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reading:

- chapter 4 of J. Edsjö, "Aspects of neutrino detection of neutralino dark matter", PhD thesis, Uppsala (1997) [hep-ph/9704384].
- P. Gondolo and G. Gelmini, "Cosmic abundances of stable particles: Improved analysis," Nucl. Phys. B **360**, 145 (1991).
- T. Bringmann, "Particle Models and the Small-Scale Structure of Dark Matter", New J. Phys. **11**, 105027 (2009). [arXiv:0903.0189 [astro-ph.CO]].

7.2 Chemical decoupling

Let us now have a closer, and more detailed, look at the chemical decoupling process. We have already encountered the collision-less Boltzmann equation in Eq.(34). Including a collision-term $C[f]$ to describe any kind of interactions that the particles may experience, it can be written as

$$\hat{L}[f] = C[f]\,, \tag{88}$$

where the Liouville operator $\hat{L}$ is the covariant generalization of the convective derivative:

$$\begin{aligned}\hat{L}[f] &= \frac{df}{d\lambda} &(89)\\ &= \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} &(90)\\ &= p^\mu\frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma}\,p^\rho p^\sigma\frac{\partial f}{\partial p^\mu} &(91)\\ &= p^0\left(\partial_t - H\bar{\mathbf{p}}\cdot\nabla_{\bar{\mathbf{p}}}\right)f\,. &(92)\end{aligned}$$

Here, λ is an affine parameter λ along a geodesic and in the last step, we assumed a flat FRW geometry and changed to local (or *comoving*) momenta $\bar{\mathbf{p}} \equiv a\mathbf{p}$ (in the following, we will, however, always consider comoving momenta and drop the bars over the $\mathbf{p}$ for simplicity). The phase-space distribution function $f(x^\mu, p^\mu)$ is normalized such that the number density of a species with g_i internal degrees of freedom is given by $n_i = g_i\int d^3p/(2\pi)^3 f$.

The collision term for the annihilation process is given by

$$\begin{aligned}C &= \frac{1}{2g_\chi}\int\frac{d^3\tilde{p}}{(2\pi)^3 2\tilde{E}}\int\frac{d^3k}{(2\pi)^3 2\omega}\int\frac{d^3\tilde{k}}{(2\pi)^3 2\tilde{\omega}}(2\pi)^4\delta^{(4)}(\tilde{p}+p-\tilde{k}-k)\\ &\quad\left[|\mathcal{M}|^2_{\bar{\chi}\chi\leftarrow\bar{f}f}\,g(\omega)g(\tilde{\omega}) - |\mathcal{M}|^2_{\bar{\chi}\chi\rightarrow\bar{f}f}\,f(E)f(\tilde{E})\right]\,, \end{aligned}\tag{93}$$

where $k^\mu = (\omega, \mathbf{k})$ and $\tilde{k}^\mu = (\tilde{\omega}, \tilde{\mathbf{k}})$ are the 4-momenta of the SM particles f and $g = g_{\rm eq} = \left(e^{\omega/T} \pm 1\right)^{-1}$ their distribution functions (with a minus-sign for bosons and a plus-sign for fermions).[12] In this expression, $|\mathcal{M}|^2$ refers to the matrix element squared, summed over all possible SM particles f, including their internal (e.g. spin) degrees of freedom, and also *summed* over the internal degrees of freedom of the DM particles χ.

