

Phase transitions in field theory and cosmology

1. recap: PTs in statistical mechanics

2. vacuum transitions

3. field theory at $T > 0$

4. Bubble nucleation (+ gravitational waves)

5. Symmetries (+ topological defects)

additional reading:

• Hindmarsh, Lüben, Lumma & Pauli, arxiv:2008.09136

• Croon, 2307.00068

1. recap: PTs in statistical mechanics

"Phase" $\leadsto$ distinct macroscopic property

"order parameter" $\leadsto$ characterizes phases, especially close to transition
(often 0 in one phase, $\neq 0$ in the other)

e.g. M in ferromagnet, ΔS for fluid/gas, ...

"1st order": discontinuous change of order parameter

$\leadsto$ absorption/release of latent heat $L = T \Delta S$

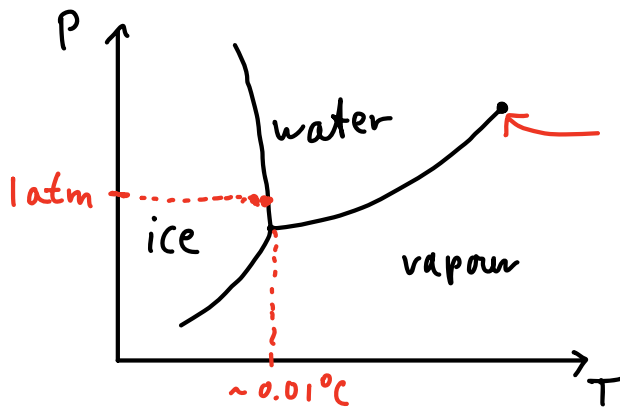
[latent heat $\equiv$ change of specific enthalpy; $\Delta H = \Delta U + P \Delta V$]

"2nd order": continuous change (but derivative typically discont.)

$\leadsto$ characterized by critical exponents, e.g. $\chi_{\text{con}}^{-1} \propto (T - T_c)^2$

" ∞ order": continuous, but break no symmetries

E.g. H_2O :



@ critical point ($T_c \approx 374^\circ C$),
transition turns from 1st to 2nd order
 $T > T_c$: "supercritical fluid"

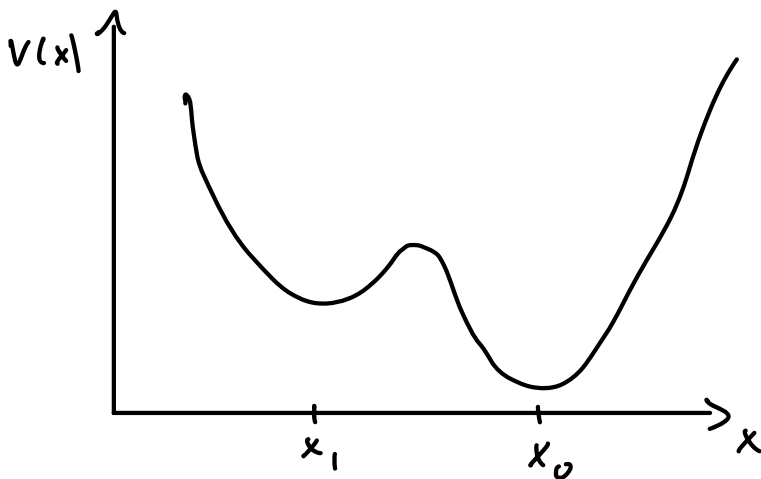
Remark: "Ehrenfest classification" was used, based on
Gibbs free energy, $dG = VdP - SdT + \mu dN$:

n^{th} order: $\Leftrightarrow \frac{\partial^n G}{\partial T^n}$ changes discontinuously, at phase boundary
still: latent heat = barrier to change only for $n=1$

2. vacuum transitions

vacuum = ground state = state of lowest energy
 $\rightarrow$ global vs. local

1D | a) classical mechanics



e. o. m. for $x(t)$:

$$\Delta x \equiv x - x_{1,0} ; |\Delta x| \ll 1$$

$$\ddot{x} + \frac{1}{m} \frac{\partial V}{\partial x} = -\gamma \dot{x} \quad | \Rightarrow \frac{V(x)}{m} = c_{1,0} + \frac{1}{2} \omega_{1,0}^2 (x - x_{1,0})^2 + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow \Delta \ddot{x} + \omega^2 \Delta x = -\gamma \Delta \dot{x} + \mathcal{O}(\Delta x^2) : \text{damped harmonic oscillator}$$

$$\Rightarrow x(t \rightarrow \infty) = \begin{cases} x_1 \\ x_0 \end{cases} \quad (\text{depending on initial conditions})$$

