
Introduction to Cosmological Inflation

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ABSTRACT

These lectures will introduce the central ideas of cosmological inflation and explore what lies beyond the standard model of cosmology. We will start with a short overview of the key cosmological concepts, before turning to the horizon and flatness problems that motivate the need for a rapid early expansion of the universe. Using a simple single-field inflationary model, we will then highlight the key features of inflation and the calculations required to connect theory with observations. Finally, we will briefly explore further implications of inflation, including the possible formation of primordial black holes and the cosmological particle production.

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DISCLAIMER

These notes are based *heavily* on the books and notes of Daniel Baumann – both his textbook *Cosmology* [1] and notes and book on inflation [2, 3]. Many of the figures are taken wholesale from Ref. [1], though sometimes I have made small modifications.

In addition, I’ve included discussions from other resources such as

- Dodelson and Schmidt’s *Modern Cosmology* [4]
- Carroll’s *Spacetime and Geometry* [5]
- Ned Wright’s Cosmology Tutorials ([link](#))

There *certainly* are errors in these notes. Any input is welcome.

NOTATION CONVENTIONS

First and foremost we will be using *natural units* throughout this text, i.e.,

$$\hbar = c = k_B = 1$$

If you are unfamiliar with this choice of units, I hope you will soon come to realize utility of this – and naturalness! The key point here is to measure everything in units of energy. For example,

$$1 \text{ meter} \simeq 5.07 \times 10^{15} \text{ GeV}^{-1},$$

$$1 \text{ second} \simeq 1.52 \times 10^{24} \text{ GeV}^{-1},$$

$$1^\circ \text{ C} \simeq 2.36 \times 10^{-11} \text{ GeV},$$

$$1 \text{ g} \simeq 5.61 \times 10^{23} \text{ GeV}.$$

Notice here that (1) distance and time are put on equal footing as required by special relativity and (2) energy-like things – temperature, mass, etc. – have the units of energy.

Eventually, we must restore our calculations to have units which are useful for experimentalists. To do so, one can simply use a conversion table. Prof. Timothy Wiser has written an application called [GUTCalc](#) which can easily do this for you.

We will stick with the *mostly pluses* metric signature, $\text{diag}(-1, +1, +1, +1)$, so particle physicists beware.

Part I

Lecture 1: Cosmology and motivation for inflation

1. INTRODUCTION

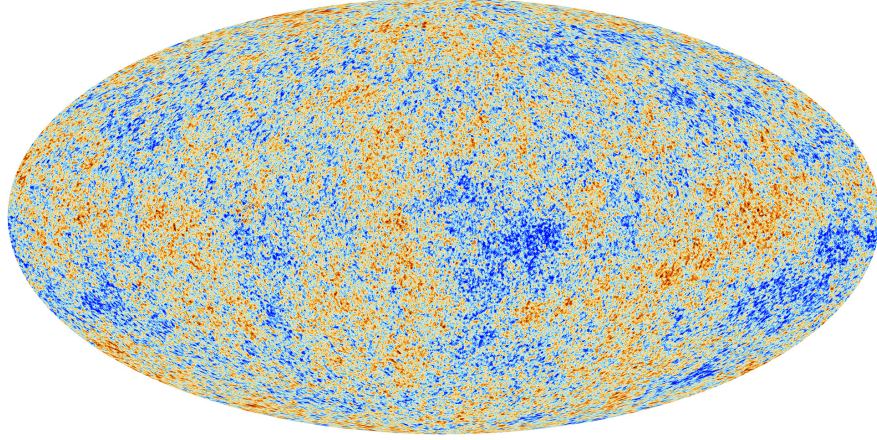


Figure 1. Credit: ESA and the Planck Collaboration

The goal of these notes is to introduce the key features of cosmological inflation and, crucially, to connect these features to observation. The most important cosmological observation to date is the measurement of the *cosmic microwave background* (CMB). This light is the earliest radiation we can hope to observe. It comes from the moment when the Universe became sufficiently cool and dilute that photons decoupled from the abundant charged particles. These photons have traveled for approximately 380,000 years before reaching our detectors, forming the image shown in Fig. 1.

Fig. 1 shows the temperature deviations in the remarkably uniform CMB. The deviations are

$$\frac{\delta T}{\bar{T}_0} \lesssim 10^{-4} \quad (1.1)$$

around the average value of 2.73 K. The structure of the deviations is examined statistically. Describing the statistical distribution of the CMB temperature fluctuations is a fundamental criterion for a successful cosmological model. As we will see, the inflationary paradigm does a remarkable job in this respect – despite the fact that its original motivating principles lay elsewhere.

In order to describe the statistical properties of the CMB, let's first define a few crucial quantities. First is the fractional temperature fluctuation $\Theta(\hat{\mathbf{n}})$,

$$T(\hat{\mathbf{n}}) = \bar{T}_0[1 + \Theta(\hat{\mathbf{n}})] \iff \Theta(\hat{\mathbf{n}}) = \delta T(\hat{\mathbf{n}})/\bar{T}_0. \quad (1.2)$$

Here $\hat{\mathbf{n}}$ is the unit vector pointing from us, the observer, to a given point in the sky. A CMB photon which reaches us from a given point in the sky is propagating antiparallel to $\hat{\mathbf{n}}$.

We can now define the *two-point* correlation function,

$$C(\theta) \equiv \langle \Theta(\hat{\mathbf{n}}) \Theta(\hat{\mathbf{n}}') \rangle \quad (1.3)$$

where $\cos \theta = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'$. The average here is a spatial over an infinite volume¹. The fact that $C(\theta)$ depends only on the angle between $\mathbf{n}$ and $\mathbf{n}'$ is a consequence of statistical isotropy.

It is natural to decompose $C(\theta)$ in a basis of Legendre polynomials,

$$C(\theta) = \sum_{\ell} C_{\ell} P_{\ell}(\cos \theta), \quad \text{where} \quad C_{\ell} = 2\pi \int_{-1}^1 d \cos \theta C(\theta) P_{\ell}(\cos \theta). \quad (1.4)$$

I will now state, without proof, the key equation which links the temperature fluctuations to inflationary theory:

$$C_{\ell} = 4\pi \int d \ln k \Delta_{\mathcal{R}}^2(k) |\mathcal{T}_{\ell}(k)|^2 \quad (1.5)$$

There are two key ingredients here:

- $\Delta_{\mathcal{R}}^2(k)$ is the *comoving curvature power spectrum*. It is a *statistical quantity* which describes the initial fluctuations that existed in the early universe. In other words, it establishes the initial conditions.
- $\mathcal{T}_{\ell}(k)$ is the *transfer function*. It is a deterministic quantity, which describes the evolution of the universe given a set of initial conditions.

In the end, we will use the inflationary paradigm to determine $\Delta_{\mathcal{R}}^2(k)$. Observations of the CMB have found that $\Delta_{\mathcal{R}}^2(k)$ is nearly scale-invariant, i.e., it is very-close to being independent of k . This feature will be a key requirement for any model of early universe physics, and will naturally arise in inflationary models.

2. MINIMAL VIABLE BACKGROUND

Unfortunately, there isn't sufficient time to cover all of the relevant background material. A better (but still incomplete) attempt has been made in the Appendices, but this itself is a crash course. Therefore, to make sure we can focus on inflation, these notes will present the minimum amount of background possible. Questions are always welcome.

2.1. From general relativity to the Hubble parameter

The framework of modern cosmology is general relativity, with Einstein's equations in the center stage,

$$\underbrace{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{curvature}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{energy-momentum}} \quad (2.1)$$

These equations describe how energy (matter) and momentum alter space-time and vice versa. Here, $R_{\mu\nu}$, R are characterize the curvature and are a function of the *metric*, denoted $g_{\mu\nu}(x)$. The metric is the key quantity

¹ Technically speaking, it is an ensemble average. However, we assume ergodicity.

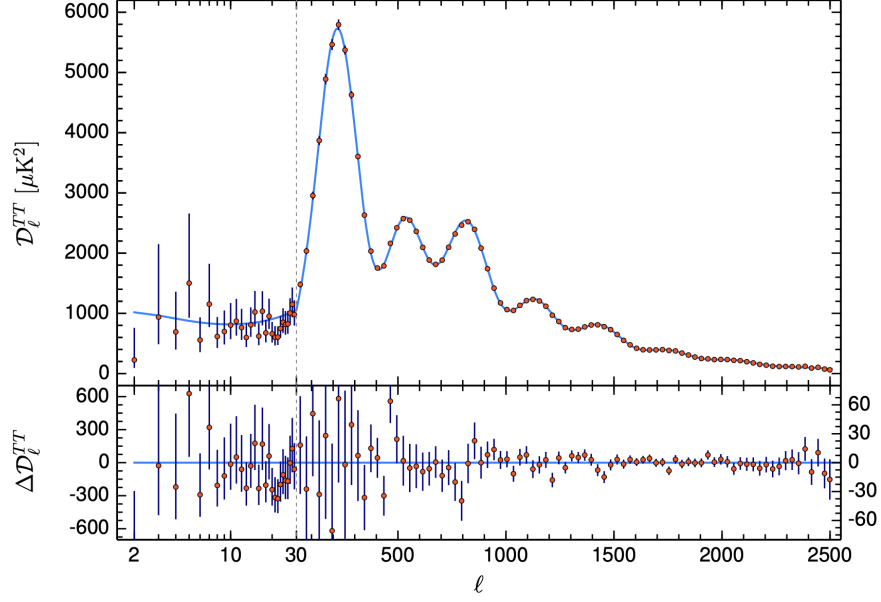


Figure 2. The CMB power spectrum as presented in Ref. [6]. Here, $\mathcal{D}_\ell = \ell(\ell + 1)C_\ell T_0^2/2\pi$.

of interest for the left-hand-side.

The right-hand-side features the energy-momentum tensor. This quantity describes how matter, energy and momentum are distributed in space-time.

Fundamentally, Eq. (2.1) is a tensor equation, i.e., it is a short way writing many equations each with $\{\mu, \nu\} = (0, 1, 2, 3)$. Furthermore Einsteins equations are, in general, partial differential equations – and coupled at that. This makes finding a solution for the metric very nontrivial. To proceed, we make a fundamental assumption which dictates the precise form of the metric.

Cosmological principle

Viewed on a sufficiently large scale, the properties of the universe are the same for all observers. [7]

In other words, the Universe is *homogeneous* and *isotropic* at sufficiently large scales.

In simpler terms, this means that there is no special place in the Universe – the cosmos are more-or-less the same everywhere. Of course, the center of the Sun is very different from the inside of your bedroom which is why this statement qualifies “on sufficiently large scales.”

For a sense of the meaning of *homogeneous* and *isotropic* see Fig. 3. On somewhat more specific terms, we can define homogeneity as invariance to translations and isotropy as invariance to rotations. The CMB displays excellent agreement with the cosmological principle, but many other experimental surveys have also demonstrated that it holds on large scales.

The cosmological principle greatly reduces the complexity of Einstein’s equations. In particular, these requirements imply that the manifold and metric describing the Universe are *maximally symmetric*. However, our Universe isn’t quite homogeneous and isotropic – at least not in time.

