

Dark Matter Thermal Relic Density Computations

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11th March 2026

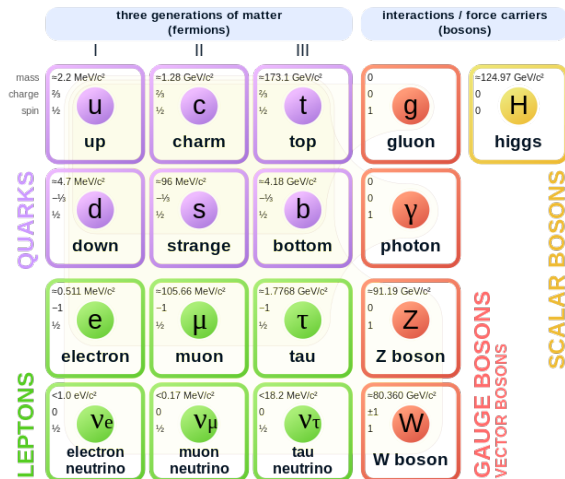
References

- <https://theses.hal.science/tel-04754204> and references therein
- Kolb, Turner: “The early universe”
- Peter, Uzan: “Primordial cosmology”
- Wagoner: “The early universe”
- Peebles: “Principles of physical cosmology”

The Standard Model

- describes the known phenomena until the $\sim \text{TeV}$ scale
- stable particles: e, ν, γ, p
- fails in explaining the matter-antimatter asymmetry

Standard Model of Elementary Particles



General Relativity

- It is the theory of gravity
- Used in the standard cosmological model
- Gravity as a geometrical effect
- It allows us to write the FLRW metric:

$$g_{\mu\nu} = \text{diag}\left(-1, \frac{a^2}{1 - kr^2}, a^2 r^2, a^2 r^2 \sin^2(\theta)\right)$$

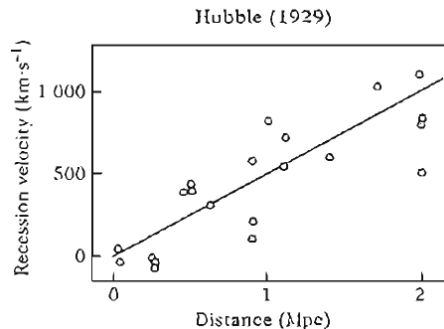
- It allows a non-static universe

The Λ CDM model

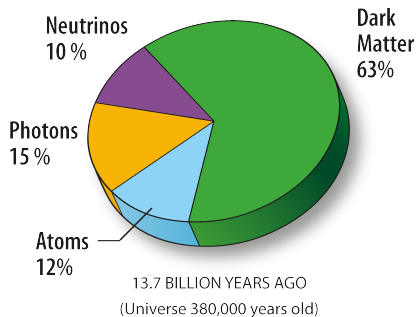
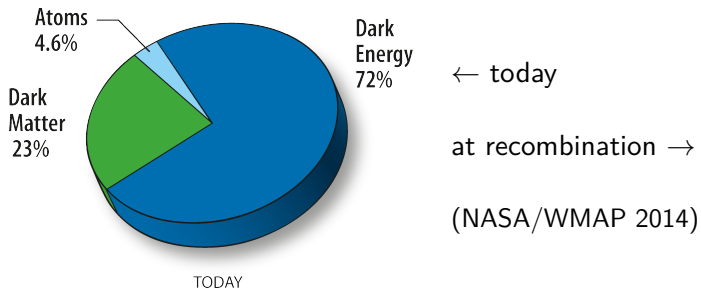
1929: Hubble observes the expansion of the Universe
→ we cannot use Newtonian gravity at large scales

The standard cosmological model relies on these assumptions:

- 1 GR describes gravity
- 2 The SM describes the particle content of the universe up to a certain scale
- 3 The **cosmological principle**: i.e. the universe is homogeneous and isotropic at large scales
- 4 A parameter Λ is needed in the Einstein equation to obtain a consistent value for the expansion velocity of the universe



The energy content of the Universe



Evidences for dark matter at different scales:

- Gravitational lensing on galaxy clusters
- The bullet cluster observations
- Impossibility to explain the growth of structures
- The rotation curves of the galaxies

Components of the Cosmological Fluid

Each component obeys $P = w\rho$, giving $\rho_w \propto a^{-3(1+w)}$:

$$\rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4}, \quad \rho_\Lambda = \text{const.}$$

Normalising to the critical density ρ_0^c , the Planck 2018 values are:

$$\Omega_m = 0.311, \quad \Omega_b h^2 = 0.0224, \quad \Omega_c h^2 = 0.120, \quad \Omega_\Lambda = 0.685$$

Dark matter evidence

$\Omega_m \gg \Omega_b$: most cold matter is **dark** (electromagnetically invisible).
 $\Omega_c h^2 = 0.120$ is the target for any DM production mechanism.

