

GEOMETRIC PHASES.

①

1) The adiabatic theorem in QM.

We have a time-dependent Hamiltonian $H(t)$ with instantaneous eigenstates

$$H(t) |n(t)\rangle = E_n(t) |n(t)\rangle$$

The state can be expanded in this basis

$$|\psi(t)\rangle = \sum_n C_n(t) e^{-i \int_0^t E_n(t') dt'} |n(t)\rangle$$

The phase factor $e^{-i \int_0^t E_n dt'}$ is extracted for convenience recalling the time evolution under a time-independent Hamiltonian: $|\psi(t)\rangle = \sum_n C_n e^{-i E_n t} |n\rangle$

$$i \partial_t |\psi(t)\rangle = i \sum_n \left[\dot{C}_n e^{-i \phi_n} |n\rangle + C_n e^{-i \phi_n} \frac{d}{dt} |n\rangle - i E_n C_n e^{-i \phi_n} |n\rangle \right]$$

$$= H|\psi\rangle = \sum_n C_n e^{-i \phi_n} E_n |n\rangle$$

$$\Rightarrow \sum_n \left[\dot{C}_n e^{-i \phi_n} |n\rangle + C_n e^{-i \phi_n} \frac{d}{dt} |n\rangle \right] = 0$$

$$\sum_n \left[\dot{C}_n e^{-i \phi_n} \underbrace{\langle k|n\rangle}_{\delta_{kn}} + C_n e^{-i \phi_n} \langle k| \frac{d}{dt} |n\rangle \right] = 0$$

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$$\dot{c}_k = -e^{i\phi_k} \sum_n c_n e^{-i\phi_n} \langle k | \frac{d}{dt} | n \rangle$$

$$= -c_k \langle k | \frac{d}{dt} | k \rangle - \sum_{n \neq k} c_n \langle k | \frac{d}{dt} | n \rangle e^{i(\phi_n - \phi_k)}$$

$$H(t) |u(t)\rangle = \bar{E}_n(t) |u(t)\rangle \Rightarrow \frac{dH}{dt} |u\rangle + H \frac{d}{dt} |u\rangle = \frac{d\bar{E}_n}{dt} |u\rangle + \bar{E}_n \frac{d}{dt} |u\rangle$$

$$\Rightarrow \underbrace{\langle k | \frac{dH}{dt} | n \rangle}_{\bar{E}_k \langle k |} + \underbrace{\langle k | H \frac{d}{dt} | n \rangle}_{= \bar{E}_n \langle k | \frac{d}{dt} | n \rangle} = \frac{d\bar{E}_n}{dt} \underbrace{\langle k | n \rangle}_{= \delta_{kn}} + \bar{E}_n \langle k | \frac{d}{dt} | n \rangle$$

for $k \neq n$: $\langle k | \frac{d}{dt} | n \rangle (\bar{E}_k - \bar{E}_n) = - \langle k | \frac{dH}{dt} | n \rangle$

$$\Rightarrow \langle k | \frac{d}{dt} | n \rangle = \frac{1}{\bar{E}_n - \bar{E}_k} \langle k | \frac{dH}{dt} | n \rangle$$

$$\Rightarrow \dot{c}_k = -c_k \langle k | \frac{d}{dt} | k \rangle - \sum_{n \neq k} \frac{c_n}{\bar{E}_n - \bar{E}_k} \langle k | \frac{dH}{dt} | n \rangle e^{i(\phi_k - \phi_n)}$$

For adiabatic change $\frac{\langle k | \frac{dH}{dt} | n \rangle}{\bar{E}_n - \bar{E}_k} \rightarrow 0$

$$\Rightarrow \dot{c}_k = -c_k \langle k | \frac{d}{dt} | k \rangle = -c_k \langle k | \dot{k} \rangle$$