CP invariance implies $|\mathcal{M}|^2_{\bar\chi\chi\to\bar f f} = |\mathcal{M}|^2_{\bar\chi\chi\leftarrow\bar f f}$ and in thermal equilibrium annihilation and creation processes should happen with the same frequency. This means that we can replace $g(\omega)g(\tilde\omega)$ in the above equation with $f_{\rm eq}(E)f_{\rm eq}(\tilde E)$:

$$\begin{aligned} C &= \frac{1}{2g_\chi}\int\frac{d^3\tilde p}{(2\pi)^3 2\tilde E}\int\frac{d^3k}{(2\pi)^3 2\omega}\int\frac{d^3\tilde k}{(2\pi)^3 2\tilde\omega}(2\pi)^4\delta^{(4)}(\tilde p + p - \tilde k - k) \\ &\quad \times |\mathcal{M}|^2_{\bar\chi\chi\to\bar f f}\left[f_{\rm eq}(E)f_{\rm eq}(\tilde E) - f(E)f(\tilde E)\right] \quad (94) \\ &= g_{\bar\chi}E\int\frac{d^3\tilde p}{(2\pi)^3}\,v_{\rm M\o l}\sigma_{\bar\chi\chi\to\bar f f}\left[f_{\rm eq}(E)f_{\rm eq}(\tilde E) - f(E)f(\tilde E)\right], \quad (95)\end{aligned}$$

where $v_{\rm M\o l}\equiv(E\tilde E)^{-1}\sqrt{(p\cdot\tilde p)^2 - m_\chi^4}$ is the velocity of one DM particle in the rest frame of the other. Integrating the full Boltzmann equation (88) over $\int d^3p\, g_\chi/\left[(2\pi)^3E\right]$ then results in

$$\dot n_\chi + 3Hn_\chi = -\langle\sigma v\rangle\left(n_\chi n_{\bar\chi} - n_\chi^{\rm eq}n_{\bar\chi}^{\rm eq}\right), \quad (96)$$

where the thermal averaged annihilation cross section defined as

$$\langle\sigma v\rangle_{\bar\chi\chi\to\bar f f} = \frac{g_\chi\bar g_\chi}{n_\chi^{\rm eq}n_{\bar\chi}^{\rm eq}}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3\tilde p}{(2\pi)^3}\,v_{\rm M\o l}\sigma_{\bar\chi\chi\to\bar f f}f_{\rm eq}(E)f_{\rm eq}(\tilde E)\,. \quad (97)$$

In arriving at this result, we had to assume that $f(E)\propto f_{\rm eq}(E)$, with a factor of proportionality that describes an effective chemical potential which may depend on T (but not E); this is motivated by the fact that the much more abundant *scattering* processes of DM with SM particles still keep the DM particles in kinetic, but not chemical equilibrium (see the following section). Very often, DM is assumed to be its own antiparticle, $\chi = \bar\chi$, in which case we recover the familiar expression Eq. (86); note that there is no additional factor of $\frac{1}{2}$ for identical *initial state* particles.

[12] In-medium effects can be taken into account by adding Pauli blocking (or Bose enhancement) factors: $|\mathcal{M}|^2_{\bar\chi\chi\to\bar f f} = |\mathcal{M}|^{2({\rm vac})}_{\bar\chi\chi\to\bar f f}[1\mp g(\omega)][1\mp g(\tilde\omega)]$ and $|\mathcal{M}|^2_{\bar\chi\chi\leftarrow\bar f f} = |\mathcal{M}|^{2({\rm vac})}_{\bar\chi\chi\to\bar f f}[1\mp f(E)][1\mp f(\tilde E)]$. For non-relativistic DM, however, $E\approx m_\chi + p^2/(2m_\chi)\gg T$ and thus $f\ll 1$; momentum conservation then also enforces $\omega,\tilde\omega > m_\chi$ and thus $g\ll 1$.