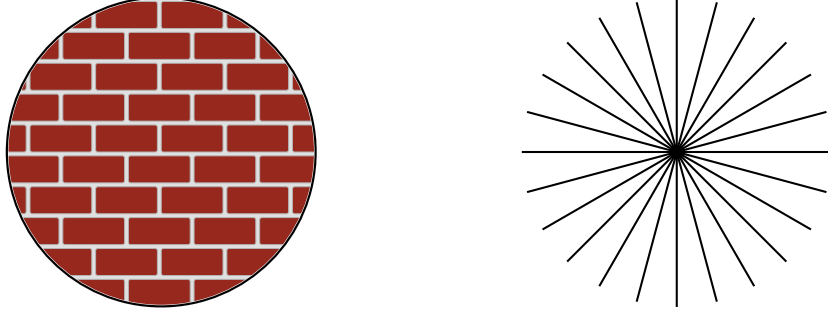


Figure 3. (Left) Homogeneous (but not isotropic) universe. (Right) Isotropic (but not homogeneous) universe. Examples taken from Ned Wright's Cosmology Tutorial.

To describe mathematically a spatially, maximally symmetric, time evolving space-time we can use the Friedmann–Lemaître–Robertson–Walker metric (FLRW) metric,

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right). \quad (2.2)$$

Here, $a(t)$ is the *scale factor* which describes the size of a space-time slice at a given time t . This is a *key quantity* of interest in our discussion as it describes the expansion history of the universe. The parameter κ characterizes the spatial curvature. Observationally $\kappa \simeq 0$ and so, but we will maintain it for now. In particular $\kappa = k/R_0^2$ where $k = \{0, \pm 1\}$ and R_0 is the curvature scale today.

In the metric as written, the scale factor has been normalized so that

$$a(t = t_0) = 1 \quad (2.3)$$

where t_0 is the present day age of the Universe.

Another important quantity is the logarithmic rate of change of the scale factor which is called the *Hubble parameter*,

$$H(t) \equiv \frac{1}{a} \frac{da}{dt} = \frac{d \ln a}{dt}. \quad (2.4)$$

The present day value of the Hubble parameter is the Hubble constant, denoted H_0 . Another crucial quantity is the conformal time η which is defined through

$$d\eta \equiv \frac{dt}{a}. \quad (2.5)$$

This definition has the advantage of taking transforming the metric in Eq. (A.6) to a form where we can factor out scale factor from all terms. Computations are often computed in conformal time rather than in physical time. Regardless – in order to proceed, we need to understand how general relativity dictates the evolution of the H and therefore, of $a(t)$.

2.2. The Friedmann equation

Here we will simply state the (first) Friedmann equation which describes how energy densities impact the evolution of this history of the universe,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_i \rho_i(a) - \frac{\kappa}{a^2}. \quad (2.6)$$

Here, ρ_i is the energy density (energy per volume) of a given *fluid*. This equation is derived in Appendix A and is a direct consequence of one of Einstein's equations ($\mu = \nu = 0$).

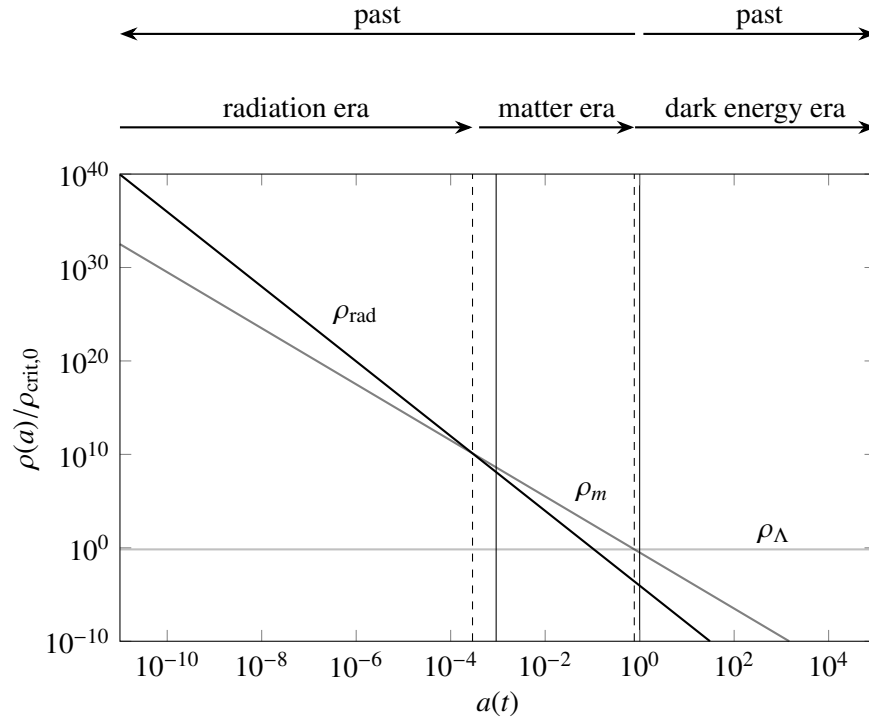


Figure 4.

Different types of matter scale differently (though understandably) with the scale factor,

$$\text{non-relativistic matter: } \rho_m \propto a^{-3}, \quad (2.7a)$$

$$\text{relativistic matter: } \rho_{\text{rad}} \propto a^{-4}, \quad (2.7b)$$

$$\text{cosmological constant: } \rho_\Lambda \propto \text{const.} \quad (2.7c)$$

In general,

$$\rho \propto a^{-3(1+w)}, \quad (2.8)$$

where $w = P/\rho$ and P is the pressure of a given fluid. This is further explained in Appendix A.

So, the name of the game is to model the universe using different components which enter via the Friedmann equation. You can then explore the implications of these components on the expansion history by (numerically) solving Eq. (2.6).

The Standard Model of Cosmology, Λ CDM, contains three components – radiation, a cosmological constant, baryons (visible matter) and dark matter. The latter two components are both non-relativistic and can be grouped together in the Friedmann equation.

Let us now write things in a more simplified form. To do so, we will define the *critical density* ρ_{crit} as

$$\rho_{\text{crit}} \equiv 3M_{\text{Pl}}^2 H^2 \quad (2.9)$$

where M_{Pl} is the *reduced* Planck mass defined as $M_{\text{Pl}} = \sqrt{8\pi G} \simeq 2.44 \times 10^{18}$ GeV. The present-day value of critical density is denoted $\rho_{\text{crit},0}$. We can use this to define the density parameter,

$$\Omega_{X,0} \equiv \frac{\rho_{X,0}}{\rho_{\text{crit},0}}. \quad (2.10)$$

This enables us to write the Friedmann equation as,

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\text{rad},0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{k,0} a^{-2} + \Omega_{\Lambda} \quad (2.11)$$

where we have defined $\Omega_{k,0} = -\kappa/H_0^2$.

2.3. The “Horizon”

Unfortunately, there are many different quantities in cosmology which are called the *horizon*. To make matters worse, we often say “the horizon” to refer to a plethora of different quantities. For a detailed discussion, refer to Ref. [8].

Particle Horizon

The particle horizon, $d_h(\eta)$, is the distance light can have traveled from $t = t_i$ to a given time t .

The conformal time and the particle horizon are related via

$$d_h(\eta) \equiv \eta - \eta_i = \int_{t_i}^t \frac{dt}{a(t)} = \int_{\ln a_i}^{\ln a} (aH)^{-1} d \ln a. \quad (2.12)$$

The final equality of Eq. (2.12) shows that the particle horizon is the logarithmic integral of another key quantity – the *comoving Hubble radius* $(aH)^{-1}$.

Comoving Hubble Radius

The comoving Hubble radius $(aH)^{-1}$ is the distance a particle can travel in one expansion time. In this way, it acts as measure of whether or not particles can interact “now”. Note that,

$$(aH)^{-1} \propto \begin{cases} a & \text{radiation domination,} \\ a^{1/2} & \text{matter domination.} \end{cases} \quad (2.13)$$

In other words for ordinary energy densities, $(aH)^{-1}$ is monotonically increasing. Therefore, the largest contributions to Eq. (2.12) will come from *late times*.

3. FLATNESS PROBLEM

In order to describe the flatness problem, we rearrange Eq. (2.6) to give

$$\Omega_k(t) = \frac{\rho_{\text{crit}} - \rho_{\text{tot}}}{\rho_{\text{crit}}} = \left(\frac{a_0 H_0}{aH} \right)^2 \Omega_{k,0}. \quad (3.1)$$

Here, $\rho_{\text{crit}}(t) = 3M_{\text{Pl}}^2 H^2(t)$ and ρ_{tot} includes all of the different components. Observations have placed

$$|\Omega_{k,0}| < 0.005. \quad (3.2)$$

Since $(aH)^{-1}$ is monotonically increasing for conventional energy densities the curvature parameter was even smaller at earlier times.

If we ignore the period of dark energy domination one finds for $\Omega_k(t)$,

$$\Omega_k(t) = \frac{|\Omega_{k,0}|}{\Omega_{m,0}} \frac{a^2}{a + a_{\text{eq}}}. \quad (3.3)$$

At matter-radiation equality when $a_{\text{eq}} \simeq 1/(1 + 3400)$,

$$|\Omega_k(t_{\text{eq}})| = \frac{|\Omega_{k,0}|}{\Omega_{m,0}} \frac{a_{\text{eq}}}{2} < 10^{-6}. \quad (3.4)$$

We can extrapolate further by using the fact that,

$$H^2 \simeq \frac{H_{\text{eq}}^2}{2} \left(\frac{a_{\text{eq}}}{a} \right)^4 \quad (3.5)$$

and deduce

$$\Omega_k(t \lesssim t_{\text{eq}}) \simeq 2\Omega_k(t_{\text{eq}}) \left(\frac{a}{a_{\text{eq}}} \right)^2. \quad (3.6)$$

From here we can see that the curvature density will tend to zero as $a \rightarrow 0$. As always, us physicists don't like when something in nature is zero for no particularly good reason. This is the *flatness problem*.

Flatness Problem

The energy density of the Universe is very close to the critical value. This requires that the energy density in the primordial universe *were even closer* to criticality. How were the initial conditions so-close to this special circumstance?

Note: Since the curvature in the history of the Universe is so negligible, we will ignore this term in the remainder of these notes.

4. HORIZON PROBLEM

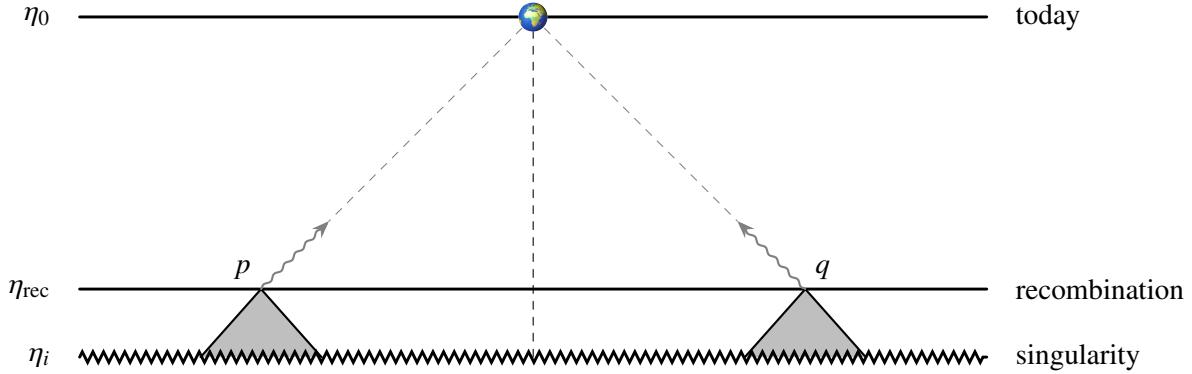


Figure 5. Reproduction of Fig. 4.2 of Ref. [1]

Let's ignore dark energy and curvature such that the Hubble parameter can simply be expressed as

$$H^2 = H_0^2 [\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4}], \quad \Omega_{r,0} = a_{\text{eq}} \Omega_{m,0} \quad (4.1)$$

We can then neatly write,

$$a^2 H^2 = a^{-2} H_0^2 [\Omega_{m,0} a + \Omega_{r,0}] \implies (aH)^{-1} = \frac{1}{\sqrt{\Omega_{m,0}}} H_0^{-1} \frac{a}{\sqrt{a + a_{\text{eq}}}}. \quad (4.2)$$

The particle horizon can be found analytically,

$$d_h(\eta) = \int_0^a (aH)^{-1} d \ln a = \frac{2}{\sqrt{\Omega_{m,0}}} H_0^{-1} (\sqrt{a + a_{\text{eq}}} - \sqrt{a_{\text{eq}}}). \quad (4.3)$$

Given that $a_{\text{eq}} \sim 3400^{-1}$ and $a_{\text{CMB}} \sim 1100^{-1}$ then,

$$d_h(\eta_0) \simeq \frac{2}{\sqrt{\Omega_{m,0}}} H_0^{-1}, \quad (4.4a)$$

$$d_h(\eta_{\text{CMB}}) \simeq 0.0175 \times d_h(\eta_0). \quad (4.4b)$$

There is, therefore, a large difference between the particle horizon today and at recombination.