From normalization, $\langle k | k \rangle = 1 \Rightarrow \frac{\langle \dot{k} | k \rangle + \langle k | \dot{k} \rangle}{\langle k | \dot{k} \rangle} = 0$

$$\Rightarrow \text{Re} \langle k | \dot{k} \rangle = 0$$

$\Rightarrow c_k$ will rotate in the complex plane

Define $C_k(t) = C_k(0) e^{i\theta_k(t)}$

(3)

$$\dot{C}_k = i\dot{\theta}_k \underbrace{C_k(0)}_{C_k} e^{i\theta_k(t)} = -C_k \langle k | \dot{k} \rangle$$

$$\Rightarrow \dot{\theta}_k = i \langle k | \dot{k} \rangle \in \mathbb{R} \quad \theta_k \text{ is real.}$$

In the adiabatic limit there are no transitions between the eigenstates. In particular, if we start in an energy eigenstate we will remain in this state as it adiabatically changes in time.

$$|\Psi(t)\rangle = \sum_n C_n(0) e^{i\theta_n(t)} e^{-i \int_0^t E_n(t') dt'} |n(t)\rangle$$

The phases $\theta_n(t)$ depend on choice of phase of the basis vectors $|n(t)\rangle$ (gauge).

We can choose $|\bar{n}(t)\rangle = e^{i\theta_n(t)} |n(t)\rangle$

$$\langle \bar{n} | \frac{d}{dt} | \bar{n} \rangle = \langle n | e^{-i\theta_n} i\dot{\theta}_n e^{i\theta_n} | n \rangle + \underbrace{\langle n | \frac{d}{dt} | n \rangle}_{-i\dot{\theta}_n} = 0$$

$\Rightarrow$ In the gauge $|\bar{n}\rangle$ the phases $\dot{\theta}_n = 0$.

But if we return to the same state the phase may be different from the initial.

2) Berry's phase.

(4)

Let the Hamiltonian depend on some parameter $x \in M$ (M is a manifold). The time dependence is given by a path x_t on M .

For each x we have $H(x)|u(x)\rangle = E_n(x)|u(x)\rangle$

Assume discrete spectrum and that we stay in eigenstate $|\psi(t)\rangle = |u(x_t)\rangle$

Let x_t be a closed cycle, so at time T , $x_T = x_0$

Then

$$|\psi(T)\rangle = e^{-i \int_0^T E_n(x_t) dt} e^{i \theta_n(T)} |u(x_0)\rangle$$

$$\dot{\theta}_n = i \langle u(x_t) | \frac{d}{dt} |u(x_t)\rangle = i \langle u(x_t) | \frac{\partial}{\partial x} |u(x)\rangle \frac{dx}{dt}$$

$$\theta_n(T) = i \int_0^T dt \langle u(x_t) | \frac{\partial}{\partial x} |u(x_t)\rangle \frac{dx}{dt} dt = i \oint_C \langle u(x) | \frac{\partial}{\partial x} |u(x)\rangle dx$$

This quantity is gauge invariant.

$$|\tilde{u}(x)\rangle = e^{i\alpha_n(x)} |u(x)\rangle$$

$$\oint_C dx \langle \tilde{u} | \frac{\partial}{\partial x} \tilde{u} \rangle = \oint_C dx \langle u | e^{-i\alpha_n} (e^{i\alpha_n} \frac{\partial}{\partial x} u) + i \frac{\partial \alpha_n}{\partial x} e^{i\alpha_n} u \rangle$$

$$= \oint_C dx \langle u | \frac{\partial}{\partial x} u \rangle + i \underbrace{\oint_C dx \frac{\partial \alpha_n}{\partial x}}_{=0 \text{ for } C \text{ closed curve}}$$

3) Geometry of Berry's phase I

(5)

$$\Theta_n(T) = i \int_C dx^\mu \langle n | \partial_\mu | n \rangle$$

This depends on the curve C but not on the parametrization x_t as long as it is adiabatic. It also does not depend on the energy $E_n(x)$.