Coannihilations

In general, one expects to find many new, e.g. supersymmetric, particles rather close in mass to the lightest particle which will be the DM. Since these particles will eventually decay to the DM particle (which is assumed to be protected against decay by an internal symmetry), their initial abundances also contribute to the DM relic density today and we have to study the full set of (coupled) Boltzmann equations that govern their evolution. To do so, we need to compute both the total annihilation and inclusive scattering cross sections, but also the inclusive decay rates and the 'relative velocities':

$$\sigma_{ij} = \sum_X \sigma(\chi_i \chi_j \to X) \tag{98}$$

$$\sigma'_{Xij} = \sum_Y \sigma(\chi_i X \to \chi_j Y) \tag{99}$$

$$\Gamma_{ij} = \sum_X \Gamma(\chi_i \to \chi_j X) \tag{100}$$

$$v_{ij} = \frac{\sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}}{E_i E_j} . \tag{101}$$

In the above expressions, X and Y denote all (sets of) standard model particles that appear in the interactions. In complete analogy to the case of a single annihilation mode, one can now derive the following set of Boltzmann equations for so-called *coannihilations*:

$$\begin{aligned} \dot{n}_i + 3Hn_i = & - \sum_{j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{\rm eq} n_j^{\rm eq} \right) \\ & - \sum_X \sum_{j \neq i} n_X^{\rm eq} [\langle \sigma'_{Xij} v_{ij} \rangle \left(n_i - n_i^{\rm eq} \right) - \langle \sigma'_{Xji} v_{ij} \rangle \left(n_j - n_j^{\rm eq} \right)] \\ & - \sum_{j \neq i} [\Gamma_{ij} \left(n_i - n_i^{\rm eq} \right) - \Gamma_{ji} \left(n_j - n_j^{\rm eq} \right)] . \end{aligned} \tag{102}$$

Relatively shortly after freeze-out, all particles will decay into the DM particles χ and we are thus only interested in the total number density $n \equiv \sum_{i=1}^N n_i$. Summing Eq. (102) over i then results in

$$\dot{n} + 3Hn = - \sum_{i,j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{\rm eq} n_j^{\rm eq} \right) . \tag{103}$$

Note that the terms on the second and third lines in Eq. (102) do not change the total number n of (heavy) particles and thus cancel in the sum. There is, however, an important consequence of inclusive scattering: Since $n_X \ll n_i$, these processes happen sufficiently often to keep the χ_i distributions in (or very close to) chemical equilibrium with respect to each other, i.e. all χ_i should have a very similar chemical potential. As a consequence, we can to a good approximation assume that

$$\frac{n_i}{n} \simeq \frac{n_i^{\mathrm{eq}}}{n^{\mathrm{eq}}}\,, \tag{104}$$

where

$$n_{\mathrm{eq}} \equiv \sum_i n_i^{\mathrm{eq}} \simeq \sum_i g_i \int \frac{dp}{(2\pi)^3} e^{-\frac{E_i}{T}} = \frac{T}{2\pi^2} \sum_i g_i m_i^2 K_2\left(\frac{m_i}{T}\right)\,. \tag{105}$$

This results in

$$\dot{n} + 3Hn = -\langle \sigma_{\mathrm{eff}} v \rangle \left(n^2 - n_{\mathrm{eq}}^2\right) \tag{106}$$

which is of the same form as for the case of no co-annihilations, Eq. (86), but with the annihilation rate replaced by an effective annihilation rate

$$\langle \sigma_{\mathrm{eff}} v \rangle = \sum_{ij} \langle \sigma_{ij} v_{ij} \rangle \frac{n_i^{\mathrm{eq}}}{n^{\mathrm{eq}}} \frac{n_j^{\mathrm{eq}}}{n^{\mathrm{eq}}}\,. \tag{107}$$

Because of the Boltzmann suppression factors of $n_i^{\mathrm{eq}}/n^{\mathrm{eq}} \sim \exp[-(m_i - m_\chi)/T]$, we thus expect that only particles rather close in mass to the dark matter particle contribute to the effective annihilation rate.

