## 3. Fermion fields

Clifford algebra : $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$

$\Rightarrow \Sigma^{MN} \equiv \frac{i}{4}[\Gamma^M, \Gamma^N]$ satisfy Lorentz algebra (= "Spin $\frac{1}{2}$ rep." of L.A.)

; $k \geq 2$

- $d = 2k$ (even) : • $\Gamma^N$ are $2^k \times 2^k$ matrices, [in fundamental rep.]

  constructed from Pauli $\sigma^i$ [algorithm for $k \to k+1$]

  - $\Gamma \equiv i^k \Gamma^0 \Gamma^1 \ldots \Gamma^{2k-1}$ "$\equiv \gamma^5$" in 4D

    $\to$ eigenvalue $\pm 1$ "chirality"

  - $\{\Gamma, \Gamma^N\} = 0 \Rightarrow [\Gamma, \Sigma^{MN}] = 0$

    $\Rightarrow$ rep. is reducible (as rep. of Lorentz algebra)

    $$\underline{2}^{k}_{\text{Dirac}} = \underset{\{\psi | \Gamma\psi = \psi\}}{\underline{2}^{k-1}_{\text{weyl}}} + \underset{\{\psi | \Gamma\psi = -\psi\}}{\underline{2}^{k-1\,\prime}_{\text{weyl}}}$$

    projection operator $P_{R,L} \equiv \frac{1}{2}(1 \pm \Gamma)$ [in 4D : positive chirality = helicity = "right-handed" & $+ \to -$, right $\to$ left]

    $\Rightarrow \psi = P_R \psi + P_L \psi \equiv \psi_R + \psi_L$

- $d = 2k+1$ (odd) : use $\{\Gamma^N\}$ for $d = 2k$ and add $\Gamma \equiv \Gamma^{2k}$

  $\{\Gamma, \Gamma^N\} \neq 0$ (only for $N = 2k$!)

  $\Rightarrow [\Gamma, \Sigma^{MN}] \neq 0$

  $\rightsquigarrow$ rep. is irreducible

consequences : chiral fermions

- must be massless (as in 4D, but valid for any d) :

  $\mathcal{L} \supset m \bar{\psi}\psi = m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$

  i.e. a mass term would mix chiral states / transform $L \leftrightarrow R$

- cannot exist in odd dimensions :

  "L" and "R" are related by Lorentz trafos !

Q : How to obtain chiral fermions in 4D ?

[NB: even for $d \geq 6$, 5D Lorentz trafos would mix eigenstates of $\frac{1 \pm \gamma^T}{2}$ !]

A : compactify on an orbifold !

e.g. in 5D : $\psi = \psi_R + \psi_L$

mix under 5D Lorentz trafos

do not " " 4D " "

$\rightsquigarrow$ $\psi_R$, $\psi_L$ can have different parities under $y \to -y$

(i.e. different properties under orbifold projections)

e.g. SM-like fermion "SU(2) singlet" (= "right-handed" <u>in 4D</u> )

$$\psi_s = \frac{1}{\sqrt{2\pi R}} \psi_{s_R}^{(0)} + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left( \underline{\psi_{s_R}^{(n)}} \cos \frac{ny}{R} + \underline{\psi_{s_L}^{(n)}} \sin \frac{ny}{R} \right)$$

<u>bottom line</u> : for every chiral fermion (in 4D) there are <u>two</u> massive fermions at each KK level ! (in a 5D theory)

## 4. Some historical notes

- Nordström (1914), Kaluza (1921) & Klein (1926)

  ⇝ consider gravity in 5D (in vacuum, no matter Lagrangian!):

$$S = \frac{1}{16\pi \hat{G}} \int d^5x \sqrt{|\hat{g}|}\, \underbrace{R^{(5)}}_{= R^{MN} R_{MN}}$$

$$= \frac{1}{16\pi \hat{G}} \int d^5x \sqrt{|\hat{g}|} \{ \underbrace{R^{\mu\nu} R_{\mu\nu}}_{R^{(4)}} + 2 R^{\mu 5} R_{\mu 5} + R^{55} R_{55} \}$$

- $A_\mu \equiv (8\pi \hat{G})^{-\frac{1}{2}} \hat{g}_{5\mu}$

  N.B.: not the same "R" !!!

  \+ assumptions: i) $y \sim y + 2\pi R$ Klein's "cylinder condition"

  ii) $\partial_y \hat{g}_{MN} \simeq 0$

  (⇒ no KK tower ⇐) R very small

  iii) $\hat{g}_{55} = \text{const.}$ ← rather ad hoc!

  (...) ⌜ii + iii ⇒ $R_{55} = 0$⌟

$$S = \int d^4x \{ \underbrace{\frac{1}{16\pi G}}_{= \hat{G}/(2\pi R)} R^{(4)} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \} \quad : \text{unification of GR \& EM!}$$

- follow-up work:
  - non-Abelian gauge groups possible for $d > 5$, depending on internal geometry
  - $G \supset SU(3) \times SU(2) \times U(1)$ possible for $d \geq 11$
  - various issues...

    quantizeability ($s > 2$ states for $d > 11$), renormalizability, chirality ..

