Theory for lateral stability and magnetic stiffness in a high-$T_c$ superconductor-magnet levitation system

T. H. Johansen and H. Bratsberg

Department of Physics, University of Oslo, P.O. Box 1048, Blindern, 0316 Oslo 3, Norway

(Received 11 May 1993; accepted for publication 2 June 1993)

A quantitative first-order theory for the lateral force between a permanent magnet and a type-II superconductor is presented. The levitation configuration discussed is that of a long rectangular bar magnet placed above a finite-sized rectangular superconductor. The central issues of stability and stiffness (elastic spring constant) associated with lateral vibrations are discussed. Closed-form expressions for both the force and stiffness are derived, thus bringing out clearly the significance of both geometrical dimensions and the magnetic response of the superconductor. It is assumed that the superconductor is either a sintered granular or consists of grains embedded in a nonactive matrix (composite) so that only intragranular shielding currents are important. The predicted behavior as a function of levitation height agrees very well with existing experimental results.

I. INTRODUCTION

High-$T_c$ superconducting materials are today showing a vast potential for a wide range of applications, one of which being magnetic levitation systems such as frictionless bearings for high-speed rotors and noncontacting transport systems. The practical use of such devices relies crucially upon the stability of the levitation, a feature that is an intrinsic part of the magnet-superconductor interaction. As explained qualitatively by Brandt, the lateral stability, i.e., the fact that a small magnet levitating over a flat superconductor of finite size does not fall off to one side when subject to small mechanical perturbations, is due to pinning of magnetic flux lines inside the type-II superconductor.

Experimentally, it has been shown that the characteristics of the lateral force during oscillatory motion has a strong amplitude dependence. For large amplitudes Johansen et al. found that even at very low frequencies (0.1 Hz) there is a considerable energy loss per cycle, whereas this loss essentially vanishes when the amplitude falls below a certain value. The crossover to elastic behavior was shown to depend on the levitation height, i.e., the magnitude of the magnetic field experienced by the superconductor.

Perhaps the most important consequence of the interaction becoming essentially elastic for small amplitudes is that such oscillations will decay very slowly with time. In practice, due to uncontrollable perturbations small-amplitude vibrations will always be present in these types of levitation systems. There have been several communications reporting on experiments measuring the natural frequencies of the vibrational modes of levitated magnets.

On the theoretical side very few workers have addressed the problem of understanding the characteristics of the lateral restoring force at a quantitative level. Davis considered the force between a long magnet and an infinitely large slab of superconductor in the case of volume currents, i.e., the weak intergranular couplings were assumed to be effective barriers for flux penetration, and the shielding currents were allowed to flow through the entire superconductor volume. The calculation was based upon Bean's critical-state model. Later Yang et al. extended these results by using a more realistic magnetic-field profile. This improved the fit to experimental results, but still the quantitative agreement between theory and experiment is far from satisfactory.

In this article we consider the opposite case where the applied field is assumed to leak through the intergranular space, thereby surrounding and penetrating each individual grain. Thus, we restrict the supercurrents to flow only within each grain. For sintered ceramic superconductors one expects such a model to be valid only at relatively high fields. However, the strong NdFeB magnets commonly used today have a remanent induction larger than 1 T and therefore at most realistic magnet-superconductor distances the applied field is considerable. In fact, the assumption of broken intergranular links at such field intensities is supported both by ac susceptibility measurements and by direct mapping of the field trapped by sintered superconductors. Furthermore, since the levitation phenomenon does not require a percolating path of zero resistance across the material, as in other application such as wires, one can benefit from producing composites of superconducting grains within a nonactive matrix. Improved mechanical and thermal properties can be obtained using epoxy or metal matrix composites without sacrificing load-bearing and stabilizing capabilities. The treatment presented here applies directly to levitation using such composites.

II. MODEL

We assume that the grain sizes are much larger than the London penetration depth, and that Bean’s model ap-
plies to the superconducting grains individually. In order to make the calculations analytically tractable we will consider a simplified model system consisting of a large collection of decoupled cylindrical grains. For a long cylinder in a parallel field $B_z$, the flux penetrates as shown in Fig. 1. The internal flux density gradient, which is due to pinning, is by Ampere's law associated with an azimuthal current. According to Bean's model this current has the critical magnitude $j_c$, which is taken to be field independent. Reaching a certain value of the applied field, $B^* = \mu_0 j_c R$, the thickness $\delta$ of the current-carrying layer equals the grain radius $R$. The grain magnetization has then the saturation value of $-\frac{1}{2} j_c R$. The magnetization curve for a monotonically increasing applied field is shown below in Fig. 1.