Integrating the Boltzmann equation

Let us now bring the Boltzmann equation into a form which is more convenient for integration. Introducing the ratio of the number density to the entropy density,

$$Y = \frac{n}{s}\,, \tag{108}$$

and using $\partial_t(a^3 s) = 0$, the left-hand side of Eq. (106) becomes

$$\partial_t \left[a^{-3}(a^3 s)Y\right] + 3HsY = s\dot{Y}\,. \tag{109}$$

Next, we change the independent variable from t to the dimensionless quantity

$$x \equiv m_\chi/T\,. \tag{110}$$

Exloiting again entropy conservation, and $s = (2\pi^2/45) g_{\mathrm{eff}}^s T^3$, we then find

$$0 = a^{-3}\partial_t\left(a^3 s\right) = 3Hs + \frac{\dot{g}_{\mathrm{eff}}^s}{g_{\mathrm{eff}}^s} s + 3\frac{\dot{T}}{T} s = 3s\left[H - \left(1 + \frac{T}{3g_{\mathrm{eff}}^s}\frac{dg_{\mathrm{eff}}^s}{dT}\right)\frac{\dot{x}}{x}\right]\,. \tag{111}$$

By using the first Friedmann equation, this allows us to rewrite Eq. (106) as

$$\frac{dY}{dx} = -\sqrt{\frac{\pi}{45G}}\frac{g_*^{1/2} m_\chi}{x^2}\langle\sigma_{\rm eff} v\rangle\left(Y^2 - Y_{\rm eq}^2\right)\,, \tag{112}$$

where $Y_{\rm eq}$ is given by

$$Y_{\rm eq} = \frac{n_{\rm eq}}{s} = \frac{45x^2}{4\pi^4 g_{\rm eff}^s(T)}\sum_i g_i\left(\frac{m_i}{m_\chi}\right)^2 K_2\left(x\frac{m_i}{m_\chi}\right) \tag{113}$$

and

$$g_*^{1/2} \equiv \frac{g_{\rm eff}^s}{\sqrt{g_{\rm eff}}}\left(1 + \frac{T}{3g_{\rm eff}^s}\frac{dg_{\rm eff}^s}{dT}\right)\,. \tag{114}$$

There exist very nice and efficient approximate ways of solving Eq. (112) analytically – which work well as long as $\langle\sigma_{\rm eff} v\rangle(T)$ takes a relatively simple form. A thorough discussion of solution strategies, be it analytical or numerical, goes beyond the scope of these lectures, so let us simply denote by Y^0 the result of integrating Eq. (112) from $x = 0$ to $x_0 = m_\chi/T_0$, where T_0 is the photon temperature of the Universe today. The DM relic density is then obtained as

$$\Omega_\chi = \rho_\chi^0/\rho_c \tag{115}$$

$$= m_\chi s_0 Y_0/\rho_c \tag{116}$$

$$= 2.755\times 10^{10} h^{-2}\left(\frac{m_\chi}{100\,\mathrm{GeV}}\right) Y_0\,. \tag{117}$$

7.3 Thermal decoupling

As already mentioned, WIMPs stay in thermal contact with the heat bath of SM particles quite long after chemical decoupling, i.e. after number density changing processes have ceased (which typically happens around $T_{\rm cd} \sim m_\chi/25$: scattering events between WIMPs and SM particles are much more frequent than WIMP annihilations simply because SM particles are much more abundant (SM particle annihilations to WIMPs, on the other hand, are kinematically possible only in the heavily suppressed tail of the energy distribution). Only when even these processes stop to be efficient after kinetic, or *thermal*, decoupling there are no longer any interactions between WIMPs and SM particles and the former have completely decoupled from the thermal bath.