We define $a_\mu^{(n)} = -i \langle n | \partial_\mu | n \rangle$

These are the coordinates of the 1-form

$$a^{(n)} = i \langle n | \partial_\mu | n \rangle dx^\mu \quad \text{on } M.$$

$$\text{Then } \Theta_n = \int_C a^{(n)} = \int_C a_\mu^{(n)} dx^\mu$$

When C is a closed curve we can use Stokes theorem and write.

$$\Theta_n = \int_\Sigma F^{(n)} \quad \text{where } F^{(n)} = dA^{(n)} \quad \text{and } \partial \Sigma = C$$

$$F_{\mu\nu}^{(n)} = 2 \partial_{[\mu} a_{\nu]}^{(n)} = \partial_\mu a_\nu^{(n)} - \partial_\nu a_\mu^{(n)} \quad \left(F_{\mu\nu} = \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu}) \right)$$

We know that $\langle n | \partial_\mu | n \rangle$ is imaginary, so $a_\mu^{(n)}$ is real.

$$\Rightarrow a_\mu^{(n)} = -\text{Im} \langle n | \partial_\mu | n \rangle$$

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$$F_{\mu\nu}^{(u)} = -i \int_u [\langle \partial_\mu u | \partial_\nu u \rangle - \langle \partial_\nu u | \partial_\mu u \rangle]$$

Under gauge-transformation $|\bar{u}\rangle = e^{i\alpha_u} |u\rangle$

$$\bar{a}_\mu = i \langle \bar{u} | \partial_\mu | \bar{u} \rangle = i \langle \bar{u} | i \partial_\mu \alpha_u | \bar{u} \rangle + i \underbrace{\langle \bar{u} |}_{e^{-i\alpha_u} \langle u |} e^{i\alpha_u} \partial_\mu | u \rangle$$

$$= i \langle u | \partial_\mu u \rangle - \partial_\mu \alpha_u = a_\mu - \partial_\mu \alpha_u$$

Same as vector potential A_μ in electromagnetism.

$$\bar{F}_{\mu\nu} = \partial_\mu \bar{a}_\nu - \partial_\nu \bar{a}_\mu = \partial_\mu a_\nu - \partial_\mu \partial_\nu \alpha_u - \partial_\nu a_\mu + \partial_\nu \partial_\mu \alpha_u = F_{\mu\nu}$$

$F_{\mu\nu}$ is gauge inv.

Berry phase $\Theta_n(\mathcal{C}) = \text{flux of } \bar{F}^{(u)} \text{ through } \mathcal{C}$
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Example. Spin- $\frac{1}{2}$ in slowly changing magnetic field.

$$H = -\frac{1}{2} \vec{B} \cdot \vec{\sigma}$$

$$\vec{B} = B (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$|+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}$$

$$\partial_\theta |+\rangle = \begin{pmatrix} -\frac{1}{2} \sin \frac{\theta}{2} \\ \frac{1}{2} e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}$$

$$\partial_\phi |+\rangle = \begin{pmatrix} 0 \\ i e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad \partial_B |+\rangle = 0$$

$$a_\theta^{(+)} = i \langle + | \partial_\theta | + \rangle = \frac{i}{2} (\langle e^{-i\phi} |) \begin{pmatrix} -\frac{1}{2} \sin \frac{\theta}{2} \\ \frac{1}{2} e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix} = \frac{i}{2} (-\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) = 0$$

$$a_\phi^{(+)} = i \langle + | \partial_\phi | + \rangle = i (\langle e^{-i\phi} |) \begin{pmatrix} 0 \\ i e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} = -\sin^2 \frac{\theta}{2}$$

$$a_B^{(+)} = 0$$

$$F_{\theta\theta}^{(+)} = F_{B\theta}^{(+)} = F_{\phi\theta}^{(+)} = F_{\theta\phi}^{(+)} = F_{B\phi}^{(+)} = 0$$

$$F_{\theta\phi}^{(+)} = \partial_\theta a_\phi = -\sin \frac{\theta}{2} \cos \frac{\theta}{2} = -\frac{1}{2} \sin \theta$$

$$F_{\phi\theta}^{(+)} = \frac{1}{2} \sin \theta$$

3D parameter space

(66)