It is crucial to locate on this diagram the operating point of a typical grain. An estimate using an intragranular $j_c \approx 10^4$ A/cm$^2$ and a grain size of $R = 10$ $\mu$m, giving $B^* \approx 10^{-3}$ T, indicates that it is realistic to consider initial conditions of saturated grains, i.e., when the magnet is brought to its levitation position (in such a way that the field increases everywhere in the sample) it generates shielding currents that fill the grains entirely.

When a grain later experiences a field reduction due to magnet displacement flux pours out of the grain establishing in an outer layer a flux density gradient of opposite sign. This layer will contain a $j_c$ circulating in the reverse direction, giving a positive addition to the grain magnetization. For small field reductions Bean's model gives

$$\mu_0 \Delta M_{\text{grain}} = |\Delta B_z|$$

i.e., the same for any long cylinder.

III. LEVITATION CONFIGURATION

To avoid the complications of a strongly nonuniform applied field we will consider the magnet-superconductor configuration shown in Fig. 2. The superconductor has a rectangular shape with dimensions $2b \times l_x \times t_b$ in the $x$, $y$, and $z$ directions, respectively. The bar magnet is initially brought into a symmetric position above the superconductor. The magnet is taken to be long in the $y$ direction ($>l_y$), in which case the applied field $B_z$, experienced by the superconductor is close to that of an infinitely long bar magnet:

![FIG. 1. Upper: flux penetration and current distribution for a long cylindrical grain in a parallel field raised from 0 to $B_z$. Lower: magnetization curve according to Bean's model. The flow of the states of grains 1 and 2 during small field reversals is indicated.](image1)

![FIG. 2. Magnet-superconductor levitation configuration seen from above (upper figure) and viewed along the horizontal $y$ axis (lower figure). The magnet is polarized in the vertical direction.](image2)
Here $2a$ and $t_a$ are the cross-sectional dimensions of the magnet, and $B_r$ is its remanent induction.

From magnetostatics\textsuperscript{18} the force on the superconductor is given by the integral over its volume

$$F = \int_{\text{vol}} (M \cdot \nabla) B_z \, dV,$$

where $M$ is its magnetization. Of course, the force acting on the magnet has the same magnitude, but is oppositely directed.

In the present geometry having $b = a$ it is a good approximation to focus only on the vertical component of the magnetization, i.e., $M \approx M(x,z) \hat{z}$. The lateral force can then be written as

$$F_x = \int_{-b}^{b} M \frac{\partial B_z}{\partial x} \, dV = I_y \int_{-b}^{b} dx \int_{-b}^{b} M \frac{\partial B_z}{\partial x},$$

(3)

where $\nabla \times B_z = 0$ was used.

One sees immediately that $F_x$ vanishes initially because along the $x$ axis $M$ is symmetric whereas the field gradient is antisymmetric. For convenience, in what follows we let $g(x)$ denote the antisymmetric profile of the gradient $\partial B_z/\partial x$.

**IV. LATERAL STABILITY**

We now calculate $F_x$ when the magnet is displaced a small distance $d$ to one side (see Fig. 3). The lateral shift in the field profile implies that $B_z$ increases for $x > 0$ and decreases for $x < 0$. To first order the field change is given by

$$\Delta B_z = B_z(x - d) - B_z(x) = -g(x)d.$$  

This will cause an asymmetry in $M(x)$ which we can evaluate as follows.

Consider two grains, no. 1 and no. 2, located symmetrically on each side of $x = 0$. Initially, they both experience the same applied field. After the magnet displacement, however, grain no. 1 has acquired a positive additional magnetization whereas grain no. 2 remains in saturation. Their trajectories, 1 $\rightarrow$ 1' and 2 $\rightarrow$ 2' on the magnetization curve in Fig. 1 is indicated.

The resulting smoothened magnetization versus $x$ at a certain $z$ is shown in Fig. 4 (upper). The additional magnetization for $x < 0$ equals

$$\Delta M = \frac{f_s}{\mu_0} |\Delta B_z| = \frac{f_s}{\mu_0} g(x)d,$$

where $f_s$ is the volume fraction of the superconducting material.

$$F_x = I_y \int_{0}^{b} dz \left( \int_{-b}^{b} dx \Delta M g(x) + \int_{-b}^{b} dx M \Delta g'(x) \right)$$

$$= I_y \int_{-b}^{b} dx \left( \int_{-b}^{b} \left[ g(x) \right]^2 dx - 2M \Delta g(b) \right) d$$

$$= (\kappa_1 - \kappa_2)d.$$  

(4)

The force consists of two terms: one that is restoring (force constant $\kappa_1$), and one (proportional to $\kappa_2$) that tends to destabilize the levitation. The restoring term originates from the hysteretic nature of the magnetization, and is therefore a consequence of flux pinning. The destabilizing term is due to the superconductor possessing a negative...
FIG. 4. Magnetization profile after a lateral displacement of the magnet (above), and after completing one displacement cycle (below).

overall magnetization. This corresponds essentially to the same kind of instability experienced when trying to levitate one permanent magnet above another with equal poles pointing towards each other.