To describe kinetic decoupling, it suffices to look at the second moment of the Boltzmann equation because higher orders in p^2/m_χ^2 are heavily suppressed. Integrating the left-hand side of Eq. (88) over $\int d^3p\, g_\chi p^2/\left[(2\pi)^3 E\right]$ results in

$$\int \frac{d^3p}{(2\pi)^3}\, g_\chi \mathbf{p}^2\left(\partial_t - H\mathbf{p}\cdot\nabla_{\mathbf{p}}\right) f = 3m_\chi\partial_t(T_\chi n_\chi) + 15 m_\chi H T_\chi n_\chi \tag{118}$$

$$= 3m_\chi n_\chi\left(\dot{T}_\chi + 2H\right)\,, \tag{119}$$

where in the last step we have assumed $\dot{n}_\chi = -3Hn_\chi$, i.e. chemical freeze-out has already taken place. We have also introduced a WIMP 'temperature' T_χ defined by

$$g_\chi \int \frac{d^3p}{(2\pi)^3} \mathbf{p}^2 f(\mathbf{p}) \equiv 3\, m_\chi T_\chi n_\chi \, . \tag{120}$$

Note that this *definition* does not require any assumptions about the form of $f(\mathbf{p})$, but simply provides a convenient means of characterizing the deviation from thermal equilibrium (for which $T_\chi = T$ holds).

The collision term for elastic scattering $\chi f \leftrightarrow \chi f$ is given by

$$\begin{aligned} C \;\; = \;\; & \frac{1}{2g_\chi} \int \frac{d^3k}{(2\pi)^3 2\omega} \int \frac{d^3\tilde{k}}{(2\pi)^3 2\tilde{\omega}} \int \frac{d^3\tilde{p}}{(2\pi)^3 2\tilde{E}} (2\pi)^4 \delta^{(4)}(\tilde{p} + \tilde{k} - p - k) \\ & \times |\mathcal{M}|^2 \left[(1 \mp g^\pm)(\omega)\, g^\pm(\tilde{\omega}) f(\tilde{\mathbf{p}}) - (1 \mp g^\pm)(\tilde{\omega})\, g^\pm(\omega) f(\mathbf{p}) \right] \, , \end{aligned} \tag{121}$$

where the summation over all SM particles in thermal equilibrium is again not shown explicitly. While we will assume that the g are thermal distributions as before, no assumptions about the χ distribution function $f(\mathbf{p})$ are necessary; as long as the WIMPs are much less abundant than their scattering partners, however, Pauli suppression factors for f can safely be neglected – as has been done in in the above expression. After chemical freezeout, one typically has $\omega \sim T \ll m_\chi$. For kinematical reasons, the average momentum transferred during the scattering events is thus small, compared to the rest mass, so the collision term can be expanded as $C = \sum_{j=0}^{\infty} C^j$, where the coefficients are defined by a Taylor expansion in $(\tilde{\mathbf{k}} - \mathbf{k})$. Formally, this can be written by replacing the 3D Dirac delta-function in Eq. (121) by

$$\delta^{(3)}(\tilde{\mathbf{p}} - \mathbf{p} + \tilde{\mathbf{k}} - \mathbf{k}) = \sum_{j=0}^{\infty} \frac{1}{j!} \left[(\tilde{\mathbf{k}} - \mathbf{k}) \cdot \nabla_{\tilde{\mathbf{p}}} \right]^j \delta^{(3)}(\tilde{\mathbf{p}} - \mathbf{p}) \, , \tag{122}$$

which is defined as usual in terms of integration by parts. After a lengthy calculation[13], including only terms of the lowest non-vanishing order in $\mathbf{p}^2/E^2$ and ω/m_χ, one finds the following expression for the collision integral:

$$C \simeq C^1 + C^2 \simeq c(T) m_\chi^2 \Big[m_\chi T \Delta_{\mathbf{p}} + \mathbf{p} \cdot \nabla_{\mathbf{p}} + 3 \Big] \, f(\mathbf{p}) \, , \tag{123}$$

where

$$c(T) = \sum_i \frac{1}{12 g_\chi (2\pi)^3 m_\chi^4 T} \int dk\, k^5 \omega^{-1}\, g^\pm \left(1 \mp g^\pm\right) |\mathcal{M}|^2_{\substack{t=0 \\ s = m_\chi^2 + 2m_\chi \omega + m_\ell^2}} \, . \tag{124}$$

[13] For details, see the third reference listed above.