$$\Theta_n = i \int_C dx^\mu \langle u | \partial_\mu | u \rangle = \int_C dx^\mu A_\mu \quad A_\mu = i \langle u | \partial_\mu | u \rangle$$

$$= \int_S dS \cdot \underbrace{\vec{\nabla} \times \vec{A}}_{\vec{V}}$$

$$(\vec{\nabla} \times \vec{A})_i = i \epsilon_{ijk} \partial_j \langle u | \partial_k | u \rangle$$

$$= i \epsilon_{ijk} \langle \partial_j u | \partial_k u \rangle + i \epsilon_{ijk} \underbrace{\langle u | \partial_j \partial_k | u \rangle}_{\rightarrow 0}$$

$$= i \langle \partial_j u | \times | \partial_k u \rangle_i$$

We insert $H = \sum_m |m\rangle \langle m|$ to get

$$V_i = (\vec{\nabla} \times \vec{A})_i = i \sum_m \epsilon_{ijk} \langle \partial_j u | m \rangle \langle m | \partial_k u \rangle$$

For the term $m=u$ we use $\partial_j \langle u | u \rangle = \langle \partial_j u | u \rangle + \langle u | \partial_j u \rangle = 0$

$$\Rightarrow \langle \partial_j u | u \rangle = -\langle u | \partial_j u \rangle$$

$$\epsilon_{ijk} \langle \partial_j u | u \rangle \langle u | \partial_k u \rangle = -\epsilon_{ijk} \langle u | \partial_j u \rangle \langle u | \partial_k u \rangle = 0$$

For $m \neq u$: $H |m\rangle = E_m |m\rangle$

$$\partial_k H |m\rangle + H | \partial_k m \rangle = \partial_k E_m |m\rangle + E_m | \partial_k m \rangle$$

$$\langle u | \partial_k H | m \rangle + \underbrace{\langle u | H | \partial_k m \rangle}_{E_m \langle u | \partial_k m \rangle} = \partial_k E_m \underbrace{\delta_{mu}}_{=0, m \neq u} + E_m \langle u | \partial_k m \rangle$$

$$\Rightarrow \langle u | \partial_k m \rangle = \frac{\langle u | \partial_k H | m \rangle}{E_m - E_u}$$

$$\Rightarrow \vec{V}_u = i \sum_{m \neq u} \frac{\langle u | \vec{\nabla} H | u \rangle \times \langle u | \vec{\nabla} H | m \rangle}{(E_m - E_u)^2} = -i \sum_{m \neq u} \frac{\langle u | \vec{\nabla} H | m \rangle \times \langle m | \vec{\nabla} H | u \rangle}{|E_m - E_u|^2}$$

$$\langle u | \partial_\mu | u \rangle \text{ is imaginary} = i \cdot \text{Im} \langle u | \partial_\mu | u \rangle$$

Beispiel. Spin $\frac{1}{2}$: ungestört. $H = \frac{1}{2} \vec{B} \cdot \vec{\sigma}$

$$\nabla H = \nabla_B H = \frac{1}{2} \vec{\sigma}$$

$$H | \pm \rangle = \pm \frac{1}{2} |\vec{B}| | \pm \rangle$$

Diese sind Eigenzustände von

Für ∇_ϕ tragen wir $\langle - | \vec{\sigma} | + \rangle$

$$| + \rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}$$

$$| - \rangle = \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

$$\langle - | \sigma_x | + \rangle = \langle - | \begin{pmatrix} e^{i\phi} & 0 \\ 0 & -e^{i\phi} \end{pmatrix} | + \rangle = -e^{2i\phi} s^2 + c^2$$

$$\langle - | = (-e^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2})$$

$$\langle - | \sigma_y | + \rangle = i \langle - | \begin{pmatrix} -e^{i\phi} & 1 \\ 1 & e^{i\phi} \end{pmatrix} | + \rangle = i e^{2i\phi} s^2 + c^2$$