Horizon Problem

We know that the CMB is remarkably uniform in temperature. Yet, if the particle horizon today is much larger than it was at recombination, this implies that regions of the Universe which were never in causal contact nevertheless exhibit nearly the same temperature. How is this possible?

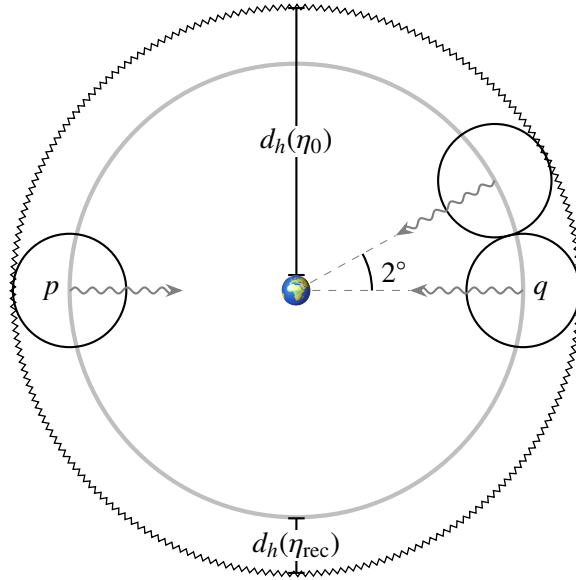


Figure 6. Based on Fig. 4.2 of Ref. [1]

An illustration of the horizon problem can be found in Fig. 5. From here you can see that some photons which we observe today could never have been in causal contact. It is also possible to quantify the angular regions in the sky that one naively expects to be the same temperature. Using Fig. 6 as a guide along with Eq. (4.4) gives

$$\theta_h = 2 \times \frac{d_h(\eta_{\text{rec}})}{d_h(\eta_0) - d_h(\eta_{\text{rec}})} \simeq 0.036 \text{ rad} \simeq 2.0^\circ. \quad (4.5)$$

5. AN INFLATIONARY SOLUTION

5.1. Argument for a decreasing comoving Hubble radius

The primary issue is that the particle horizon at recombination is too small. Let's parameterize the comoving Hubble radius as

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}, \quad (5.1)$$

where, again, $w = P/\rho$ is the equation of state parameter. Then,

$$d_h(a) = \frac{2H_0^{-1}}{(1+3w)} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)} \right]. \quad (5.2)$$

If $w > -1/3$ then $d_h(a)$ is dominated at late times for $a_i \ll 1$. In this particular case, the comoving Hubble radius and the particle horizon are not too dissimilar. If, at early times, the equation of state parameter is $w < -1/3$ then

$$\eta_i \equiv \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \xrightarrow{a_i \rightarrow 0} -\infty, \quad (w < -1/3). \quad (5.3)$$

From here we can conclude that if there was a period that the Hubble radius was *decreasing with time* then we will obtain a large, early-time contribution to $d_h(\eta)$. This “pushes” the singularity back so that there is sufficient time for particles to interact – see Fig. 7.

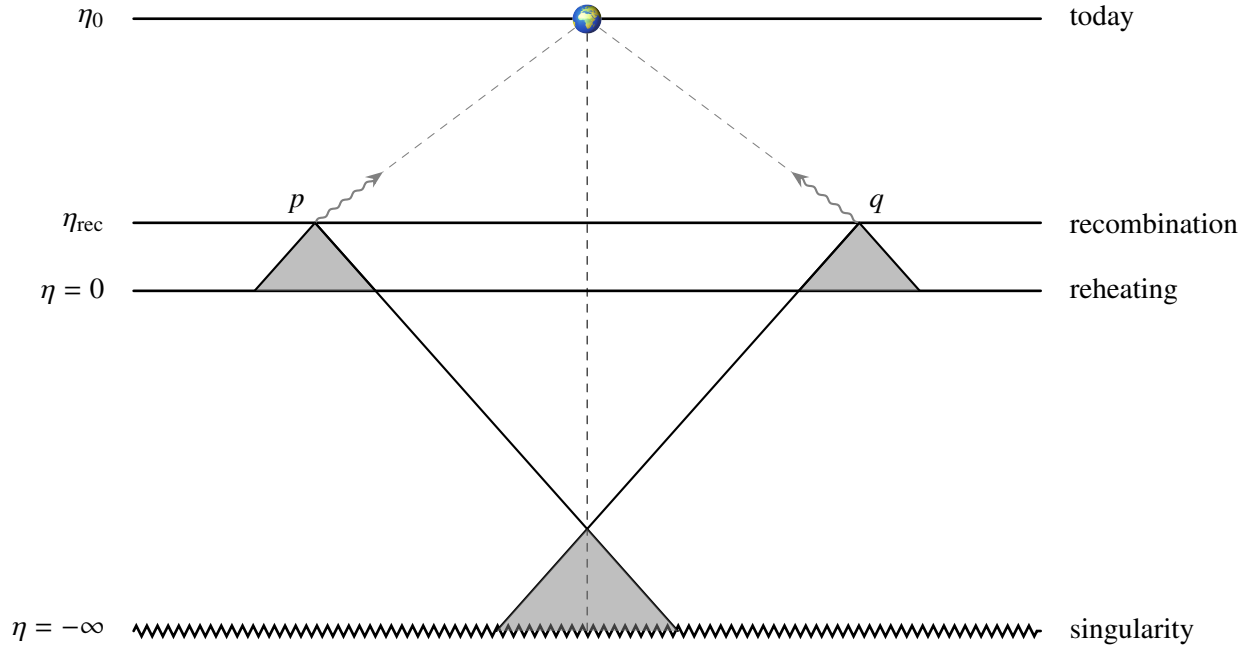


Figure 7. Reproduction of Fig. 4.4 of Ref. [1]

5.2. How much inflation do we need?

It is natural to ask how much inflation is needed to solve the problems we discussed above. For simplicity, we will require that,

$$(a_0 H_0)^{-1} < (a_i H_i)^{-1} \quad (5.4)$$

where the subscript i signifies the beginning of inflation. Figure 8 shows a rough history of the Universe under the inflationary paradigm. From here we see that the above requirement is identical to saying that models which are entering the horizon today must have been within the Hubble radius at the time of inflation.

We will define the total number of e -folds as

$$N_{\text{tot}} \equiv \ln(a_e/a_i) \quad (5.5)$$

where a_e is the scale factor at the end of inflation. The end of the inflationary period leaves the Universe nearly devoid of particles. In order to repopulate the thermal bath an era of *reheating* must occur. This will repopulate the thermal plasma and set the *reheating temperature* T_{RH} . Therefore, the hot Big Bang starts in earnest when $a = a_{\text{RH}}$.

The ratio of comoving Hubble radii from $t = t_{\text{RH}}$ to now is

$$\frac{a_0 H_0}{a_{\text{RH}} H_{\text{RH}}} = \frac{a_0}{a_{\text{RH}}} \left(\frac{a_{\text{RH}}}{a_0} \right)^2 = \frac{a_{\text{RH}}}{a_0} \sim \frac{T_0}{T_{\text{RH}}} \sim 10^{-28} \left(\frac{10^{15} \text{ GeV}}{T_{\text{RH}}} \right). \quad (5.6)$$

In the above estimation we assumed that the Universe is radiation dominated throughout its entire history – which is obviously wrong. Regardless, we will be taking a logarithm so this will be largely unimportant.

It follows then that,

$$(a_i H_i)^{-1} > (a_0 H_0)^{-1} \sim 10^{28} \left(\frac{T_{\text{RH}}}{10^{15} \text{ GeV}} \right) (a_e H_e)^{-1} \quad (5.7)$$

where we have assumed instantaneous reheating i.e., $(a_e H_e)^{-1} \sim (a_{\text{RH}} H_{\text{RH}})^{-1}$.

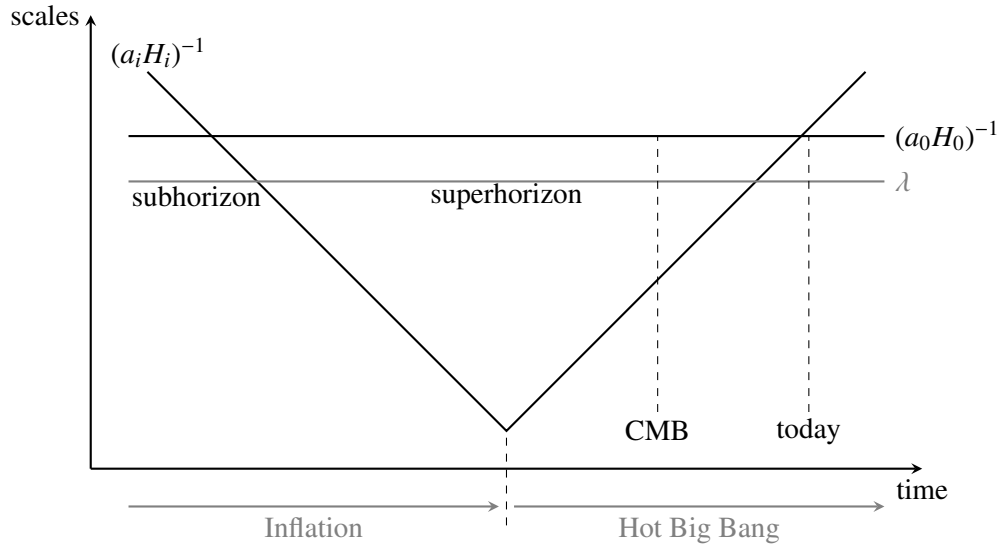


Figure 8. Reproduction of Fig. 4.7 of Ref. [1]

From here we obtain,

$$N_{\text{tot}} \equiv \ln(a_e/a_i) \gtrsim 64 + \ln(T_{\text{RH}}/10^{15} \text{ GeV}). \quad (5.8)$$

Part II

Lecture 2: Single field inflation and beyond

6. SINGLE FIELD INFLATION

6.1. Slow-roll parameters

To begin we will define parameters which will describe the evolution of the Hubble parameter throughout the evolutionary process.

Looking at the time evolution of the comoving Hubble rate,

$$\frac{d}{dt}(aH)^{-1} = \frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1 - \varepsilon), \quad (6.1)$$

where we have defined the *slow-roll parameter*

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d \ln H}{dN}. \quad (6.2)$$

Here, $N \equiv \ln a$ which is an incredibly useful measure of time, as we will see.