Boltzmann Equation in a FLRW Universe

The phase-space density f evolves according to:

$$L[f] = C[f],$$

where

- $L = \frac{d}{dt}$ is the Liouville operator
- C is the collision operator:
number of particles per phase space volume which are lost or gained per unit of time under collisions with other particles

We can notice that

- L depends on the space-time
- C contains all the information on the dynamics

Boltzmann Equation in a FLRW Universe

In a homogeneous and isotropic universe:

- there can't be dependency on spatial coordinates
- nor a privileged direction

Boltzmann Equation in a FLRW Universe

In a homogeneous and isotropic universe:

- there can't be dependency on spatial coordinates
- nor a privileged direction
- f depends only on energy and time

Then

$$L[f] = \frac{df}{dt} = \frac{\partial f}{\partial t} - H \frac{|\mathbf{p}|^2}{E} \frac{\partial f}{\partial E}$$

The Boltzmann equation now reads as

$$\frac{\partial f}{\partial t} - H \frac{|\mathbf{p}|^2}{E} \frac{\partial f}{\partial E} = C[f]$$

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We cannot solve in this form, so we integrate over momenta and sum over helicity d.o.f. Considering more species, labelled with an index i , and integrating it by parts, we get:

$$g_i \int \frac{d^3 p_i}{(2\pi)^3} L[f_i] = \frac{1}{a^3} \frac{d}{dt} (a^3 n_i) = \dot{n}_i + 3H n_i.$$

The $3H n_i$ term accounts for dilution due to cosmic expansion.

The Collision Term

We assume that the most relevant contributions come from:

- $2 \rightarrow 2$ inelastic reactions (annihilation / co-annihilation / co-scattering)
- Decays

$$C[f] = C_{2 \rightarrow 2}[f] + C_{\text{decay}}[f]$$

The term for $2 \rightarrow 2$ reactions gets contributions only from inelastic processes

Fixed a species “ i ”, for a process $i, j \leftrightarrow k, l$:

$$\begin{aligned}
 g_i \int \frac{d^3 p_i}{(2\pi)^3} C_{2 \rightarrow 2}[f_i] = & - \sum_{j,k,l} \int \left[f_i f_j (1 \pm f_k)(1 \pm f_l) |\mathcal{M}_{ij \rightarrow kl}|^2 \right. \\
 & \left. - f_k f_l (1 \pm f_i)(1 \pm f_j) |\mathcal{M}_{kl \rightarrow ij}|^2 \right] \\
 & \times (2\pi)^4 \delta^{(4)}(p_i + p_j - p_k - p_l) \\
 & \times \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2E_i} \frac{d^3 \mathbf{p}_j}{(2\pi)^3 2E_j} \frac{d^3 \mathbf{p}_k}{(2\pi)^3 2E_k} \frac{d^3 \mathbf{p}_l}{(2\pi)^3 2E_l}
 \end{aligned}$$

where the $+$ ($-$) apply to bosons (fermions)

With some hypotheses:

H1 Maxwell–Boltzmann statistics: $1 \pm f_i \approx 1$
(no Fermi degeneracy / Bose–Einstein condensation)

H2 CP invariance: $|\mathcal{M}_{ij \rightarrow kl}|^2 = |\mathcal{M}_{kl \rightarrow ij}|^2$

the previous expression can be simplified in:

$$g_i \int \frac{d^3 p_i}{(2\pi)^3} C_{2 \rightarrow 2}[f_i] = - \sum_{j,k,l} \int \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2E_i} \frac{d^3 \mathbf{p}_j}{(2\pi)^3 2E_j} \frac{d^3 \mathbf{p}_k}{(2\pi)^3 2E_k} \frac{d^3 \mathbf{p}_l}{(2\pi)^3 2E_l} \\ \times (2\pi)^4 \delta^{(4)}(p_i + p_j - p_k - p_l) [f_i f_j - f_k f_l] |\mathcal{M}_{ij \rightarrow kl}|^2$$

Neither k or l are equal to i , so we further assume that:

H3 k, l quickly enter thermal equilibrium $\Rightarrow f_{k,l} = f_{k,l}^{\text{eq}}$