- renewed interest with string theory (70's - 80's)

point particles

closed strings

open strings

idea: finite extent ⇝ cure UV (= short-distance) problems

- consistent quantization possible only in $d = 11$ (Minkowski)
  - → discrete oscillation patterns: massless / light modes + KK towers
- open strings always attached to $d < 11$ objects ("branes")
  - ⇝ various massless vector (gauge) fields in spectrum
- closed strings propagate freely in entire $d = 11$ space ("bulk")
  - ⇝ always contain massless spin-2 field ⇝ gravity

- "large ED" scenario by Arkani-Hamed, Dimopoulos & Dvali (1998)

may explain hierarchy problem: (i.e. why is gravity so much weaker than other forces?)

$$\left.\begin{array}{l}\text{– only gravity propagates in EDs}\\ \text{– e.g. } d=6 + R\sim \text{mm}\end{array}\right\} \Rightarrow M_{pl} = 10^{19}\,\text{GeV} \Leftrightarrow \hat{M}_{pl} \sim 10^{2}\,\text{GeV} \sim m_{EW}$$

- Randall & Sundrum (1999)
  - RS1: "warped" geometry allows solution of hierarchy also w/ small EDs

    (anti-de Sitter space in bulk: $ds^2 = e^{-2\underbrace{k}_{\sim M_{pl}} r_c |\phi|} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 dr^2$)
  - RS2: Even infinitely large EDs can give Newton's law in 4D!

    (but this no longer solves the hierarchy problem)

- ... [lots of (fading) activity; ADD + RS1/RS2 among most-cited theory papers of all times]

## 5. "Universal" extra dimensions

Can we put the entire SM in 5D?

Appelquist, Cheng & Dobrescu (2001): yes! If compactified on $S^1/\mathbb{Z}_2$ ...

- fermions: recall $G \supset SU(2)_L$ ~> doublets, e.g. $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$
  + singlets, e.g. $e_R$

  -> both singlets and doublets (= different chiralities in 4D) come with two KK towers each

  $\hat{=}$ right-/left-handed sfermions in SUSY

- (massless) vectors – for $U(1), SU(2), SU(3)$ – come with each their KK tower

  $\hat{=}$ gauginos

- scalars: Higgs doublet $\phi \sim \begin{pmatrix} \chi^2 + i\chi^1 \\ H - i\chi^3 \end{pmatrix}$ give mass to $W^\pm, Z$ in SM

  $H$ " const + h

  ~> $\chi_1^{(n)}, \chi_2^{(n)}, \chi_3^{(n)}, h^{(n)}$ @ KK level

  + $A_5^{(n)}$: 1x for $U(1)$
  3x " $SU(2)$ '$W^\pm/Z$'
  8x " $SU(3)$ "gluons"

  } same quantum numbers as $\chi_i^{(n)}$
  => mix!

  16 scalar states at each KK level
  − 12 scalar d.o.f. ('goldstone modes') to give mass to $A_\mu^{(n)}$

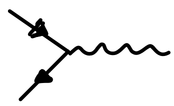
  4 physical scalars for each KK level

$\hat{=}$ non-trivial Higgs sector in MSSM:

2 Higgs doublets => 5 physical scalars + 4 Higgsinos

still: UED sometimes referred to as "bosonic supersymmetry"

# interactions

e.g. 

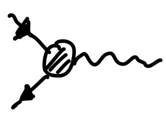
$$S \supset -\hat{e}\int d^4x\, dy\; A_\mu \bar{\psi}_5 \gamma^\mu \psi_5$$

N.B.: Dirac, not Weyl, i.e. $\psi_5 = \psi_{5,L} + \psi_{5,R}$

$$\vdots$$

$$\simeq -e \sum_{n,k,\ell} \int d^4x\; A_\mu^{(n)} \bar{\psi}_5^{(k)} \gamma^\mu \psi_5^{(\ell)} \times \underbrace{\left\{ \delta_{n,k+\ell} + \delta_{n,k-\ell} + \delta_{n,\ell-k} \right\}}_{\text{"KK-number conservation"}}$$

why? conservation of $p_5$!

what about  ?

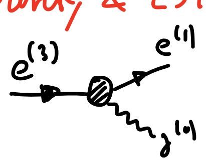
⇝ can <u>not</u> expect KK-number conservation!

why? $p_5$ <u>not</u> conserved @ $y = 0, \pi R$

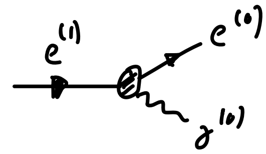
But $y \leftrightarrow -y$ still symmetry ⟺ "KK parity" $(-1)^n$ is conserved

⇒ lightest KK particle (LKP) is absolutely stable

c.f. R-parity & LSP in SUSY

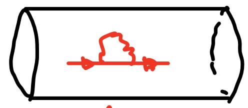
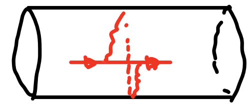
e.g. 



but 



not possible

⇒ spectrum of states phenomenologically very important mass contributions : $\frac{1}{R}$ , $m_{EW}$ ,  , 

$\frac{1}{R}$: same for all state

$m_{EW}$: "tree-level" SM masses, from EW symmetry breaking

first diagram: requires additional counterterms at orbifold boundaries → dependence on cutoff scale $\Lambda$

second diagram: finite + usual 4D infinities

result : LKP = $\gamma^{(1)} \simeq B^{(1)}$ because Weinberg angle than at KK-0 level !

→ • would be thermally produced in early universe
• = account for entire dark matter density (observed today) if $R^{-1} \sim 1.4$ TeV

current constraint from LHC : $R^{-1} \gtrsim 1.7$ TeV [2110.00500]

→ † for KK DM ...

⌜caveats apply : this is only for "minimal" VED.
c.t. on orbifold fixpoints allow in principle for "arbitrary"
$\delta m$ ⇒ required $R^{-1}$ may be factor 2-3 higher⌟