By differentiation of $B_z(x,z)$ in Eq. (2) one obtains the gradient function $g$, and thereby one can evaluate the integrals contained in $K_1$ and $K_2$. Figure 5 shows $K_1$ and $K_2$ as functions of levitation height $h$ where we have used the parameter values $a = t_x = b = 5$ mm, $l_x = 15$ mm, $B_z = 1.2$ T, $M_{sat} = 10$ A/cm, and $f_x = 0.6$. The value for $M_{sat}$ is taken from Ref. 16, where they measured the trapped field of a sintered YBa$_2$Cu$_3$O$_x$ cylindrical disk magnetized axially in a strong applied field. It was found that the profile of the trapped field can be well reproduced by a surface current, $J_z = 10$ A/cm, representing the outermost part of numerous small intragranular current loops. Within Bean’s model each grain will (after removal of the applied field) be in saturation, and hence $M_{sat} = J_z$.

Two cases are included in the graph: (i) $b = a$ as solid lines; and (ii) $b = a/2$ as dashed lines. In the former case, when the superconductor has the same lateral width as the magnet, one sees that for all practical $h$ values the force constant $K_1$ always exceeds $K_2$. Hence, the levitation is stable. However, at some relatively large distance the curves for $K_1$ and $K_2$ will intersect (not included in the plot). Beyond this point the lateral force does not provide stability. On the other hand, it is doubtful that the present model applies to such distances since the magnetizing field is then strongly reduced.

In case (ii), where the width of the superconductor is only half of that of the magnet, the crossover from stable to unstable conditions occurs at a smaller height. With the parameters chosen in this example we find stable levitation only up to $h = 7$ mm.

V. LATERAL STIFFNESS

We proceed to find the spring constant (magnetic stiffness) associated with vibrational motion. One then needs to consider what happens in terms of magnetization when the magnet motion is reversed after first being displaced a small amplitude $d_0$. With reference to Figs. 1 and 3, one sees that the state of a grain located at $x < 0$, such as grain no. 1, flows reversibly back from $1'$ toward $1$, since in Bean’s model such a minor loop is to first order nonhysteretic. A grain at $x > 0$, such as grain no. 2, will instead experience a corresponding field reduction, and hence will flow from $2'$ and toward a new point $2''$. It is now important to note that as long as the reversed motion, and also subsequent oscillations, do not exceed the virgin interval, $0 < d < d_0$, the response of each grain will be reversible and linear. To find the lateral stiffness, i.e., the slope of the linear $F_x$ vs $d$ relation, it suffices to calculate $F_x$ at one position in addition to $d_0$. It is then most convenient to choose $d = 0$. With the magnet back to its starting point the field $B_z$ has increased to its original magnitude for $x < 0$, and the memory of its magnetic history is erased. This yields the new magnetization profile shown in Fig. 4 (below), which has the same shape as in Fig. 4 (above) except now the positive $\Delta M = -f/\mu_0 g(x) d_0$ is added to the right-hand side of $x = 0$. Note also that now the field gradient is the fully antisymmetric $g(x)$. The force, Eq. (3), therefore contains only one term, namely

$$F_x(d=0) = - K_1 d_0.$$  

Figure 6 (upper) illustrates schematically the behavior of $F_x(d)$ for a case where flux pinning provides stable levitation (positive initial slope). The new equilibrium position $d_{eq}$, around which free oscillations will take place is seen as the intersection point where $F_x = 0$. This occurs at

FIG. 5. Force constants $K_1$ and $K_2$ as functions of levitation distance for (i) superconductor and magnet having same width (solid lines), and (ii) magnet twice as wide as the superconductor (dashed lines). The lateral force is proportional to the difference $K_1 - K_2$. 

FIG. 6 (upper) illustrates schematically the behavior of $F_x(d)$ for a case where flux pinning provides stable levitation (positive initial slope). The new equilibrium position $d_{eq}$, around which free oscillations will take place is seen as the intersection point where $F_x = 0$. This occurs at...
FIG. 6. Lateral force $F_x$ on the superconductor vs magnet displacement $d$.

$d_{eq} = \frac{d_0}{2 - \frac{K_2}{K_1}}$.