Now we can apply the detailed balance (invariance of the measure of the phase space):

$$f_i^{\text{eq}} f_j^{\text{eq}} = f_k^{\text{eq}} f_l^{\text{eq}}$$

$$g_i \int \frac{d^3 p_i}{(2\pi)^3} C_{2 \rightarrow 2}^{\text{ann-coan}}[f_i] = - \sum_{j,k,l} \int \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2E_i} \frac{d^3 \mathbf{p}_j}{(2\pi)^3 2E_j} \frac{d^3 \mathbf{p}_k}{(2\pi)^3 2E_k} \frac{d^3 \mathbf{p}_l}{(2\pi)^3 2E_l} (2\pi)^4 \\ \times \delta^{(4)}(p_i + p_j - p_k - p_l) \left(f_i f_j - f_i^{\text{eq}} f_j^{\text{eq}} \right) |\mathcal{M}_{ij \rightarrow kl}|^2$$

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- we recall the expression for the unpolarised cross-section

$$\int \frac{d^3\mathbf{p}_k}{(2\pi)^3 2E_k} \frac{d^3\mathbf{p}_l}{(2\pi)^3 2E_l} (2\pi)^4 \delta^{(4)}(p_i + p_j - p_k - p_l) |\mathcal{M}_{ij \rightarrow kl}|^2 = 4F_{ij} g_i g_j \sigma_{i,j \rightarrow k,l}$$

with

$$F_{ij} = \sqrt{p_i^2 p_j^2 - m_i^2 m_j^2}$$

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So:

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Since we're interested in the species i , we need to sum over the final states, defining:

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Now we recall that

$$n_i = g_i \int \frac{d^3 p_i}{(2\pi)^3} f_i$$

and we introduce

$$(v_{\text{Mø1}})_{ij} = \frac{F_{ij}}{E_i E_j}$$

which has the property to keep the term $(v_{\text{Mø1}})_{ij} n_i n_j$ a Lorentz invariant

We get:

$$\begin{aligned}
 g_i \int \frac{d^3 p_i}{(2\pi)^3} C_{2 \rightarrow 2}^{\text{ann-coan}}[f_i] &= - \sum_j \int \frac{d^3 p_i}{(2\pi)^3} g_i \frac{d^3 p_j}{(2\pi)^3} g_j \left(f_i f_j - f_i^{\text{eq}} f_j^{\text{eq}} \right) \sigma_{ij}(v_{\text{Mø1}})_{ij} \\
 &= - \sum_j \int \left(dn_i dn_j - dn_i^{\text{eq}} dn_j^{\text{eq}} \right) \sigma_{ij}(v_{\text{Mø1}})_{ij}
 \end{aligned}$$

Now, we

- Use H0 (kinetic equilibrium)
- use the fact that distributions in kinetic equilibrium are proportional to the ones in chemical equilibrium, with a factor independent on the momenta

So we can define

$$\langle \sigma_{ij}(v_{\text{Mø1}})_{ij} \rangle = \frac{\int dn_i^{\text{eq}} dn_j^{\text{eq}} \sigma_{ij}(v_{\text{Mø1}})_{ij}}{\int dn_i^{\text{eq}} dn_j^{\text{eq}}} = \frac{\int dn_i dn_j \sigma_{ij}(v_{\text{Mø1}})_{ij}}{\int dn_i dn_j}$$

Neglecting the decay term (DM is stable), we get

$$\dot{n}_i + 3Hn_i = - \sum_j \langle \sigma_{ij}(v_{\text{Mø1}})_{ij} \rangle (n_i n_j - n_i^{\text{eq}} n_j^{\text{eq}})$$

Case $i = j$

This is the case of a model with just one particle BSM:

$$\dot{n} + 3Hn = -\langle\sigma v\rangle\left(n^2 - (n^{\text{eq}})^2\right)$$

We can notice that

- the larger the H , the faster n decreases
- n decreases faster the more it is different from n^{eq}
- n decreases faster the larger is $\langle\sigma v\rangle$

Case of generic i

For a more general model, considering

- a $\mathbb{Z}_2$ symmetry
- more BSM fields
- co-scattering and decays in $C[f]$

one arrives to the same equation of before:

$$\dot{n}_i + 3Hn_i = - \sum_{j \in \{\text{BSM fields}\}} \langle \sigma_{ij}(v_{\text{Mø}})_{ij} \rangle \left(n_i n_j - n_i^{\text{eq}} n_j^{\text{eq}} \right)$$

because the new terms compensate, and give a null contribution

Back to $i = j$

Qualitatively, we can refer to the case $i = j$

$$\dot{n} + 3Hn = -\langle\sigma v\rangle\left(n^2 - (n^{\text{eq}})^2\right)$$

It is clear that if i is in thermal equilibrium at all times, then

$$n = n^{\text{eq}}$$

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The freeze-out scenario

For temperature in today's universe $T \lesssim m$:

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After that, its number density is only affected by the Hubble expansion

Disentangling the effects

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Recalling that $n \propto a^{-3}$, and that $s \propto a^{-3}$, one can define the yield

$$Y \equiv \frac{n}{s}$$

and we get the equation

$$\dot{Y} = -s \langle \sigma v \rangle \left(Y^2 - (Y_{\text{eq}})^2 \right)$$

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Getting

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle}{xH} \left(Y^2 - (Y_{\text{eq}})^2 \right)$$

Or,

$$\frac{x}{Y_{\text{eq}}} \frac{dY}{dx} = -\frac{\Gamma_{\text{eq}}}{H} \left(\left(\frac{Y}{Y_{\text{eq}}} \right)^2 - 1 \right)$$

with $\Gamma_{\text{eq}} = n_{\text{eq}} \langle \sigma v \rangle$ contains all the information about the dynamics:

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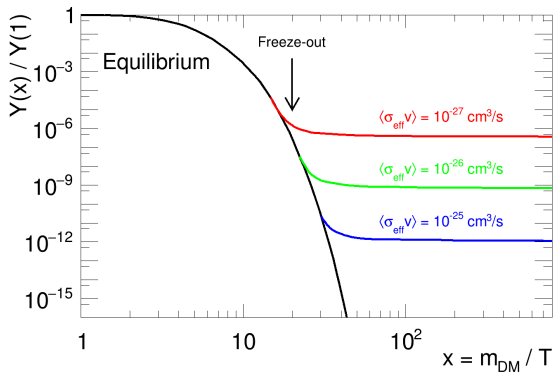
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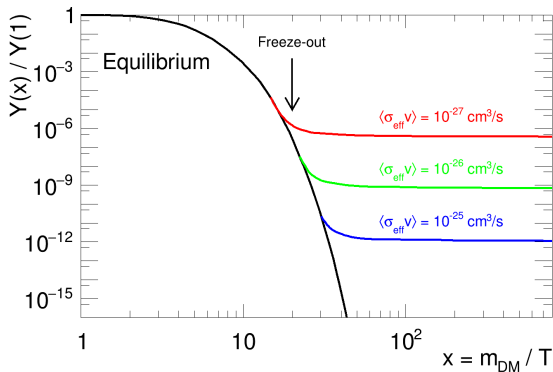
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- $\Gamma_{\text{eq}}/H \ll 1$: the Hubble expansion dominates
(sooner or later there will be too few particles per unit of volume)
the derivative of Y is zero, so Y remains constant



by increasing $\langle \sigma v \rangle$, particles remain in equilibrium for a longer time



by increasing $\langle \sigma v \rangle$, particles remain in equilibrium for a longer time and their final abundance is less

Some approximations

$$\frac{dY}{dx} = - \left(\frac{45}{\pi} G \right)^{-1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma v \rangle \left(Y^2 - (Y_{\text{eq}})^2 \right)$$

$g_*^{1/2}$: degrees of freedom parameter from s

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Approximate for of x_F (from $\Gamma_{\text{eq}} = H$):

$$x_F^{1/2} e^{-x_F} = \sqrt{\frac{4\pi^3 g_{\text{eff}}(x_F)}{45}} \frac{(2\pi)^{3/2}}{g_i M_P m \langle \sigma v \rangle (x_F)}$$

solving it iteratively, one finds $x_F \in [20, 30]$

Some approximations

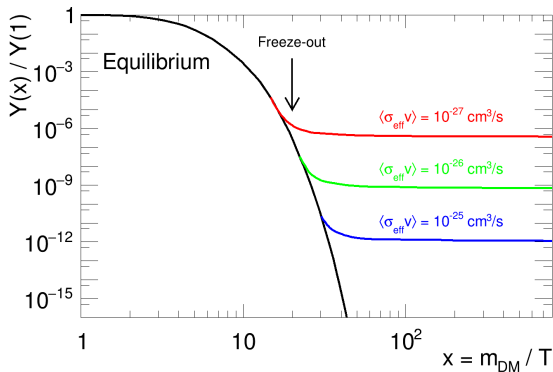
and

$$Y_{\text{eq}}(x_F) = \sqrt{\frac{45g_{\text{eff}}(x_F)}{\pi}} \frac{x_F}{h_{\text{eff}}(x_F) M_P m \langle \sigma v \rangle (x_F)}$$

One can also estimate

$$\Omega h^2 \approx 2.76 \times 10^8 Y_0 \frac{m}{\text{GeV}} \propto \frac{1}{\langle \sigma v \rangle (x_F)}$$

where $Y_0 \approx Y_{\text{eq}}(x_F)$ is the value of Y today



for $x < x_{\text{fo}}$

- $Y \approx Y_{\text{eq}}$
- Y evolves close to the equilibrium curve because the interaction rate is large enough to allow it

What is a WIMP?