For clarity, the sum over all SM scattering partners i has here been made explicit. For relativistic SM particles, the above integral can be solved analytically if (as usually is the case) $|\mathcal{M}|^2 \propto (\omega/m_\chi)^n$ in this expression, resulting in $c(T) \propto T^{4+n}$. Note also that the first moment of Eq. (123) just vanishes – as it should since we are considering scattering processes that do not change the number density of dark matter particles.

The second moment of the Boltzmann equation, on the contrary, reads

$$\begin{aligned}
\dot{T}_\chi + 2HT_\chi &= \frac{g_\chi}{3m_\chi n_\chi}\int \frac{d^3p}{(2\pi)^3 E}\,\mathbf{p}^2 C[f(\mathbf{p})] \\
&\simeq \frac{g_\chi}{3n_\chi}c(T)\int \frac{d^3p}{(2\pi)^3}\mathbf{p}^2\Big[m_\chi T\Delta_\mathbf{p} + \mathbf{p}\cdot\nabla_\mathbf{p} + 3\Big]f(\mathbf{p}) \\
&= \frac{g_\chi}{3n_\chi}c(T)\int \frac{d^3p}{(2\pi)^3}\Big[6m_\chi T - 2\mathbf{p}^2\Big]f(\mathbf{p}) \\
&= 2m_\chi c(T)[T - T_\chi]\,.
\end{aligned} \tag{125}$$

Analogous to the treatment of the first moment of the Boltzmann equation, one may now introduce the dimensionless quantity

$$y \equiv \frac{m_\chi T_\chi}{s^{2/3}} \propto T_\chi a^2 \tag{126}$$

and re-write the left-hand side of Eq. (125) as $\dot{T}_\chi + 2HT_\chi = m_\chi^{-1}s^{2/3}\dot{y}$. In exactly the same fashion as in Eqs. (110-112), we then arrive at

$$\frac{1}{y}\frac{dy}{dx} = \frac{2m_\chi\left(1 + \frac{T}{3g^s_\mathrm{eff}}\frac{dg^s_\mathrm{eff}}{dT}\right)c(T)}{xH}\left(1 - \frac{T_\chi}{T}\right)\,. \tag{127}$$

Inspection of this equation shows the expected asymptotic behaviour for the WIMP temperature: at early times, or large T, the term in front of the right-hand side is much larger than unity (recall that $H \propto T^2$ and $c(T) \propto T^{4+n}$, typically with $n = 2$); the solution $T_\chi = T$ thus provides a strong attractor of the differential equation, i.e. the system is very efficiently kept in thermal equilibrium. When T becomes small, the WIMPs finally decouple completely from the thermal bath: the pre-factor becomes negligibly small and y stays constant, i.e. $T_\chi \propto s^{2/3} \propto a^{-2}$ – which simply reflects the redshift of the WIMP momenta. Since the transition between the two regimes happens on a rather short timescale, the *kinetic decoupling temperature* is naturally defined by equating these two asymptotic behaviours, as if the decoupling process were indeed to occur instantaneously:

$$x_\mathrm{kd} = \frac{m_\chi}{T_\mathrm{kd}} \equiv y|_{x\to\infty} \times \left.\frac{s^{2/3}}{T^2}\right|_{T=T_\mathrm{kd}}\,. \tag{128}$$

With the formalism presented above, one can calculate the kinetic decoupling temperature to an accuracy of $\mathcal{O}(x_{\rm kd}^{-1})$, for any WIMP candidate that was non-relativistic at chemical freeze-out and for which we have $x_{\rm cd} \ll x_{\rm kd}$. The kinetic decoupling temperature for, e.g., neutralino DM lies between a few MeV and a few GeV, with $x_{\rm kd}$ between 200 and almost 10^5 – a much larger range than for $x_{\rm cd}$ which falls into the range $20 \lesssim x_{\rm cd} \lesssim 28$ for neutralinos with the correct relic density.