

Quantum Field Theory at Finite Temperature

Bror Hjemgaard

University of Oslo

November 9, 2022

Overview

- Motivate thermal QFT
- "Derive" the formalism
- Interpret
- Relate to physical observables

Thermal QFT: Idea and Initial Challenges

Idea: Can we use the powerful and accurate framework of QFT to model thermal systems, like plasmas?

Challenges:

- "Normal" QFT is built around the existence of free states
 - A plasma has no "free states"
 - Can we even perform perturbation?
- Thermal systems are statistical ensembles $\Rightarrow$ processes of individual particles are not the interesting ones
 - Scattering?
 - Decay rates?
 - Feynman diagrams?
- A plasma is defined in a specific reference frame $\Rightarrow$ loss of Lorentz invariance

Building a Thermal QFT

Goal: Create a statistical QFT model.

Starting point: partition function for a thermalized system

$$\mathcal{Z}(\beta) = \text{Tr}\{e^{-\beta\hat{H}}\} = \int dq \langle q | e^{-\beta\hat{H}} | q \rangle, \quad \beta \equiv \frac{1}{T}.$$

Compare to probability amplitude of a state $|\phi_a\rangle$ at $t = 0$ transitioning into a state $|\phi_b\rangle$ at $t = t'$:

$$\langle \phi_b | e^{-i\hat{H}t'} | \phi_a \rangle = \int_{\phi(t=0)=\phi_a}^{\phi(t=t')=\phi_b} \mathcal{D}\phi \exp \left\{ i \int_0^{t'} dt \int d^3x \mathcal{L} \right\}$$

Idea: Wick rotation change variables $t \mapsto -i\tau$. Take $t' = -i\beta$ and write $\mathcal{Z}(\beta)$ as an integral over transition amplitudes $|\phi(\tau = 0)\rangle \mapsto |\phi(\tau = \beta)\rangle$.

Building a Thermal QFT

$$\mathcal{Z}(\beta) = \int d\phi \langle \phi | e^{-\beta \hat{H}} | \phi \rangle = \int_{\phi(\tau=0)=\phi}^{\phi(\tau=\beta)=\phi} \mathcal{D}\phi \exp \left\{ - \int_0^\beta d\tau \int d^3\mathbf{x} \mathcal{L} \right\}.$$

Observation: The partition function resembles a generating functional of a Euclidean QFT:

$$Z_E[j] = \int \mathcal{D}\phi \exp \left\{ - \int d^4x (\mathcal{L}_E + j\phi) \right\},$$

evaluated at $j = 0$ while imposing:

- The imaginary-time is bounded: $\tau \in [0, \beta]$.
- Bosonic fields are periodic in imaginary-time: $\phi(\tau = \beta) = \phi(\tau = 0)$.
- Fermionic fields are *anti*-periodic: $\psi(\tau = \beta) = -\psi(\tau = 0)$

Building a Thermal QFT

$$\mathcal{Z}(\beta) = Z_E$$

partition function = generating functional

LHS yields statistical observables

- Average energy: $E = \frac{1}{\mathcal{Z}} \text{Tr}\{\hat{H}e^{-\beta\hat{H}}\},$
- Free energy: $F = -T \ln \mathcal{Z},$
- Entropy: $S = -\frac{\partial F}{\partial T},$

RHS is calculated with QFT

- Fields
- Perturbation
- Feynman diagrams

Combined, they form a thermal theory expressed in the formalism of QFT.

Imaginary-time Formalism

Working with thermal QFT thus effectively equates to working with a Euclidean (imaginary-time) field theory:

- Non-singular propagator: $\Delta(k_0, \mathbf{k}) = \frac{1}{k_0^2 + \mathbf{k}^2 + m^2}$
 - $\Rightarrow$ All propagation is described by a single propagator, $\Delta(k_0, \mathbf{k})$.
- Periodic/anti-periodic boson/fermion fields
- Imaginary time-domain is bounded: $\tau \in [0, \beta]$
 - $\Rightarrow$ Discrete energy spectrum

Matsubara Frequencies

These discrete energy values are called the *Matsubara frequencies*:

$$\omega_n \equiv \begin{cases} \frac{2\pi}{\beta} n & \text{bosons} \\ \frac{2\pi}{\beta} \left(n + \frac{1}{2}\right) & \text{fermions} \end{cases}, \quad n \in \mathbb{Z}$$

Consequence: energy phase-space integrals become sums over discrete ω_n .

e.g: Imaginary-time propagator:

$$\Delta(\tau, \mathbf{x}) = T \sum_{\omega_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{-\tau \omega_n - i \mathbf{x} \cdot \mathbf{k}} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}$$

Matsubara Sums

Problem:

How to solve a general Matsubara sum

$$T \sum_{\omega_n} f(\omega_n) T \sum_{\omega_n} f(\omega_n) \mapsto \int ?? T \sum_{\omega_n} f(\omega_n) \mapsto \int \frac{dk_0}{2\pi} f(k_0) + ??$$

integrals sums

←
Easy

→
Difficult

Residue theorem for $g : \mathbb{C} \mapsto \mathbb{C}$ with poles $z_n \in \Gamma \subseteq \mathbb{C}$:

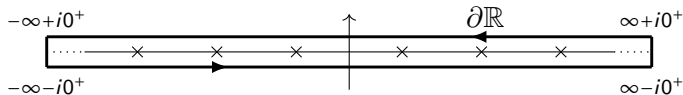
$$\sum_{z_n \in \Gamma} \text{Res}\{g; z_n\} = \frac{1}{2\pi i} \oint_{\partial\Gamma} g(z) dz, \quad \text{where } \partial\Gamma \text{ bounds } \Gamma.$$

Solution: Find a g with poles $z_n = \omega_n$ s.t. $\text{Res}\{g; z_n\} = T \cdot f(\omega_n)$.