$\leadsto$ particle can be trapped in "false vacuum" at $x = x_1$

b) quantum mechanics

• particle will (eventually) tunnel to true vacuum:

$$VEV \equiv \langle x \rangle (t \rightarrow \infty) = x_0$$

• tunneling probability: (from WKB: $\psi(x) = A(x) e^{i\varphi(x)}$)

fast variation $\downarrow$
slow variation: $\frac{A''}{A} \ll (\varphi')^2, p^2$

$$T \approx e^{-2 \int_{x_1}^{x_0} dx \underbrace{\sqrt{2m(V-E)}}_{=p}} \sim e^{-2 \underbrace{\Delta x}_{\text{width}} \underbrace{\sqrt{m \Delta V}}_{\text{height of barrier}}}$$

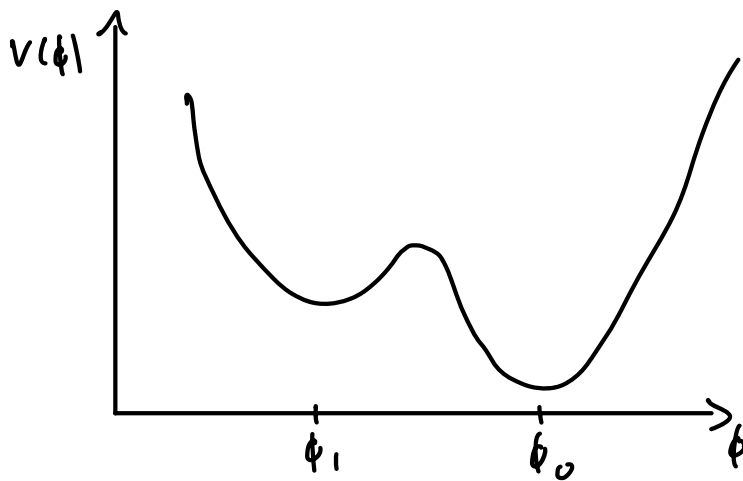
• tunneling rate:

$$\Gamma = \omega T$$

characteristic / "attempt" frequency:

$$\omega \approx \omega_1 = \sqrt{V''(x_1)/m}$$

∞D | $X \rightarrow q_i \equiv \phi(x)$: field theory!



e.o.m. : $\square \phi + V'(\phi) = -\Gamma \dot{\phi}$ (+ shear terms: $s(x) \nabla^2 \phi, \dots$)

$\Rightarrow \langle \phi \rangle = \phi_0$

↑
"decay" or "friction" term from background dynamics ($\rightarrow$ cosmology) or coupling to other fields (e.g. particle decay $\phi \rightarrow \dots$)

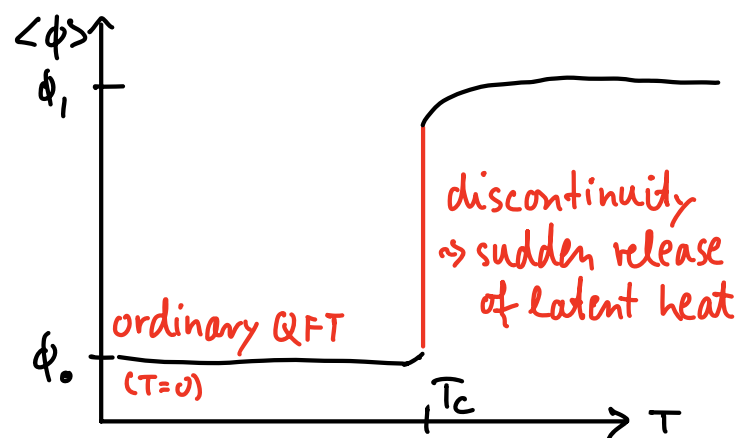
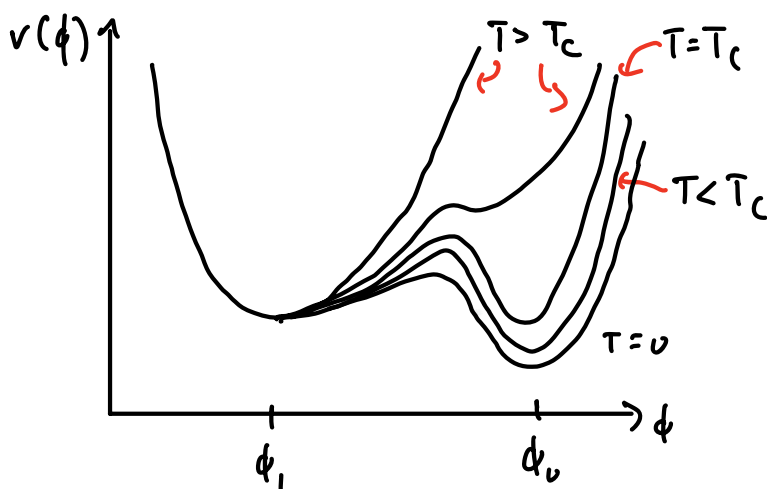
- rate calculation: much more complicated than in QM, see reviews