For a shrinking Hubble radius $\varepsilon < 1$. In the limit that $\varepsilon \rightarrow 0$, then $H = \text{const.}$ and the space-time is *de Sitter space* with the scale factor evolving as

$$a \propto e^{Ht}. \quad (6.3)$$

In addition to ε , we can define

$$\kappa \equiv \frac{d \ln \varepsilon}{dN} = \frac{\dot{\varepsilon}}{H\varepsilon} \quad (6.4)$$

which describes the evolution of ε itself. So long as $|\kappa| < 1$, then the change of ε per e -fold is small and inflation continues.

6.2. Scalar field dynamics

In its simplest form, inflation can be implemented by examining the dynamics of a scalar field, ϕ , with a potential $V(\phi)$ with a particular set of characteristics. Associated with this scalar field, called the *inflaton*, are an energy density ρ_ϕ and pressure, P_ϕ which will produce the correct equation of state parameter to drive a period of inflation.

Recall that the action for a scalar field in flat space-time is

$$S = \int d^4x \left[\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right], \quad (6.5a)$$

$$= \int dt d^3x \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right], \quad (6.5b)$$

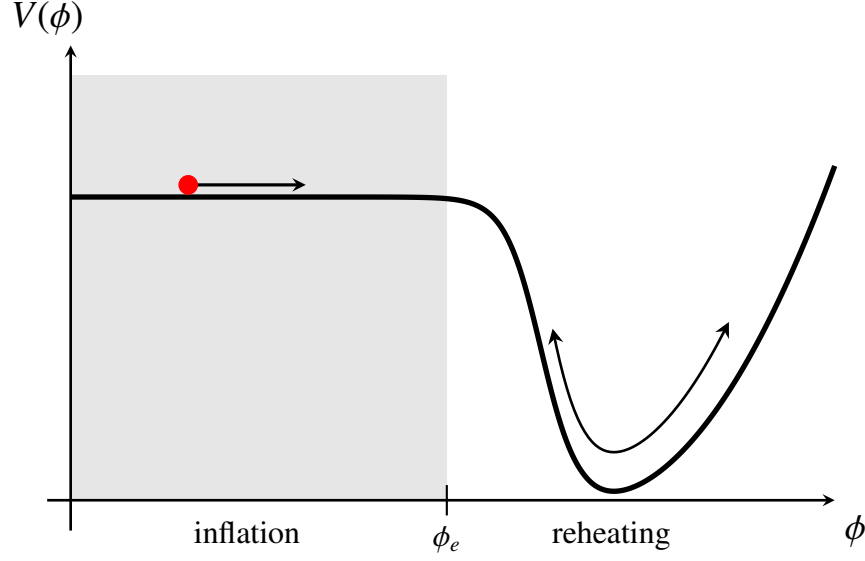


Figure 9. Reproduction of Fig. 4.8 of Ref. [1]

which, via the Euler-Lagrange equation, gives the Klein-Gordon equation

$$\ddot{\phi} - \nabla^2 \phi = -\frac{dV}{d\phi}. \quad (6.6)$$

To derive the same result in the FLRW metric, one can simply take $dx \rightarrow a(t)dx$ and recompute the variation of the action with respect to $\delta\phi$. This gives,

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \quad (6.7)$$

where we have ignored spatial variation, i.e. $\phi = \phi(t)$. The above equation is identical to that which arises in elementary mechanics describing the motion of a particle with potential V . Note that here, however, we have a friction term which is proportional to the Hubble factor.

As for the energy density, we can imagine that it is simply the sum of kinetic plus potential energy,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (6.8)$$

One can deduce the pressure from the continuity equation,

$$\dot{\rho}_\phi = -3H(\rho_\phi + P_\phi), \quad (6.9)$$

and by taking a time derivative of Eq. (6.8) and using the Klein-Gordon equation. This leads to,

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (6.10)$$

such the equation of state parameter w_ϕ is given by

$$w_\phi = P_\phi/\rho_\phi = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \quad (6.11)$$

When the kinetic energy is small, then

$$w_\phi \approx -1, \quad \dot{\phi} \ll 1 \quad (6.12)$$

or, in other words, the field acts like a cosmological constant when the potential energy density dominates. This sources a period of exponential expansion as desired.

6.3. Slow-roll inflation

Let's now connect the scalar field model to the slow roll parameters we discussed before. Let's consider the case where the scalar field dominates the energy density of the Universe such that,

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right], \quad (6.13)$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}. \quad (6.14)$$

Given a potential, one can solve the above equation (numerically, of course) to obtain the “background evolution” during the inflationary period. We will show some solutions in due time. The above two equations can be combined to give,

$$\dot{H} = -\frac{1}{2} \frac{\dot{\phi}^2}{M_{\text{Pl}}^2}, \quad (6.15)$$

such that,

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2 H^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V}. \quad (6.16)$$

From here we again see the importance of the condition $\dot{\phi} \ll 1$. We can also derive the second slow roll parameter,

$$\kappa = \frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta), \quad (6.17)$$

where,

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \quad (6.18)$$

Therefore, we want both $\{\dot{\phi}, \ddot{\phi}\} \ll 1$.

Slow-roll approximation

In order to solve the background equation of motion, we need to determine the initial conditions. To do so, we will approximate

$$H^2 \approx \frac{V}{3M_{\text{Pl}}^2} \quad (6.19)$$

as well as the Klein-Gordon equation,

$$3H\dot{\phi} \approx -\frac{dV}{d\phi}. \quad (6.20)$$

Under these approximations,

$$\varepsilon \approx \varepsilon_V \equiv \frac{M_{\text{Pl}}^2}{2} \left(\frac{1}{V} \frac{dV}{d\phi} \right)^2. \quad (6.21)$$

We will also define a second slow-roll parameter

$$\eta_V \equiv M_{\text{Pl}}^2 \frac{V_{,\phi\phi}}{V} \quad (6.22)$$

With this definition, note that the total number of e -folds which take place during a period inflation starting when $a = a_i$ and ending when $a = a_e$ is,

$$N_{\text{tot}} = \int_{a_i}^{a_e} d \ln a = \int_{t_i}^{t_e} H(t) dt = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi. \quad (6.23)$$

The starting and ending times are defined when $\varepsilon(t_i) = \varepsilon(t_e) = 1$. The above expression can be approximated in the slow-roll approximation as

$$N_{\text{tot}} \approx \int_{\phi_i}^{\phi_e} \frac{1}{\sqrt{2\varepsilon_V}} \frac{|d\phi|}{M_{\text{Pl}}}. \quad (6.24)$$

Using Eq. (6.21), we can estimate the ϕ_e which gives $\varepsilon_V = 1$. Given this value, we can then obtain ϕ_i by computing Eq. (6.24), having fixed $N_{\text{tot}} \simeq 60$.

Concrete example: quadratic potential

Let's take the potential to be

$$V(\phi) = \frac{1}{2} m_\phi^2 \phi^2. \quad (6.25)$$

This model is no longer consistent with observation, but we can use it to explore the ideas we just developed. In particular, note that

$$\varepsilon_V(\phi) = 2 \left(\frac{M_{\text{Pl}}}{\phi} \right)^2. \quad (6.26)$$

Therefore the slow-roll condition is satisfied when

$$\phi > \sqrt{2} M_{\text{Pl}} \equiv \phi_e. \quad (6.27)$$

What remains is to find ϕ_i . In particular,

$$N_{\text{tot}} = \int_{\phi_e}^{\phi_i} \frac{d\phi}{M_{\text{Pl}}} \frac{1}{\sqrt{2\varepsilon_V}} = \frac{\phi^2}{4M_{\text{Pl}}^2} \Big|_{\phi_e}^{\phi_i} = \frac{\phi_i^2}{4M_{\text{Pl}}^2} - \frac{1}{2}. \quad (6.28)$$

For $N_{\text{tot}} > 60$ we require that

$$\phi_i > 2\sqrt{60} M_{\text{Pl}} \approx 15 M_{\text{Pl}}. \quad (6.29)$$

Note however that the quadratic model is largely ruled due to measurements of the scalar-to-tensor ratio r . This is something we will return to later.

7. QUANTUM FLUCTUATIONS

Cosmological inflation is perhaps the greatest example of the implications of quantum mechanics on macroscopic scales. The very structure of the Universe – the formation of planets, stars and galaxies are a consequence of the inhomogeneities which were established by quantum mechanical fluctuations of the inflaton. In this sense, inflation sets the initial conditions for which structure can build off of.

To describe how cosmological inflation sets the initial conditions with any kind of detail, we would have to use *cosmological perturbation theory*. This topic is a few seminars itself, and so we will only be able to get a general idea of the required calculations.

7.1. Lightning round: cosmological perturbation theory

The idea of cosmological perturbation theory is to perturb the FLRW metric and the energy-momentum tensor and to explore the dynamics of the perturbations. In particular,

$$g_{\mu\nu}(t, \mathbf{x}) \rightarrow g_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}), \quad (7.1a)$$

$$\rho(t, \mathbf{x}) \rightarrow \bar{\rho}(t) + \delta\rho(t, \mathbf{x}). \quad (7.1b)$$

Note that the cosmological principle is not strictly obeyed here and there *is* a spatial dependence in the perturbations. In making the above expansion, we are assuming that the perturbations are small.

For the metric, $g_{\mu\nu}$, we will write the perturbations as follows

$$ds^2 = a^2(\eta) \left[-[1 + 2\Psi(t, \mathbf{x})] d\eta^2 + [1 - 2\Phi(t, \mathbf{x})] \delta_{ij} dx^i dx^j \right]. \quad (7.2)$$

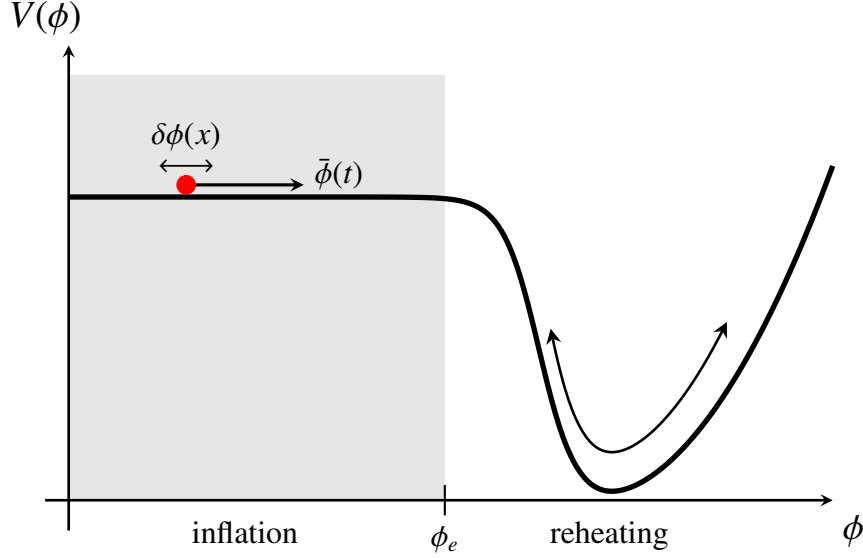


Figure 10. Reproduction of Fig. 8.1 of Ref. [1]

This is choice called the conformal Newtonian gauge. The sudden appearance of the word “gauge” is related to the fact that form of the metric perturbations depend on your choice of coordinates. All of the physical observables will, naturally, be independent of this.

The conformal Newtonian gauge is convenient because Ψ ($= \Phi$ for our needs) is more-or-less the same as gravitational potential in Newtonian limit. Furthermore, the metric is diagonal which makes some calculations simpler.