$$\langle - | \sigma_z | + \rangle = \langle - | \begin{pmatrix} c & 0 \\ 0 & -e^{i\phi} s \end{pmatrix} | + \rangle = -e^{2i\phi} cs - e^{i\phi} c^2 = -2e^{i\phi} cs$$

$$\langle - | \nabla H | + \rangle \times \langle + | \nabla H | - \rangle = \frac{1}{4} (-e^{2i\phi} s^2 + c^2, i e^{2i\phi} s^2 + c^2, -2e^{i\phi} cs)$$

$$\times (-e^{-2i\phi} s^2 + c^2, -i e^{-2i\phi} s^2 + c^2, -2e^{-i\phi} cs)$$

$$= \frac{1}{4} (-2i e^{i\phi} cs^3 - 2i e^{i\phi} c^3 s - 2i e^{-i\phi} cs^3 + 2i e^{-i\phi} c^3 s,$$

$$2e^{-i\phi} cs^3 - 2e^{i\phi} c^3 s - 2e^{i\phi} cs^3 + 2e^{-i\phi} c^3 s,$$

$$i s^4 - e^{2i\phi} c^2 s^2 - i e^{-2i\phi} c^2 s^2 + c^4 + i s^4 + i e^{2i\phi} c^2 s^2 + e^{-2i\phi} c^2 s^2 - c^4)$$

$$= \frac{1}{4} (-2i(e^{i\phi} + e^{-i\phi})cs^3 + 2(e^{i\phi} - e^{-i\phi})c^3 s,$$

$$-2(e^{i\phi} - e^{-i\phi})cs^3 + 2(e^{i\phi} - e^{-i\phi})c^3 s,$$

$$2i s^4 -$$

$$|-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$\langle -| = \begin{pmatrix} -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

6d

$$\sigma_x |+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ e^{i\phi}s \end{pmatrix} = \begin{pmatrix} e^{i\phi}s \\ c \end{pmatrix}$$

$$\langle -|\sigma_x|+\rangle = -e^{i\phi}s^2 + e^{-i\phi}c^2$$

$$\sigma_y |+\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c \\ e^{i\phi}s \end{pmatrix} = \begin{pmatrix} -ie^{i\phi}s \\ ic \end{pmatrix}$$

$$\langle -|\sigma_y|+\rangle = ie^{i\phi}s^2 + ie^{-i\phi}c^2$$

$$\sigma_z |+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c \\ e^{i\phi}s \end{pmatrix} = \begin{pmatrix} c \\ -e^{i\phi}s \end{pmatrix}$$

$$\langle -|\sigma_z|+\rangle = -cs - cs = -2cs.$$

$$\vec{U} = \langle -|\vec{\sigma}|+\rangle = (-e^{i\phi}s^2 + e^{-i\phi}c^2, ie^{i\phi}s^2 + ie^{-i\phi}c^2, -2cs)$$

$$\vec{U} \times \vec{U}^* = (-e^{i\phi}s^2 + e^{-i\phi}c^2, ie^{i\phi}s^2 + ie^{-i\phi}c^2, -2cs)$$

$$\times (-e^{-i\phi}s^2 + e^{i\phi}c^2, -ie^{i\phi}s^2 - ie^{-i\phi}c^2, -2cs)$$

$$= (-2ie^{i\phi}cs^3 - 2ie^{-i\phi}c^3s - 2ie^{-i\phi}cs^3 + 2ie^{i\phi}c^3s,$$

$$2e^{-i\phi}cs^3 - 2e^{i\phi}c^3s - 2e^{i\phi}cs^3 + 2e^{-i\phi}c^3s,$$

(M)

$$\vec{U} \times \vec{U}^* = -2i\vec{n}$$

$$\vec{n} = \frac{\vec{B}}{|\vec{B}|}$$

$$\Rightarrow \vec{V}_+ = \frac{1}{4\pi} \frac{\vec{U} \times \vec{U}^*}{|\vec{B}|^2} = \frac{\vec{B}}{2|\vec{B}|^3}$$

monopole field.