Now consider $V(\phi) = V_T(\phi)$

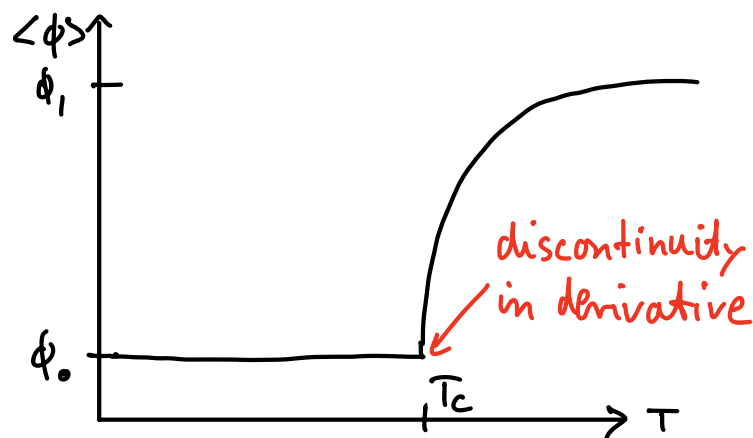
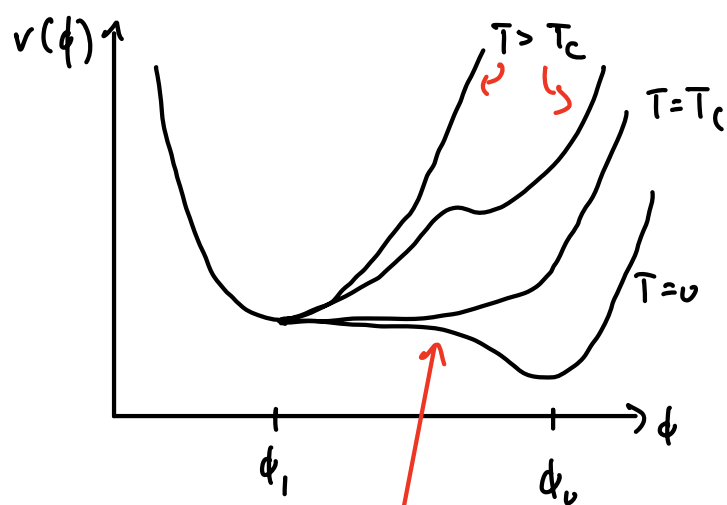
$\Rightarrow \langle \phi \rangle = \langle \phi \rangle_T$: useful order parameter to describe transitions!

NB: physics can be very different for ϕ_1 and ϕ_0 phase, cf. SM

- Example of first-order transition:



- Example of second-order transition;



no potential barrier $\rightarrow$ no latent heat release

NB: T -dependence of $v(\phi)$ expected in any (interacting) theory!

3. field theory at $T > 0$

a) "free" fields (assume thermal EQ! $\rightarrow$ "free" = "weak coupling")

$$Z(T) = \text{Tr} [e^{-\beta \hat{H}}] ; \beta \equiv \frac{1}{T}, \quad Z: \text{"partition function"}$$

$$\Rightarrow F = U - TS \\ = -\beta^{-1} \ln Z$$

[Helmholtz free energy: $\sim$ usable E for const. T & V
 $\leftrightarrow$ Gibbs $\sim$ T & P]

$$S = - \frac{\partial F}{\partial T}$$

$$E = - \frac{\partial \ln Z}{\partial \beta}$$

[D] $\hat{H}_0 = \omega (n + \frac{1}{2}) |n\rangle$ for bosons

$$\Rightarrow Z_0 = \sum_{n=0}^{\infty} \langle n | e^{-\beta \hat{H}_0} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta \omega (n + \frac{1}{2})} = e^{-\frac{\beta \omega}{2}} \sum_{n=0}^{\infty} e^{-\beta \omega n} = \frac{e^{-\beta \omega / 2}}{1 - e^{-\beta \omega}}$$