Initial conditions

Without proving, the initial conditions for superhorizon modes are²

$$\delta_\gamma = \delta_v = \frac{4}{3}\delta_c = \frac{4}{3}\delta_b = -2\Phi_i, \quad (7.3)$$

where $\delta_x = \delta\rho_x/\bar{\rho}_x$. In other words, for length scales larger than the Hubble radius, the initial perturbations is just determined by the gravitational potential.

For various reasons (like gauge independence), the gravitational potential Φ is not the best metric for establishing the initial conditions. What is instead used is the *comoving curvature perturbation* $\mathcal{R}$. This quantity is gauge independent but also related to Φ in the superhorizon limit via,

$$\Phi_i(\mathbf{k}) = \frac{2}{3}\mathcal{R}_i(\mathbf{k}) \quad (\text{superhorizon}). \quad (7.4)$$

The curvature perturbation is called as such because it is equivalent to intrinsic curvature certain gauges.

² This is only true if the perturbations are *adiabatic*, which is not something we will discuss in too much detail.

7.2. The Mukhanov-Sasaki equation

The key equation of this section is the **Mukhanov-Sasaki equation**

$$f_k''(\eta) + \left(k^2 - \frac{z''}{z}\right)f_k(\eta) = 0, \quad f = a\delta\phi, \quad z \equiv \frac{a\bar{\phi}}{\mathcal{H}} \quad (7.5)$$

The above equation is the same as a simple harmonic oscillator, however with a time-dependent frequency. The comoving curvature perturbation is related to perturbations of the inflaton via,

$$\mathcal{R} = \frac{\mathcal{H}}{\bar{\phi}'}\delta\phi = \frac{f}{z} \quad (7.6)$$

where $\mathcal{H} = aH = d \ln a / d\eta$. Note there that $\mathcal{R} \sim H\delta t$. This confirms our intuition that the curvature perturbations are a result of time delays in the end of inflation.

In order to understand quantum fluctuations of the inflaton, we first must examine the limiting solutions of Eq. (7.5).

Subhorizon solution

During the slow-roll phase, H and $\bar{\phi}'$ are approximately constant. This implies that,

$$\frac{z''}{z} \approx \frac{a''}{a} \approx 2\mathcal{H}^2. \quad (7.7)$$

At early times $\mathcal{H}^{-1} = (aH)^{-1}$ is large and therefore, we can ignore z''/z in (7.5). This leaves the standard harmonic oscillator equation,

$$f_k''(\eta) + k^2 f_k(\eta) = 0, \quad (\text{subhorizon}). \quad (7.8)$$

The solutions are $f_k(\eta) \propto e^{ik\eta}$, i.e., that which we expect from a normal harmonic oscillator.

Superhorizon solution

As the comoving horizon shrinks, modes will exit the horizon and Eq. (7.5) instead becomes,

$$f_k''(\eta) - \frac{z''}{z} f_k(\eta) = 0. \quad (7.9)$$

This has a growing mode $f \propto z$ and decaying mode $f \propto z^{-2}$ in the slow-roll regime – i.e., $z \propto a$.

7.3. Review of the quantum harmonic oscillator

An elementary problem in non-relativistic quantum mechanics is the simple harmonic oscillator. Starting with the Hamiltonian,

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \quad (7.10)$$

where $p \equiv \dot{q}$ is the conjugate momentum and we set $m = 1$ by rescaling q and p . The quantization process involves promoting the classical variables to operators $\{q, p\} \rightarrow \{\hat{q}, \hat{p}\}$, then imposing

$$[\hat{q}, \hat{p}] \equiv \hat{q} \cdot \hat{p} - \hat{p} \cdot \hat{q} = i\hbar. \quad (7.11)$$

The traditional trick is to then introduce new variables

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger), \quad (7.12)$$

$$\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger). \quad (7.13)$$

Under the commutation relation Eq. (7.11), we find

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (7.14)$$

We in turn find that,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 \quad (7.15)$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \quad (7.16)$$

The ground state is defined as

$$\hat{a}|0\rangle = 0 \quad (7.17)$$

and has energy,

$$E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\hbar\omega. \quad (7.18)$$

The states,

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad (7.19)$$

are also eigenstates of the Hamiltonian with eigenvalue,

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \quad (7.20)$$

The last point we wish to make is about the expectation value of $\hat{q}$ and $\hat{q}^2$. It is a simple exercise to show that,

$$\langle \hat{q} \rangle = 0, \quad (7.21a)$$

$$\langle \hat{q}^2 \rangle = \frac{\hbar}{2\omega}. \quad (7.21b)$$

The Heisenberg Picture

To proceed further, we will use the Heisenberg picture – i.e., the description where the operators (rather than the states) are time dependent. In particular, recall that

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{O}] \quad (7.22)$$

where $\hat{H}$ is the Hamiltonian.

We can use the above expression to derive the differential equations,

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a} \quad (7.23a)$$

$$\frac{d\hat{a}^\dagger}{dt} = +i\omega\hat{a}^\dagger. \quad (7.23b)$$

For $\omega = \text{const.}$, the solution to these equations are

$$\hat{a}(t) = \hat{a}(t_i)e^{-i\omega t} \quad (7.24a)$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(t_i)e^{+i\omega t}. \quad (7.24b)$$

The vacuum state $|0\rangle$ is defined in terms of the initial time creation and annihilation operators, i.e., $\hat{a}(t_i)|0\rangle = 0$. It follows then that,

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}} [\hat{a}(t_i)e^{-i\omega t} + \hat{a}^\dagger(t_i)e^{+i\omega t}]. \quad (7.25)$$

We can define the function,

$$q(t) = \sqrt{\frac{\hbar}{2\omega}} e^{-i\omega t}, \quad \omega = \text{const.} \quad (7.26)$$

Then, it is simple so show that

$$\langle |\hat{q}(t)|^2 \rangle = |q(t)|^2. \quad (7.27)$$

7.4. Inflationary quantum fluctuations

We now want to examine the fluctuations of ϕ , $\delta\phi$. For simplicity, we will use the slow roll approximation:

$$H = \text{const.} \quad (7.28a)$$

$$a = -(H\eta)^{-1}. \quad (7.28b)$$

The Mukhanov-Sasaki equation becomes,

$$f_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\eta^2}\right)f_{\mathbf{k}} = 0. \quad (7.29)$$

Canonical quantization

The first step is to promote the field $f(\eta, \mathbf{x})$ and its conjugate momentum $\pi(\eta, \mathbf{x})$ to operators.

$$\{f(\eta, \mathbf{x}), \pi(\eta, \mathbf{x})\} \rightarrow \{\hat{f}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x})\}$$

We then impose a commutation relation,

$$[\hat{f}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta_D(\mathbf{x} - \mathbf{x}'). \quad (7.30)$$

Going to Fourier space,

$$[\hat{f}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = \int d^3x \int d^3x' [\hat{f}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \quad (7.31)$$

$$= i \int d^3x e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \quad (7.32)$$

$$= i(2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}'). \quad (7.33)$$

We will, as before, write

$$\hat{f}_{\mathbf{k}}(\eta) = f_k(\eta)\hat{a}_{\mathbf{k}} + f_k^*(\eta)\hat{a}_{-\mathbf{k}}^\dagger. \quad (7.34)$$

Note that the operator above is Hermitian, which is why we have the particular structure above. In particular, $\hat{f}_{\mathbf{k}}(\eta)^\dagger = \hat{f}_{-\mathbf{k}}(\eta)$ which can be demonstrated from the Fourier transformation.

We will normalize

$$f_k f_k'^* - f_k' f_k^* \equiv i. \quad (7.35)$$

Then, we can deduce from Eqs. (7.30) and (7.31)

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}'). \quad (7.36)$$

We want to define the vacuum state,

$$\hat{a}_{\mathbf{k}}(\eta_i) |0\rangle = 0 \quad (7.37)$$

where $|k\eta_i| \gg 1$. In this limit, $\omega_k^2 = k^2 - z''/z \rightarrow k^2$. We can then take as an initial condition,

$$\lim_{k\eta \rightarrow \infty} f_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad (7.38)$$

which is the same solution as we derived for the simple harmonic oscillator. The solution to Mukhanov-Sasaki equation, under the slow-roll approximation

$$f_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta}. \quad (7.39)$$

This is the **Bunch-Davies mode function** while the corresponding vacuum is the **Bunch-Davies vacuum**.

It is easy to see that

$$\langle \hat{f} \rangle \equiv \langle 0 | \hat{f} | 0 \rangle = 0$$

The variance is,

$$\langle |\hat{f}|^2 \rangle \equiv \langle 0 | \hat{f}(\eta, \mathbf{0}) \hat{f}(\eta, \mathbf{0}) | 0 \rangle \quad (7.40a)$$

$$= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \langle 0 | (f_k^*(\eta) \hat{a}_{-\mathbf{k}}^\dagger + f_k(\eta) \hat{a}_{\mathbf{k}}^\dagger) (f_{k'}(\eta) \hat{a}_{\mathbf{k}'} + f_{k'}^*(\eta) \hat{a}_{-\mathbf{k}'}^\dagger) | 0 \rangle \quad (7.40b)$$

$$= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} f_k(\eta) f_{k'}^*(\eta) \langle 0 | [\hat{a}_{\mathbf{k}}, \hat{a}_{-\mathbf{k}'}^\dagger] | 0 \rangle \quad (7.40c)$$

$$= \int \frac{d^3 k}{(2\pi)^3} |f_k(\eta)|^2 \quad (7.40d)$$

$$= \int d \ln k \frac{k^3}{2\pi^2} |f_k(\eta)|^2. \quad (7.40e)$$

We can define the dimensionless power spectrum,

$$\Delta_f^2(k, \eta) \equiv \frac{k^3}{2\pi^2} |f_k(\eta)|^2. \quad (7.41)$$

Using fact that $f = a\delta\phi$, then

$$\Delta_{\delta\phi}^2(k, \eta) = \frac{\Delta_f^2(k, \eta)}{a^2(\eta)} \quad (7.42)$$

Using the Bunch-Davies mode function, we have

$$\Delta_{\delta\phi}^2(k, \eta) = \left(\frac{H}{2\pi} \right)^2 \left[1 + (k\eta)^2 \right] \xrightarrow{k\eta \rightarrow 0} \left(\frac{H}{2\pi} \right)^2. \quad (7.43)$$

Above we took the superhorizon limit $k\eta \rightarrow 0$. From here we see that the power spectrum becomes scaleless – exactly what we wanted!

Now, there is a limit to the slow-roll approximation. The Hubble parameter is not completely constant, which leads to a small scale dependence.

We accommodate the time dependence by writing,

$$\Delta_{\delta\phi}^2(k, \eta) \approx \left(\frac{H(t)}{2\pi} \right)^2 \bigg|_{k=aH(t)}. \quad (7.44)$$

Since $H(t)$ decreases during inflation (?), this implies that the amplitude will be slightly larger for long-wavelength fluctuations since these exited the horizon first.