Direct calculation:

$$H(\vec{B}) = \frac{1}{2} \vec{B} \cdot \vec{\sigma}$$

$$|+\rangle = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix}$$

(7)

$$\Theta = i \int_C dt \langle + | \frac{d}{dt} | + \rangle$$

Assume $\Theta = \cos \theta$, $\phi = \phi(t) = \phi(t)$

$$\begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} i\dot{\phi} e^{i\phi} \sin \frac{\theta}{2} \\ 0 \end{pmatrix}$$

$$= i \int_C dt i \dot{\phi} \sin^2 \frac{\theta}{2} = -\frac{1}{2} (1 - \cos \theta) \int_0^{2\pi} dt \dot{\phi} = -\pi (1 - \cos \theta)$$

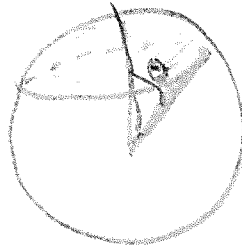
Wrong flux:

$$\Theta = \int \vec{F} = \int_0^\theta \int_0^{2\pi} d\phi \cdot \frac{1}{2} \sin \theta = -[\cos \theta - 1] \pi = \pi (1 - \cos \theta)$$

wrong sign.

From V_+ :

$$\Theta = \int_S \vec{L} \cdot \vec{B}^2 d\vec{R} \cdot \vec{n} = \frac{1}{2} \int_S d\vec{R} = \frac{1}{2} \cdot \int_0^{2\pi} d\phi \int_0^\theta d\theta \sin \theta = \pi (1 - \cos \theta)$$



For V_- or $\vec{V}_- = -\vec{V}_+$

Close to degeneracy.

$$\vec{V}_n = - \int_m \sum_{m \neq n} \frac{\langle m | \nabla H | n \rangle \times \langle n | \nabla H | m \rangle}{(E_m - E_n)^2}$$

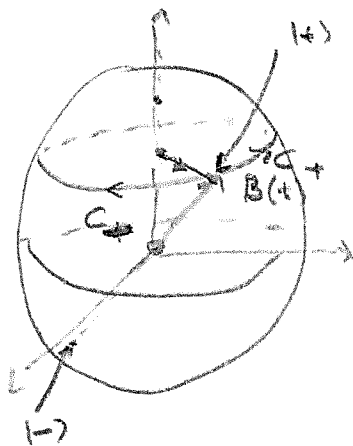
$$\Theta_n = \int_{\vec{S}, dS=c} d\vec{S} \cdot \vec{V}_n$$

If C is close to a degeneracy of n with another level, the sum over m will be dominated by this term. The H restricted to these two levels takes the form $H = \frac{1}{2} \vec{R} \cdot \vec{\sigma}$ with suitably scaled parameters $\vec{R}$.

4) Experiment to measure the Berry phase.

We have a 1-qubit quantum computer.
(TLS with full control of the Hamiltonian)

B)



Start in $B \parallel z$ and eigensate $|\uparrow\rangle$

$$\Rightarrow \text{Superposition } \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$$

$$\xrightarrow{C_-} \frac{1}{\sqrt{2}} (e^{-i\frac{\omega T}{2} + i\delta_-} |+\rangle + e^{+i\frac{\omega T}{2} - i\delta_-} |-\rangle)$$

$$= \frac{1}{\sqrt{2}} e^{-i\delta + i\delta_-} (|+\rangle + e^{2i\delta - 2i\delta_-} |-\rangle)$$

We could measure $\delta - \delta_-$ from this state
but $\delta \gg \delta_-$, so it is difficult.

$$\xrightarrow{\pi} \frac{1}{\sqrt{2}} (e^{-i\delta + i\delta_-} |-\rangle + e^{2i\delta - i\delta_-} |+\rangle)$$

$$\xrightarrow{C_+} \frac{1}{\sqrt{2}} (e^{-i\delta + i\delta_-} e^{i\delta - i\delta_+} |-\rangle + e^{i\delta - i\delta_-} e^{-i\delta + i\delta_+} |+\rangle)$$

$$= \frac{1}{\sqrt{2}} (e^{-i(\delta_- - \delta_+)} |-\rangle + e^{-i(\delta_- - \delta_+)} |+\rangle)$$