$$\Rightarrow F_0 = \frac{\omega}{2} + \beta^{-1} \ln (1 - e^{-\beta \omega})$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\downarrow \frac{e^{-\beta \omega / 2}}{1 - e^{-\beta \omega}}$$

$$\infty D \quad \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow (\square + m^2) \phi(\vec{x}, t) = 0$$

$$\Rightarrow (-\hbar^2 + \underbrace{(\vec{k}^2 + m^2)}_{\equiv \omega_{\vec{k}}^2}) \phi_{\vec{k}}(t) = 0 \quad : \text{no mode mixing!}$$

field: ∞ single harmonic oscillators, cf. QFT!

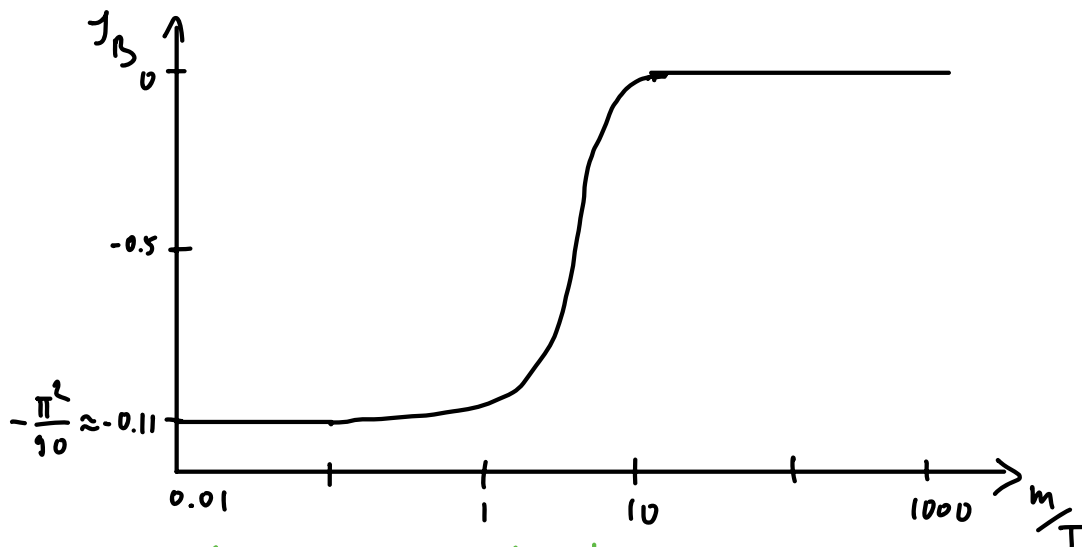
$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} (a_{\vec{k}} e^{-i\vec{k}\vec{x}} + a_{\vec{k}}^\dagger e^{i\vec{k}\vec{x}})$$

$$\mathcal{Z} = \prod_{\vec{k}} \mathcal{Z}_{1D}(T, \omega_{\vec{k}}) \quad \left| \quad \ln \prod_{\vec{k}} = \sum_{\vec{k}} \ln \rightarrow V \int \frac{d^3k}{(2\pi)^3} \ln \right.$$

$$\Rightarrow \mathcal{F} \equiv \frac{F}{V} = \underbrace{\left[\beta_0 \right]}_{\substack{\uparrow \\ \text{"}\frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} = \infty\text{"}}} + \beta^{-1} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta \omega_{\vec{k}}}) \equiv \beta_0 + T^4 \mathcal{J}_B\left(\frac{m}{T}\right)$$

$\uparrow$
" $\frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} = \infty$ " : no T-dependence

$\hookrightarrow$ regularized / renormalized in usual (QFT) way



$$\leftarrow \approx -\frac{\pi^2}{90} + \frac{1}{24} \left(\frac{m}{T}\right)^2 \quad \quad \quad \approx -\left(\frac{m}{2\pi T}\right)^{3/2} e^{-m/T} + \dots$$

$$- \frac{1}{12\pi} \left[\left(\frac{m}{T}\right)^2\right]^{3/2} + \dots$$

$\uparrow$
non-analytic in $\frac{m^2}{T^2}$!

$\nwarrow$ MB statistics $\checkmark$