Curvature Perturbations

We can use the relation between the of fluctuations to the comoving curvature perturbation, Eq. (7.6), to obtain

$$\Delta_{\mathcal{R}}^2(k) = \left(\frac{H}{\dot{\phi}} \right)^2 \Delta_{\delta\phi}^2 \quad (7.45)$$

from which we can obtain,

$$\Delta_{\mathcal{R}}^2(k) = \left(\frac{H^2}{2\pi\dot{\phi}} \right)^2 \bigg|_{k=aH} = \frac{1}{8\pi^2\epsilon} \frac{H^2}{M_{\text{Pl}}^2} \bigg|_{k=aH} \quad (7.46)$$

We define the **scalar spectral index** as

$$n_s - 1 \equiv \frac{d \ln \Delta_{\mathcal{R}}^2(k)}{d \ln k}. \quad (7.47)$$

Taking advantage of the fact that $k = aH$, then

$$n_s - 1 = \frac{d \ln \Delta_{\mathcal{R}}^2(k)}{d \ln(aH)} \quad (7.48)$$

$$\approx \frac{d \ln \Delta_{\mathcal{R}}^2(k)}{d \ln a} = \frac{2}{H} \frac{d \ln H}{dt} - \frac{1}{H} \frac{d \ln \epsilon}{dt} = 2 \frac{\dot{H}}{H^2} - \frac{\dot{\epsilon}}{H\epsilon}. \quad (7.49)$$

This allows us to conclude,

$$n_s - 1 = -2\epsilon - \kappa. \quad (7.50)$$

Finally, we can bring this all together to obtain

$$\Delta_{\mathcal{R}}^2(k) = A_s \left(\frac{k}{k_\star} \right)^{n_s-1}, \quad (7.51)$$

where the amplitude and spectral index are given by

$$A_s \equiv \frac{1}{8\pi^2} \frac{1}{\varepsilon_\star} \frac{H_\star^2}{M_{\text{Pl}}^2}, \quad (7.52a)$$

$$n_s \equiv 1 - 2\varepsilon_\star - \kappa_\star. \quad (7.52b)$$

Here, everything is evaluated at the time when a given mode $k_\star$ exited the horizon. In particular, for $k_\star = 0.05 \text{ Mpc}^{-1}$,

$$A_s = (2.098 \pm 0.023) \times 10^{-9} \quad (7.53a)$$

$$n_s = 0.965 \pm 0.004. \quad (7.53b)$$

7.5. Tensor perturbations

Up-to this point, we have only focused on scalar perturbations of the metric. There is another class of perturbations, i.e., *tensor perturbations*, which also impact the CMB. We will not dive into the details here at all other than to say that one also has an associated power spectrum for tensor modes,

$$\Delta_h^2(k) = A_t \left(\frac{k}{k_\star} \right)^{n_t}, \quad (7.54)$$

where

$$A_t = \frac{2}{\pi^2} \frac{H_\star^2}{M_{\text{Pl}}^2} \quad (7.55a)$$

$$n_t = -2\varepsilon_\star \quad (7.55b)$$

The tensor perturbations are directly related to the emission of gravitational waves. Measurement of the stochastic gravitational background will be crucial in developing our understanding of inflationary scenarios and more!

Tensor-to-scalar ratio

We mentioned the tensor perturbations to highlight the other key measure of inflationary models – the *tensor-to-scalar ratio*

$$r \equiv \frac{A_t}{A_s} = 16\varepsilon_\star \quad (7.56)$$

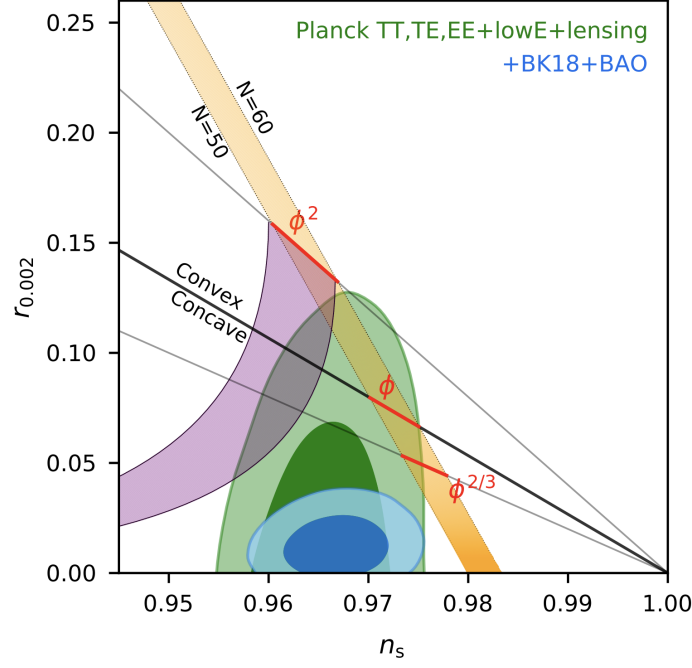


Figure 11. Tensor to Scalar ratio r for $k_\star = 0.002 \text{ Mpc}^{-1}$ as determined from CMB measurements of Planck and BICEP/Keck [9]

7.6. Slow-roll predictions

We can recast of expressions for $\{A_s, n_s\}$ in terms of the slow-roll parameters ϵ_V and η_V ,

$$A_s = \frac{1}{24\pi^2} \frac{1}{\epsilon_{V,\star}} \frac{V_\star}{M_{\text{Pl}}^4}, \quad A_t = \frac{2}{3\pi^2} \frac{V_\star}{M_{\text{Pl}}^4} \quad (7.57a)$$

$$n_s - 1 = -6\epsilon_{V,\star} + 2\eta_{V,\star}, \quad n_t = 2\epsilon_V. \quad (7.57b)$$

Concrete example: quadratic potential

For the case of $V(\phi) = m^2\phi^2/2$ we found,

$$\epsilon_V(\phi) = \eta_V(\phi) = \frac{2M_{\text{Pl}}^2}{\phi^2}$$

The number of e -folds was

$$N_{\text{tot}} \approx \frac{\phi^2}{4M_{\text{Pl}}^2}. \quad (7.58)$$

It follows that,

$$\epsilon_{V,\star} = \eta_{V,\star} \approx \frac{1}{2N_\star} \approx 0.01, \quad N_\star \approx 50. \quad (7.59)$$

We then find,

$$n_s = 1 - \frac{2}{N_\star} \approx 0.96 \quad (7.60a)$$

$$r = \frac{8}{N_\star} \approx 0.16. \quad (7.60b)$$

The tensor-to-scalar ratio, is measured to be below $r < 0.035$ – see Fig. 11.

8. ADVANCE EXAMPLE: α -ATTRACTOR T-MODEL

There are a broad class of models, denoted α -attractor models, which share simple and common solutions for n_s and r [10]

$$n_s \approx 1 - \frac{2}{N_{\text{tot}}}, \quad r \approx \alpha \cdot \frac{12}{N_{\text{tot}}^2}, \quad (8.1a)$$

where the potential is of the form

$$V(\phi) = V_0 \left[1 - \exp\left(-\sqrt{\frac{2}{3\alpha}}\phi\right) + \dots \right]. \quad (8.2)$$

Models of this type can be embedded in supergravity theories, so are not completely unfounded. One particular implementation is the T -model

$$V(\phi) = \lambda M_{\text{Pl}}^4 \left| \sqrt{6} \tanh\left(\frac{\phi}{\sqrt{6}M_{\text{Pl}}}\right) \right|^\kappa. \quad (8.3)$$

This potential is interesting because of the power κ which can lead to non-standard equation of state parameters during reheating.

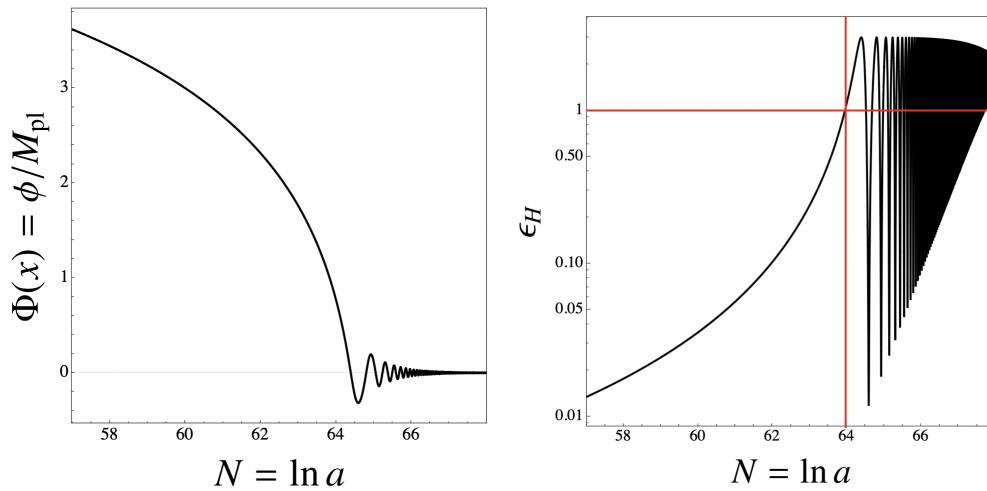


Figure 12. Solution for ϕ and the slow-roll parameter $\varepsilon = \epsilon_H$.

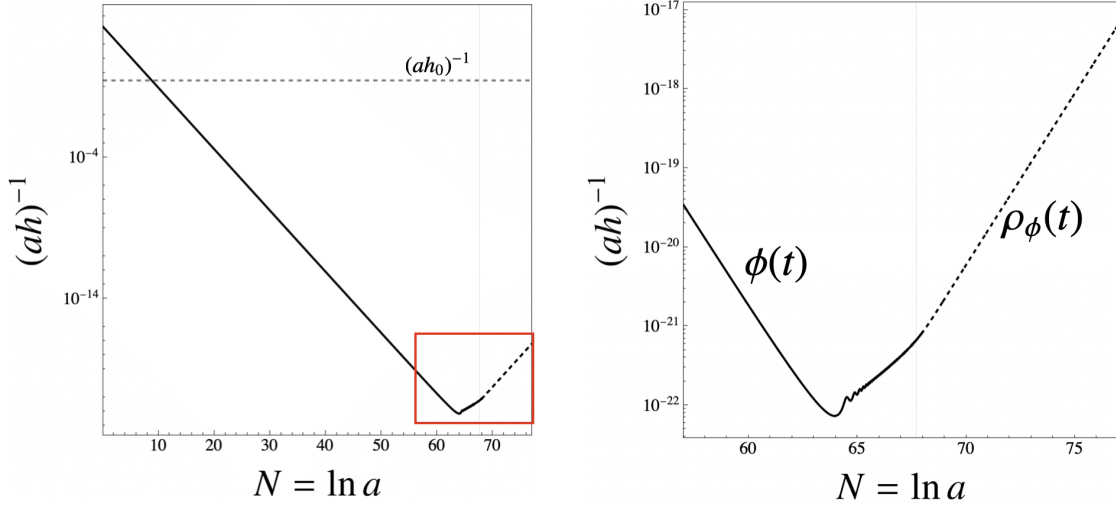


Figure 13. Solution for comoving Hubble radius for the α -attractor T -model with $\kappa = 2$. The right plot is a zoom in of the left plot. There we can more clearly see the oscillations. The dashed line corresponds to transitioning to a solution for ρ_ϕ .

Complicated potentials require numerically solving the Klein-Gordon equation and Friedmann equations. Below, I will present numerical solutions for $\kappa = 2$ and $h = H/M_{\text{Pl}}$.

With this set-up we determined $n_s \sim 0.965$ which is close to the approximate $n_s \approx 1 - 2/N_{\text{tot}} \sim 0.968$. As for the amplitude, this can be tuned by picking the overall energy scale of the potential, λ , accordingly.

9. APPLICATIONS OF CANONICAL INFLATIONARY THEORY

Here we will discuss applications of cosmological inflation which extend beyond Λ CDM. This is far from a comprehensive survey of topics, but hopefully it offers some inspiration for investigating the full potential (pun intended) of inflationary theories.

9.1. Primordial Black Holes

General principles

Primordial black holes (PBHs) are black holes which formed in the earliest moments of the history of the Universe – usually before $t \lesssim 1$ second. Consequently, any black holes forming in this era must arise from mechanisms of collapse that differ entirely from the standard stellar-collapse process responsible for astrophysical black holes.