A **Weakly Interacting Massive Particle** (WIMP) is a DM candidate with:

- Mass in the range $\sim \text{GeV} - \text{TeV}$
- Interactions with SM particles of *weak-interaction strength*
- Production mechanism: thermal freeze-out

Prototype interaction (e.g. Z -boson or new mediator exchange):

$$\chi\chi \leftrightarrow \bar{f}f, \quad WW, \quad ZZ, \dots$$

The coupling strength is set by the electroweak scale: $g \sim g_{\text{EW}}, m_\chi \sim m_W$.

The WIMP Miracle

Evaluating $\langle\sigma v\rangle$ for a particle with weak-scale mass and coupling:

$$\langle\sigma v\rangle_{\text{EW}} \sim \frac{g_{\text{EW}}^4}{16\pi^2 m_\chi^2} \sim 3 \times 10^{-26} \text{ cm}^3 \text{ s}^{-1} \quad \text{for } m_\chi \sim 100 \text{ GeV}$$

Plugging into the relic density formula:

$$\Omega_c h^2 \approx \frac{0.1 \text{ pb}}{\langle\sigma v\rangle} \approx \frac{0.1 \text{ pb}}{0.1 \text{ pb}} \approx 0.1$$

The WIMP miracle

A particle motivated entirely by **electroweak physics** (hierarchy problem, gauge unification) naturally produces *the right relic density*.

This numerical coincidence is the main appeal of the WIMP scenario.

WIMP Candidates

Supersymmetry (MSSM)

- Lightest neutralino $\tilde{\chi}_1^0$ (mixture of $\tilde{B}$, $\tilde{W}^0$, $\tilde{H}^0$)
- Stability guaranteed by R -parity ($\mathbb{Z}_2$)
- Mass range: ~ 10 GeV–3 TeV
- Most studied WIMP candidate

Singlet scalar / fermion

- Minimal WIMP-like extensions
- Higgs portal interactions

Experimental Searches for WIMPs

Direct detection

- Measure DM–nucleus recoil
- Probes spin-independent / spin-dependent cross sections

Indirect detection

- DM annihilation products: $\gamma, \bar{p}, e^+, \nu$
- Sensitive to $\langle\sigma v\rangle$ today

Collider searches

- Experiments at the LHC
- Probes DM production and mediators

Current status

No confirmed WIMP signal. Strong constraints on spin-independent cross section ($\sigma_{\text{SI}} \lesssim 10^{-47} \text{ cm}^2$ for $m_\chi \sim 100 \text{ GeV}$). Large parameter space remains viable.

Limitations of the Standard Freeze-Out Scenario

Despite the WIMP miracle, there are reasons to consider alternatives:

- **Null results** in direct, indirect and collider searches — increasing tension for vanilla WIMPs
- **Naturalness**: no BSM physics at the LHC weakens the SUSY motivation
- **Cosmological model dependence**: freeze-out result changes with non-standard $H(T)$ or entropy injection

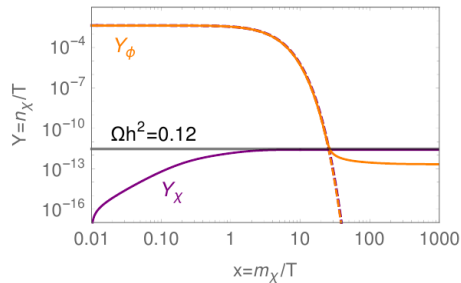
⇒ a broader perspective on DM production mechanisms is needed.

Freeze-In: The FIMP Scenario

- **Feebly Interacting Massive Particle** (FIMP): DM never reaches thermal equilibrium
- Coupling to SM is tiny: $g \sim 10^{-12}$ – 10^{-7}
- DM is gradually **produced** from SM scatterings / decays, starting from $n_\chi \approx 0$

The yield grows until $T \sim m_{\text{med}}$ or until the SM bath cools enough to suppress production

No freeze-out: DM abundance keeps accumulating; $\Omega_c h^2$ sets the coupling.



Thank you!