Note that $d_{eq} > d_0/2$ so that oscillations starting from $d=d_0$ with zero velocity will be confined within the reversible interval. It follows that the spring constant for the lateral vibrations is given by

$$K = 2K_1 - K_2 = 2l_1, sotb dz(i s,"kW12d=-M,,l g(\beta ) . (3$$

Figure 6 (lower) illustrates an unstable situation with $\kappa_1 < \kappa_2 < 2\kappa_1$. In this case a quite unusual mechanical behavior occurs. In spite of lacking an initial stabilizing force the magnet-superconductor interaction is still able to provide a restoring stiffness for vibrational relative motion; however, there is here no equilibrium point $d_{eq}$ for free oscillations.

From Eq. (5) we can quantitatively evaluate the lateral stiffness $\kappa$ as a function of levitation height. Using the same parameter values as in the previous example with $b=a$, one gets the result shown in Fig. 7. One sees that $\kappa$ depends very strongly upon the distance $h$. In fact, $\kappa(h)$ is very close to an exponential relation (as seen from the semilog plot) over the range covered in Fig. 7. It is interesting that experimentally one indeed observes a stiffness falling off exponentially with the distance. However, this empirical law was found studying the vertical force versus vertical displacement, and it is not obvious that the lateral and vertical stiffness have the same $h$ dependence.

Measurements of lateral stiffness for various heights have been performed by Basinger and co-workers. They used a cylindrical magnet 12.7 mm in diameter and a large sintered YBa$_2$Cu$_3$O$_x$ disk. Although this geometry is not the same as in the present article one should expect an order-of-magnitude agreement. Indeed, we find that their low-amplitude ($< 10 \mu m$) results, which cover magnet heights from 2 to 8 mm, agrees with our theoretical curve within a factor of 2.

In order to test the present theory in greater detail one needs results obtained from proper geometrical configurations. Such results are not available at present.

As mentioned in Sec. I, it is an experimental fact that going beyond the small-amplitude elastic regime treated in this article, one sees a significant change in the response properties. For increasing amplitudes the stiffness gets reduced and at the same time hysteresis and thereby energy dissipation becomes increasingly important. In the present model this can be understood only if the full magnetic response of Bean's model is taken into account. Large displacement amplitudes implies that the field experienced by a grain oscillates with an amplitude comparable to, or even larger than, $B^*$. Thus, instead of moving along small linear portions of the magnetization curve, the grains now follow large subloops that have considerable hysteresis. This leads to a smaller $\Delta M/\Delta B_g$, and it is intuitively clear that the effective stiffness is reduced as the amplitude grows.

Since $B^*$ is the parameter controlling the onset of nonlinear behavior in $M$ one expects that materials with large $j_c$, such as melt-textured materials, have a relatively wide...
amplitude range of elastic force interaction. A quantitative analysis of the hysteretic regime is not given here.

VI. SUMMARY

The force between a permanent magnet and a superconductor results from two distinct factors: (i) the magnetization of the superconductor, i.e., the distribution of supercurrents induced by the magnet, and (ii) the degree of inhomogeneity in the magnet's $B$ field. Since the field produced by magnets of regular shapes can be readily computed to a high precision, one should in principle be able to discuss all aspects of the levitation phenomenon once the magnetization curve of the superconductor material is at hand; however, in practice this proves to be a difficult task. This is due to the fact that nonuniform fields and also irreversible magnetization are essential ingredients. Nevertheless, in view of the vital importance for designs of levitation systems one should aim at understanding all features of the force at a quantitative level.

In this article we have discussed one particular magnet-superconductor configuration that allows an analytical treatment of the lateral (horizontal) component of the force. Using Bean's critical-state model to the individual grains of a granular or composite superconductor we have discussed two central issues, namely the degree of lateral stability and the stiffness associated with vibrational motion. Closed-form expressions for the linearized force and stiffness were derived. The force was found to consist of two terms: one that stabilizes the levitation, which is the result of flux pinning, and one that tends to destabilize, which is due to the diamagnetic response of the finite-sized superconductor. The way in which both geometrical dimensions and the magnetic response of the superconductor determines the stability is brought out in detail. A quite unusual mechanical behavior was predicted by the analysis. Since the stiffness, $K = 2k_1 - k_2$, is always larger than the initial slope, $k_1 - k_2$, of the $F_x(d)$ curve a restoring spring constant may exist even when lateral stability is absent.

As yet there are no reported experiments for the rectangular geometry discussed in this work. However, the quantitative agreement with measured stiffness values taken at different distances in a similar cylindrical configuration is striking. It is therefore reasonable to believe that the approach developed here can be very useful, if also extended to the geometries of more complex levitation systems.

ACKNOWLEDGMENTS

The authors wish to thank Professor Y. Galperin for stimulating discussions. The financial support of NAVF is gratefully acknowledged.

18 See, for example, F. C. Moon, Magneto-Solid Mechanics (Wiley, New York, 1984).