Fortunately, inflation offers nearly everything we need. To understand how a black hole could possibly form, note that the Friedmann equation can be written suggestively,

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \rho_{\text{tot}} \xrightarrow{\rho_{\text{tot}} \rightarrow M/(4\pi H^{-3}/3)} H^{-1} = 2GM. \quad (9.1)$$

You might recognize the right-hand-side as the Schwarzschild radius. The above equation tells us, in some sense, that the Universe is already kind-of like a black hole

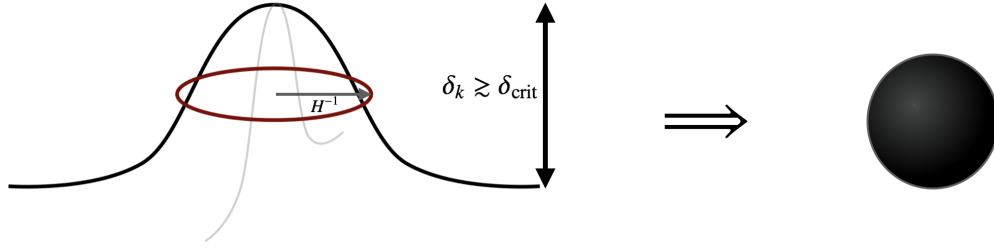


Figure 14. Visualization of PBH formation for Hubble radius-sized overdensities.

The Universe is, of course, not collapsing but expanding – this is evident by the Friedmann equation ($\kappa = 0$). One way to initiate collapse is to induce local curvature perturbations. As we have learned, this is exactly what inflation does. Once there is some effective curvature present, then $\kappa \neq 0$ locally and collapse can occur.

PBH formation requires that a Hubble-radius density perturbation, $\delta = \delta\rho/\rho$, exceed a critical threshold:

$$\delta_k \gtrsim \delta_{\text{crit}} \simeq 0.45. \quad (9.2)$$

Here k is the scale of the perturbation which enters the horizon, i.e, $k = aH$. The mass of the PBH is given by

$$M_{\text{PBH}} = \gamma \times \frac{4\pi}{3} \rho_{\text{tot}} H^{-3} \simeq 2 \times 10^5 \gamma \left(\frac{t}{1 \text{ sec}} \right) M_{\odot} \quad (9.3)$$

where γ is an efficiency factor which depends on the details of collapse.

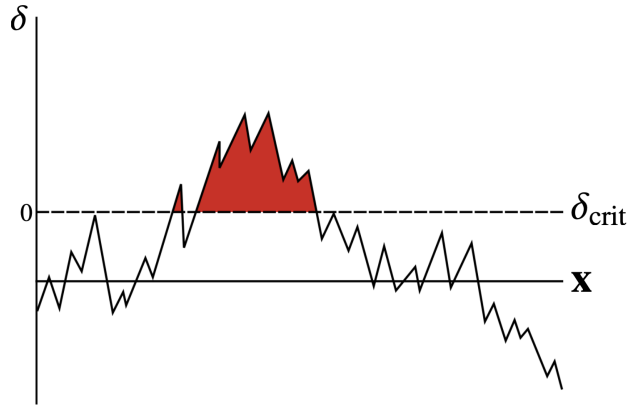


Figure 15. Visualization of overdense regions which lead to the formation of PBHs.

In order to compute the number of PBHs, we use the Press-Schechter formalism. In simple terms, we integrate the probability distribution of overdensities from the critical threshold,

$$\beta_{\text{PBH}} \equiv \frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}} = \int_{\delta_c}^{\infty} P(\delta) d\delta \sim \sigma(M_H) \exp\left(-\frac{\delta_c}{2\sigma^2(M_H)}\right) \quad (9.4)$$

where M_H is the mass of the horizon for a given scale k and, crucially,

$$\sigma^2(M_H) \sim \Delta_{\mathcal{R}}^2(k). \quad (9.5)$$

This means that the comoving curvature power spectrum at a given scale k needs to be of the order $\Delta_{\mathcal{R}}^2(k) \sim \delta_c$ to have non-negligible PBH formation.

PBH-forming potentials

A schematic of the features of the inflaton potential needed for both the CMB observations and PBH is shown in Fig. 16.

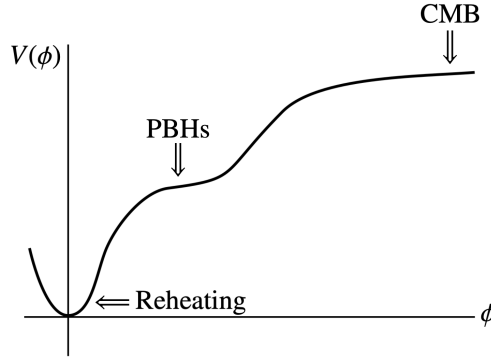


Figure 16. Schematic of the potential needed for PBH formation to occur alongside satisfying the CMB constraints.

We can see that we need a second slow-roll phase. This shouldn't be too surprising – we saw earlier that $A_s \propto 1/\epsilon$. However, now want a second enhancement of order $A_s \sim 10^{-2}$ at scales which correspond to observable PBH masses. There are a wide variety of models which are able to achieve this and many more developed each year. Figure 17, taken from Ref. [11], shows one such implementation.

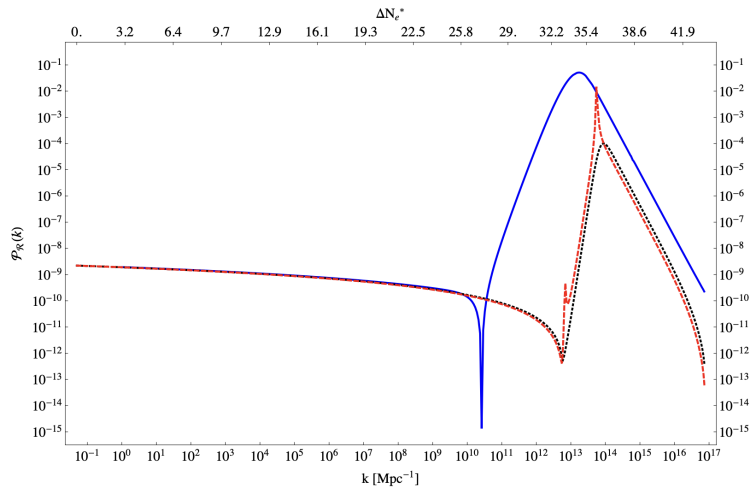


Figure 17. Schematic of the potential needed for PBH formation to occur alongside satisfying the CMB constraints.

Why study primordial black holes?

PBHs have attracted more and more attention ever since gravitational waves were first observed by LIGO. These ancient black holes may address many of the outstanding questions in particle physics and cosmology:

- PBHs could account for all or some of the observed dark matter.
- PBHs may explain a subset of the black hole merger events detected via gravitational waves.
- PBHs may seed or outright be the supermassive black holes we observe at the centers of galaxies.

There are many phenomenological implications of PBHs besides the above list. In addition, non-inflationary mechanisms of PBH formation constitute another rich sub-field and avoid the fine-tuning issues that plague inflationary scenarios of PBH formation. Regardless of their existence, PBHs have led to a better understanding of general relativity and even inspired Stephen Hawking in developing his seminal work on Hawking evaporation.

9.2. Gravitational Particle Production

Inflation can lead to the generation of particles simply by virtue of the fact that expanding the universe by ~ 60 e -folds in a short period of time is a “dramatic” event. As an example of a similar idea, let’s look at the simple square well in quantum mechanics. Let’s assume that the square well has size L such that the solution to Schrödinger’s equation yields,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}. \quad (9.6)$$

The associated energy is

$$E_n = \frac{n^2 \pi^2}{2mL^2}. \quad (9.7)$$

Let’s now imagine that we *rapidly* expand the region so that its size is $2L$. Then the wave function we just found acts as the initial condition – in particular because the wave function wouldn’t have time to smoothly transition to the new physical situation. The new eigenfunctions are,

$$\phi_n(x) = \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{2L}\right) \quad (9.8)$$

with new energies,

$$E'_n = \frac{n^2 \pi^2}{8mL^2}. \quad (9.9)$$

The solution to Schrödinger’s equation with initial condition $\phi(x, t) = \psi_{n'}(x)$ is

$$\phi(x, t) = \sum_n c_{nn'} \phi_n(x) e^{-iE_n t} \quad (9.10)$$

where

$$c_{nn'} = \int_0^{2L} \psi_{n'}(x) \phi_n^*(x) dx = \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{n'\pi}{L}x\right) \sin\left(\frac{n\pi}{2L}x\right) dx = \begin{cases} \frac{4(-1)^m \sqrt{2}n' \sin[(\frac{n}{2}+1)\pi]}{4\pi(n^2-m^2)} & n \neq n', \\ \frac{32}{9\pi^2 n^2} \sin^6\left(\frac{n\pi}{2}\right) & n = n'. \end{cases} \quad (9.11)$$

What conclusion can we draw? If we started in a lower energy state – say $n' = 1$ then the rapid expansion would *excite* higher energy modes.

This is precisely what is happening during inflation. The rapid expansion excites the vacuum, leading to the creation of particles. These particles could act as dark matter or provide some utility in other open problems like baryogenesis.

Why is it called *gravitational* particle production? First, we assume a (scalar) theory of the type

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_{\text{inflaton}} + \mathcal{L}_\Phi - \frac{1}{2} M_{\text{Pl}}^2 R \right] \quad (9.12)$$

The only link between the inflaton Lagrangian and the matter Lagrangian $\mathcal{L}_\Phi$ is their shared coupling to gravity. Schematically, we can view this coupling in two ways.

Particle physics point of view

One intuitive way of understanding particle generation is through the Feynman diagram presented in Fig. 18.

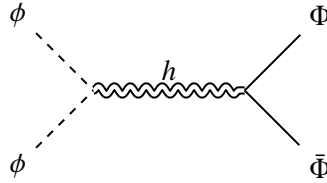


Figure 18. Graviton-mediated inflaton annihilation

There we see that two inflaton particles annihilate into a graviton leading to a particle-antiparticle pair. One can use cross section of this process to calculate the interaction rate and solve the Boltzmann equation to obtain the relic abundance.

9.3. Gravitational point of view

We could instead visualize particle creation in a similar way to Hawking radiation. Recall that the uncertainty principle allows for the spontaneous creation of particles and antiparticles. During inflation such pairs are also being generated. However, the rapid expansion is such that two particles may come into existence only to be quickly torn apart from one another. This leads to net particle number.

The specifics of the exact calculation here are complicated. In general, one must solve the *mode equation*. For scalars, this is

$$\chi_k''(\eta) + \omega_k^2(\eta) \chi_k(\eta) = 0 \quad (9.13)$$

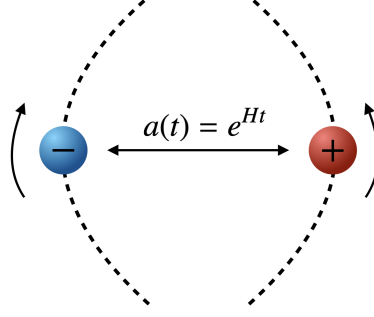


Figure 19. Visualization of particle production as a result of particle-antiparticle pairs driven away due to rapid expansion.

where the frequency depends on the specifics of the theory. Roughly speaking, χ_k can be decomposed into its negative and positive frequency components, α_k and β_k . One can then solve the mode equation, using the background inflationary solution, and extract the number density via

$$a^3 n_\Phi = \int \frac{dk}{k} a^3 n_k, \quad a^3 n_k = \frac{k^3}{2\pi^2} |\beta_k|^2. \quad (9.14)$$

Appendix A: Fundamentals of Cosmology

This appendix will contain the fundamentals required for our investigations of cosmological inflation. This is by no means a comprehensive overview of general relativity and cosmology. However, this section should a sufficient background to understand the concepts discussed in the main text.