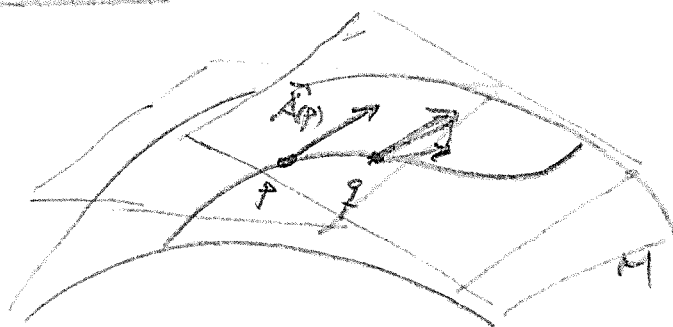
Since $\delta_- = -\delta_+$ we get $\frac{1}{\sqrt{2}} (e^{2i\delta_-} |-\rangle + e^{-2i\delta_-} |+\rangle)$

5) Geometry of Berry phase II

9

Parallel transport.

Vå ser på en flate
med en kurve



Hvor vi flytter $\vec{A}(p) \in T_p M$ til q vil den generelt ikke være
længst til flate (ikke $\in T_q M$)

$\forall$ def. den paralleltransporten som projeksjonen
ned på $T_q M$ for en infinitesimal forflytning $p \rightarrow q$.

Hvor $\vec{n}_q$ er normal til M i q er $\vec{A}_{||}(q) = \vec{A}(p) - \vec{n}_q (\vec{n}_q \cdot \vec{A}(p))$

Covariant derivative

The covariant derivative at p in direction x^μ is

$$D_\mu \vec{A} = \partial_\mu \vec{A} - \vec{n} (\vec{n} \cdot \partial_\mu \vec{A}).$$

If the vector field $\vec{A}$ is parallel transported then
covariant derivative is $D_\mu \vec{A} = 0$

Berry-Simon's connection.

For states $|n(x)\rangle$, $x \in M$ some set of parameters.

$$D_\mu |n(x)\rangle = \partial_\mu |n(x)\rangle - |n(x)\rangle \langle n(x) | \partial_\mu |n(x)\rangle$$

The time derivative along a curve x_t :

$$\frac{D|\psi\rangle}{dt} = D_\mu |\psi\rangle \cdot \dot{x}^\mu = (\partial_\mu - \langle \psi | \partial_\mu | \psi \rangle) |\psi\rangle \dot{x}^\mu$$

During adiabatic transport: $|\psi(t)\rangle = e^{i\theta(t)} |\psi(x_t)\rangle$

$$d_t |\psi\rangle = i\dot{\theta} e^{i\theta} |\psi\rangle + e^{i\theta} \frac{\partial}{\partial t} |\psi\rangle$$

$$\dot{\theta} = i \langle \psi | \partial_t | \psi \rangle \dot{x}^\mu$$

$$\downarrow = - \langle \psi | \partial_\mu | \psi \rangle \dot{x}^\mu e^{i\theta} |\psi\rangle + e^{i\theta} \partial_\mu |\psi\rangle \dot{x}^\mu$$

$$= [\partial_\mu - \langle \psi | \partial_\mu | \psi \rangle] |\psi\rangle \dot{x}^\mu = D_\mu |\psi\rangle \dot{x}^\mu = \frac{D}{dt} |\psi\rangle$$

Here we remove the dynamical phase
By $H \rightarrow H' = H - E_n(t) \mathbb{1}$
So $E_n'(t) = 0 \quad \forall t$.

$\Rightarrow$ Adiabatic change is equ. to parallel transport.

$\Rightarrow$ Berry phase = holonomy (anholonomy) of the parallel transport around a curve in M .

Holonomy.

