case for both a radiation- and a matter-dominated universe, one has $d_H = t/(1-n)$ and $H^{-1} = t/n$. During an inflationary phase, on the other hand, one has $H \approx$ *const.* and d_H grows exponentially with time.

Looking back in time, the temperature thus increases with decreasing a as

$$T \propto {g_{\rm eff}^s}^{-1/3} a^{-1}\,, \tag{15}$$

which motivates the picture of a *hot* Big Bang. Entropy conservation implies $H = -\dot{s}/(3s)$, so the first Friedmann equation can be integrated to give

$$\begin{aligned} t &= -\int \frac{s'(T)dT}{s(T)\sqrt{24\pi G\rho(T)}} &(16)\\ &\approx \sqrt{\frac{45}{16\pi^3 G g_{\rm eff}}}T^{-2} + const. &(17)\\ &\sim \left(\frac{\rm MeV}{T}\right)^2 \rm s\,, &(18)\end{aligned}$$

where $g_{\rm eff}$ denotes the effective number of degrees of freedom contributing to the energy density in radiation:

$$\rho_r = \frac{\pi^2}{30} g_{\rm eff} T^4\,. \tag{19}$$

In the second step of the above derivation, it was assumed that the effective number of degrees of freedom can be approximated as constant. Starting from a reasonably dense and hot initial state – beyond which one would have to take into account effects from unknown physics at very high energies – one can now reconstruct the thermal history of the universe.

2 Big bang nucleosynthesis

The production of light elements during the first minutes of the cosmological evolution, corresponding to temperatures down to about 0.01 MeV, is considered to be one of the most important tests of big bang cosmology since the observed abundances generally agree very well with the predicted ones. The beginning of BBN at around 1 MeV is therefore often referred to as the earliest time about which one can make statements without having to assume anything beyond currently well-understood physics. In our context, the most important aspect of BBN is that the predicted abundances of light elements strongly depend on the ratio of the baryon to photon number density,

$$\eta \equiv n_B/n_\gamma\,. \tag{20}$$

Since this quantity does not change with the expansion of the universe, and the number of photons today can be accurately determined from CMB measurements, BBN thus provides a very efficient way to determine the total baryon content Ω_B: as we will see later, this turns out to be much smaller than the total amount Ω_m of matter inferred by other observations. The successful prediction of light element abundances can also be used to put stringent constraints on the Hubble expansion rate (aka additional relativistic degrees of freedom via Eq. (19) and $H^2 = 8\pi G\rho/3$) or any form of energy injection (e.g. from decaying dark matter) during BBN.

At times $t \ll 1\,\mathrm{s}$ (or temperatures $T \gg 1\,\mathrm{MeV}$), neutrons and protons were in thermal equilibrium through the following reactions:

$$n \leftrightarrow p + e^- + \bar{\nu}_e \tag{21}$$

$$n + \nu_e \leftrightarrow p + e^- \tag{22}$$

$$n + e^+ \leftrightarrow p + \bar{\nu}_e \tag{23}$$

Their number densities were thus given by the value in thermal equilibrium,

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{1 + \exp[(E_i - \mu_i)/T]} \overset{T \ll m_i}{\simeq} g_i \left(\frac{m_i T}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{m_i}{T}}\,, \tag{24}$$

and the ratio given by

$$\frac{n_n}{n_p} \simeq e^{-\frac{\Delta m}{T}} = e^{-\frac{1.29\,\mathrm{MeV}}{T}}\,. \tag{25}$$

Once the interaction rates for the reactions (22) and (23) fall behind the expansion rate of the universe, they cannot change anymore the abundance of protons and neutrons. The value of n_n/n_p thus "freezes out" (at a value of about 1/6), thereafter only decreasing slightly due to the decay of free neutrons as described by (21).[5]

Once the temperature falls to values of the order of nuclear binding energies, nucleosynthesis can start. The first link in the chain of possible reactions is the production of deuterium d (or ^{2}H),

$$p + n \to d + \gamma\,. \tag{26}$$

Protons and neutrons leave thermal equilibrium around $0.8\,\mathrm{MeV}$.The deuterium binding energy is much higher, $E_B^d = 2.2\,\mathrm{MeV}$, so one would naively expect the above process to happen very efficiently. However, it proceeds with a rate proportional to n_B^2 and the inverse process of deuterium photodissociation ($\propto n_B n_\gamma \exp(-E_B^d/T)$) is much more effective due to the large

[5] The numerical value of 1/6 is obtained by solving the Boltzmann equation that describes the time-evolution of n_i in a FRW spacetime. We will later analyze the Boltzmann equation and describe the freeze-out process in some detail in the context of WIMP production.

photon-to-baryon ratio $1/\eta \gtrsim 10^9$ – a situation known as the *deuterium bottleneck*. At $T \sim 80\,\mathrm{keV}$, deuterium finally becomes stable against photo-dissociation and almost all neutrons end up in ^{4}He nuclei through the additional reactions

$$d + d \leftrightarrow {}^3\mathrm{He} + n \tag{27}$$

$$^3\mathrm{He} + d \leftrightarrow {}^4\mathrm{He} + p\,. \tag{28}$$

By this time, n_n/n_p has decreased to a value of about 1/7 due to neutron decay, so the resulting Helium mass fraction becomes

$$Y(^4\mathrm{He}) \equiv \frac{4n_{\mathrm{He}}}{n_n + n_p} = \frac{2n_n}{n_n + n_p} \approx 0.25\,, \tag{29}$$

which is in very good agreement with current observations. It is the rather large binding energy of ^{4}He, about 28 MeV, that makes the conversion of neutrons to ^{4}He so efficient and its abundance not too dependent of η. Deuterium and ^{3}He on the other hand, are burned to more complex nuclei until BBN eventually comes to an end due to the growing importance of the Coulomb barrier at small T and the lack of free neutrons; their relative mass abundance today is $Y(d) \sim Y(^3\mathrm{He}) \sim 10^{-5}$. The rate of these interactions increases with the amount of baryons, so the d and ^{3}He abundances decrease rather rapidly with increasing η.

Another interesting indicator for BBN is ^{7}Li, which is produced with an abundance of $Y(^7\mathrm{Li}) \sim 10^{-7}$. This happens mostly through

$$^4\mathrm{He} + {}^3\mathrm{H} \to {}^7\mathrm{Be} + \gamma\,, \qquad {}^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu_e\,, \tag{30}$$

which is more effective for higher values of η. At rather low values of η, on the other hand, ^{7}Li is destroyed by protons ($^7\mathrm{Li} + p \to {}^4\mathrm{He} + {}^4\mathrm{He}$) with an efficiency that increases with η, so the final abundance decreases with η.

Even heavier elements are produced in much smaller abundances – both because Coulomb repulsion becomes more and more important and because there are no stable elements with $A = 5$ or $A = 8$ that could serve as intermediate steps in the nuclear burning chain.