Matsubara Sums

Observation: $in_B(iz) \equiv \frac{i}{e^{i\beta z} - 1}$ has poles at $z_n = \frac{2\pi}{\beta} n = \omega_n$ and residue T .

$$\begin{aligned}
 T \sum_{\omega_n} f(\omega_n) &= \sum_{z_n = \omega_n} \text{Res}\{f \cdot in_B; z_n\} = \oint_{\partial\mathbb{R}} \frac{dz}{2\pi} f(z) n_B(iz), \\
 &= \int_{-\infty}^{+\infty} \frac{dz}{2\pi} f(z) + \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dz}{2\pi} [f(z) + f(-z)] n_B(iz). \\
 &\quad T = 0 \qquad \qquad \qquad T > 0
 \end{aligned}$$



e.g: The imaginary-time propagator (suppressing spatial components):

$$\Delta(\tau) = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} e^{-\tau k_0} \frac{1}{k_0^2 + m^2} + \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dk_0}{\pi} \frac{\cosh(\tau k_0)}{k_0^2 + m^2} n_B(ik_0)$$

Hold up... Imaginary time???

How on earth can we impose a multitude of physical qualities to our theory when it is not placed in Minkowski-space?

It is not even placed in Euclidean "spacetime", but in $S^1_\beta \times \mathbb{R}^3$

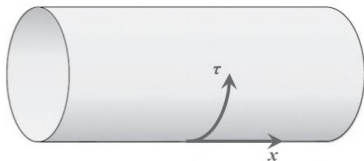


Figure: Illustration of $S^1 \times \mathbb{R}$

$$\mathcal{Z}(\beta) = Z_E$$

We should check if we can reformulate our imaginary-time thermal QFT to one of real times in Minkowski spacetime.

Constructing Real-Time Thermal QFT

Result from axiomatic QFT: If the propagator $\Delta(\tau, \mathbf{x})$ of a Euclidean QFT has the same analytic continuation to complex times as a propagator $G(t, \mathbf{x})$ from a Minkowskian QFT, then the two QFTs describe the same physics.

(Osterwalder-Schrader theorem, 1973)

Idea:

- Construct a propagator $G(z, \mathbf{x})$ defined for complex times $z \in \mathbb{C}$ which reduces to $\Delta(\tau, \mathbf{x})$ for $z = -i\tau \in -i[0, \beta]$.
- Then, $G(z, \mathbf{x})$ evaluated for $z = t \in \mathbb{R}$ will produce a Minkowskian (real-time) thermal QFT.

Result: Real-time Feynman propagator for $T \geq 0$:

$$G_F^{T \geq 0}(k) = \frac{i}{k^2 - m^2 + i0^+} + n_B(k_0) 2\pi \delta(k^2 - m^2)$$

Hold up (again)... Free plasma??

We have constructed the *free* propagators in thermal QFT in both imaginary- and real-time:

$$\Delta(\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}, \quad G_F^{T \geq 0}(k) = \frac{i}{k^2 - m^2 + i0^+} + n_B(k_0) 2\pi \delta(k^2 - m^2)$$

But, there is no such thing as a "free plasma" - it is interacting by definition!

How can we define a perturbative scheme then?

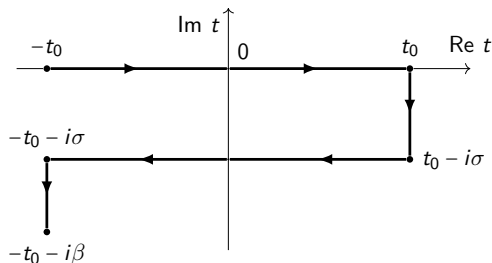
I lied

$$\Delta(\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}, \quad G_F^{T \geq 0}(k) = \frac{i}{k^2 - m^2 + i0^+} + n_B(k_0) 2\pi \delta(k^2 - m^2)$$

imaginary-time propagator ~~real-time propagator~~

$$\begin{pmatrix} \frac{i}{k^2 - m^2 + i0^+} + n_B(k_0) 2\pi \delta(k^2 - m^2) & e^{\sigma k_0} [n_B(k_0) + \theta(-k_0)] 2\pi \delta(k^2 - m^2) \\ e^{-\sigma k_0} [n_B(k_0) + \theta(k_0)] 2\pi \delta(k^2 - m^2) & \frac{-i}{k^2 - m^2 - i0^+} + n_B(k_0) 2\pi \delta(k^2 - m^2) \end{pmatrix}$$

$$\phi \mapsto \phi_1, \phi_2$$



Observables

$$\mathcal{Z}(\beta) = Z_E$$

$$\mathcal{Z}^{\text{cl}}(\beta) + \mathcal{Z}^{\text{q}}(\beta) = Z_E^{T=0} + Z_E^{T>0}$$

$\Rightarrow$ thermal gets quantum corrections / quantum gets thermal corrections

$$m = m_{\text{bare}} + \text{Re}\{\text{self-energy}\} \quad m = m_{\text{bare}} + \text{Re}\{\text{self-energy}\}$$

$$\mathbf{T} = \mathbf{0} : m_{\text{bare}} = 0 \Rightarrow m = 0$$

$$\mathbf{T} > \mathbf{0} : m_{\text{bare}} = 0 \not\Rightarrow m = 0$$

- Massive thermalized photons and gluons!
- Breaks gauge invariance

$$\Gamma = \frac{1}{m} \text{Im}\{\text{self-energy}\} \quad \Gamma = \frac{1}{m} \text{Im}\{\text{self-energy}\} \Rightarrow$$

thermal decay width

Conclusions

- Statistical QFT
- Imaginary- & real-time formalisms
- Challenges in interpretations
- Altered observables