1. Differential Geometry and General Relativity

Differential geometry is an enormous topic. Here I will just highlight the key quantities needed for our basic understanding of cosmology.

The metric

For our needs, the fundamental quantity is the **metric** $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (\text{A.1})$$

Here we are adopting the Einstein summation notation such that there is an implicit sum over repeated indices. Given a metric, we can also define the metric connection coefficients – better known as the Christoffel symbols,

$$\Gamma_{\mu\nu}^\lambda \equiv \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}), \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}. \quad (\text{A.2})$$

The Christoffel symbols describe how vectors change as they are transported along a curved manifold. Alongside this, we can define the Ricci tensor,

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{\mu\nu}^\rho - \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\rho}^\lambda \quad (\text{A.3})$$

and the Ricci scalar

$$R \equiv R^\lambda{}_\lambda. \quad (\text{A.4})$$

All of these quantities combine to give Einstein's equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (\text{A.5})$$

which relates curvature to the energy-momentum tensor $T_{\mu\nu}$ through a coupling determined by Newton's Gravitational constant.

2. Basics of cosmology

a. The scale factor

Based on the cosmological principle discussed in the main text, one can prove that the Friedmann–Lemaître–Robertson–Walker metric (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (\text{A.6})$$

is the most general spherically symmetric metric we can write down.

The *scale factor* $a(t)$ characterizes the expansion of space. In particular, Eq. (A.6) is expressed in terms of *comoving coordinates*. For fixed observers (i.e., those not subject to any external force) r, θ, ϕ remain constant.

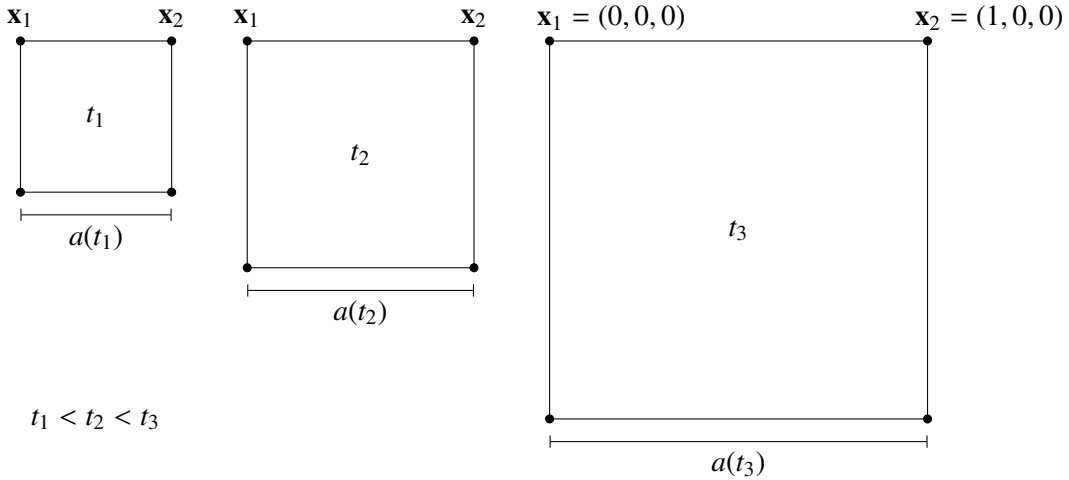


Figure 20. Reproduction of Fig. 1.1 of Ref. [4]

The physical distance, (i.e., the distance we would actually measure) between two objects is given by

$$\mathbf{r}_{\text{phy}} \equiv a(t) \mathbf{r}_{\text{comov}}. \quad (\text{A.7})$$

Two important things about the scale factor. First, we have defined $a(t)$ to be unitless. This has become the usual convention but certain, older, texts often will give $a(t)$ the unit of length instead. Second, it is conventional to normalize $a_{\text{today}} \equiv a_0 \equiv 1$. This choice is convenient in light (pun intended) of the *redshift* z

$$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}}}{a_{\text{emit}}} = \frac{1}{a_{\text{emit}}}. \quad (\text{A.8})$$

A fundamental quantity in cosmology is the *Hubble parameter*

$$H(t) \equiv \frac{1}{a} \frac{da}{dt}. \quad (\text{A.9})$$

The *Hubble constant*, H_0 , is the present-day value of Hubble parameter. It is common to write

$$H_0 = 100 h \text{ km/s/Mpc}, \quad (\text{A.10})$$

$$= 2.13 \times 10^{-41} h \text{ GeV} \quad (\text{A.11})$$

Our next goal is to examine the time evolution of the Hubble parameter.

b. Deriving the Friedmann equations

Our starting point are Einstein's equations Eq. (A.5). In order to derive the time evolution of the Hubble parameter, we must examine the 00-th component of Einstein equation. To do so, we must then compute $G_{\mu\nu}$ and specify $T_{\mu\nu}$.

The order of operations is to compute $\Gamma_{\mu\nu}^\lambda \rightarrow R_{\mu\nu} \rightarrow R \rightarrow G_{\mu\nu}$. To begin, let's write the metric as

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j. \quad (\text{A.12})$$

Computing $G^\mu{}_\nu$

The only nonzero Christoffel symbols are

$$\Gamma_{ij}^0 = a\dot{a} \gamma_{ij}, \quad (\text{A.13a})$$

$$\Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i, \quad (\text{A.13b})$$

$$\Gamma_{jk}^i = \frac{1}{2} \gamma^{il} (\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}) \quad (\text{A.13c})$$

plus those related to these by the symmetry $\Gamma_{\mu\nu}^\alpha = \Gamma_{\beta\alpha}^\mu$.

The non-vanishing components of the Ricci tensor are,

$$R_{00} = -3 \frac{\ddot{a}}{a} \quad (\text{A.14a})$$

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{k}{a^2} \right] \gamma_{ij}. \quad (\text{A.14b})$$

Finally,

$$R \equiv g^{\mu\nu} R_{\mu\nu} = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right]. \quad (\text{A.15})$$

Having assembled the necessary ingredients, it follows that (using $G^\mu{}_\nu \equiv g^{\mu\lambda} G_{\lambda\nu}$)

$$G^0{}_0 = -3 \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right], \quad (\text{A.16a})$$

$$G^i{}_j = - \left[2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \delta_j^i. \quad (\text{A.16b})$$

Computing $T^\mu{}_\nu$

In general, the energy-momentum tensor is given by

$$T^\mu{}_\nu(\mathbf{x}, t) = \frac{g_{\text{dof}}}{\sqrt{-\det g}} \int \frac{d^3 P}{(2\pi)^3} \frac{P^\mu P_\nu}{P^0} f(\mathbf{x}, \mathbf{p}, t) \quad (\text{A.17})$$

where g_{dof} is the number of internal degrees of freedom of the particle under consideration, $P^\mu = dx^\mu/d\lambda$ is the *comoving momentum* while $P_\mu = g_{\mu\nu}P^\nu$ is the *physical momentum* and f is the phase-space distribution function.

Rather than using this expression, we will instead examine the situation where $T_{\mu\nu}$ is described by a perfect fluid,

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} \quad (\text{A.18})$$

where U^μ is the relative four-vector between the fluid and the observed. Further, ρ and P are the energy density and pressure in the rest frame of the fluid.

In the frame of the comoving observer $U^\mu = (1, 0, 0, 0)$ and

$$T^\mu{}_\nu = \text{diag}(-\rho, P, P, P). \quad (\text{A.19})$$

The energy density ρ and pressure P depend only on time, as expected by the our desired homogeneous and isotropic description of the Universe. A similar rational explains why the only nonzero elements are on the diagonal.

Conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$ yields, for $\nu = 0$,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \quad (\text{A.20})$$

We can define the *equation-of-state parameter* $w \equiv P/\rho$ such that

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \implies \rho \propto a^{-3(1+w)} \quad (\text{A.21})$$

which shows how the energy density of a given fluid dilutes with the expansion of the Universe. The equation of state parameter for relativistic fluids and non-relativistic fluids can be derived from thermodynamics. For now, it is sufficient to remember that

$$w = \begin{cases} 0 & \text{non-relativistic matter,} \\ 1/3 & \text{radiation,} \\ -1 & \text{de Sitter, cosmo. constant.} \end{cases} \quad (\text{A.22})$$

These values of w lead to the scalings written in Eq. (2.7).

The Friedmann Equations

The $\{\mu, \nu\} = \{0, 0\}$ Einstein equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \quad (\text{A.23})$$

This is the first Friedmann equation or *the Friedmann equation* as will be referred to in these notes.

The $\{\mu, \nu\} = \{i, j\}$ Einstein equation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \quad (\text{A.24})$$

This is the second Friedmann equation.

Both of these equations are important. However, given the energy densities, one can solve Eq. (A.23) to obtain the time evolution of the scale factor.

c. Λ CDM and the background evolution

The Standard Model of Cosmology is the Λ CDM – it is not so much a single equation but a collection of ingredients which combine to establish a remarkably accurate description of our Universe. Foundational to Λ CDM are the components which form the total energy density of Universe. There are three primary components in Λ CDM,

ρ_m : non-relativistic matter

ρ_{rad} : relativistic matter or radiation

ρ_Λ : cosmological constant

The non-relativistic matter component is comprised of baryonic matter and dark matter while radiation comprises of photons and neutrinos. The inclusion of a non-relativistic or *cold* dark matter component and a cosmological constant Λ is what distinguishes the Λ CDM from other possible descriptions of the Universe.

In terms of curvature, it is observed that the Universe is remarkably flat. This will be addressed via cosmological inflation, but for now on we will assume that $k = 0$.

Including these components in Friedmann equation Eq. (A.23) is yields

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{\text{rad}} + \rho_\Lambda). \quad (\text{A.25})$$

We will now make some adjustments to the above equation. First, we will use the *reduced Planck mass* $M_{\text{Pl}}^2 = 1/8\pi G$ in favor of Newton's gravitational constant. Second, we will define the critical density,

$$\rho_{\text{crit}} = 3M_{\text{Pl}}^2 H^2 \quad (\text{A.26})$$

with the *present-day* critical density, $\rho_{\text{crit},0}$, being the above quantity with $H \rightarrow H_0$. Defining the *density*

parameter, Ω_X ,

$$\Omega_X \equiv \frac{\rho_X}{\rho_{\text{crit},0}} = \frac{\rho_{X,0}}{\rho_{\text{crit},0}} a^{-3(1+w)} \quad (\text{A.27})$$

allows us to express the Friedmann equation as

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{m,0} a^{-3} + \Omega_{\text{rad},0} a^{-4} + \Omega_\Lambda. \quad (\text{A.28})$$

Using the definition of the Hubble parameter, $H = \dot{a}/a$, we can determine the present-day age of the Universe

$$t = \int_0^a \frac{da'}{a' H(a')} \quad (\text{A.29a})$$

$$= H_0^{-1} \int_0^a \frac{da'}{a' \sqrt{\Omega_{m,0} a'^{-3} + \Omega_{\text{rad},0} a'^{-4} + \Omega_\Lambda}}. \quad (\text{A.29b})$$

From here you can see that the age of the Universe is proportional to H_0^{-1} . In the Λ CDM scenario, the integral is ~ 1 and so H_0 and t_0 are quite close one another.

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