Rethinking Geometry in Physics

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Outline

- Empirical view of geometry in physics: Einstein's Practical geometry.
- Limitations of practical geometry at small scales.
- Rethinking geometry for the quantum realm.

The Emergence of Practical Geometry

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- Marked the beginning of "Practical Geometry."

Einstein's Empiricism and the Emergence of Practical Geometry

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- Einstein emphasized the importance of linking geometry with physical experiments.

The birth of practical geometry: Riemann and Einstein

"I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in **removing** *certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the a priori. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the a priori in order to adjust them and put them in a serviceable condition." (Einstein,"The meaning of relativity" [4])*

Mathematician vs.Physicist view on geometry

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- Einstein advocated associating fundamental geometric concepts with physical objects and testing geometric theorems empirically based on the behavior of these objects.
- In this sense, geometry is a natural science, which Einstein called **practical geometry**.

Practical geometry

" One is ordinarily accustomed to study geometry divorced from any relation between its concepts and experience. There are advantages in isolating that which is purely logical and independent of what is, in principle, incomplete empiricism. This is satisfactory to the pure mathematician. He is satisfied if he can deduce his theorems from axioms correctly, that is, without errors of logic. The questions as to whether Euclidean geometry is true or not does not concern him. But for our purpose it is necessary to **associate the fundamental concepts of geometry** *with* **natural objects***; without such an association geometry is worthless for the physicist. The physicist is concerned with the question as to whether the theorems of geometry are true or not. " (Einstein, "The meaning of relativity" [4])*

Geometry as the oldest branch of physics

"Geometry thus completed is evidently a natural science; we may in fact regard it as the most ancient branch of physics. Its affirmations rest essentially on induction from experience, but not on logical inferences only. We will call this completed geometry "practical geometry", and shall distinguish it in what follows from "purely axiomatic geometry." (Einstein, "Geometry and experience" [3])

Practical geometry and small scale

- Riemannian geometry: distance as the "building block of geometry".

  "The whole of geometry may be founded upon this conception of distance. In the present treatment, geometry is related to actual things (rigid bodies), and its theorems are statements concerning the behaviour of these things, which may prove to be true or false." (Einstein, "The meaning of relativity")

- Not only did Riemannian geometry play a basic role in the development of general relativity but it became the central paradigm of geometry in the XXth century.

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- In these general spaces, geometric shapes like triangles cannot generally be moved without deformation, i.e., altering the lengths of its sides or its angles.
- **Riemann's key idea**: Even if we can't move shapes without squishing them, we can still measure tiny distances using a small standard length (a "rigid body" in practical geometry).

Riemannian paradigm and the concept of distance

- The distance $d(x,y)$ between two points x and y is computed by summing the lengths of the small intervals ds along a path between x and y, and then finding the smallest such length.
- $d(x,y) = \text{Inf}\left\{ \int_\gamma ds \mid \gamma \text{ is a path between } x \text{ and } y \right\}$

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  2. There is no concept of "path" or "trajectory" $\Rightarrow$ No empirical definition of spatial distance
  3. $\Rightarrow$ Einstein's practical geometry cannot be maintained as the geometrical framework in quantum scale.

Invalidity of Riemanian paradigm of geometry in the small scale

"Complications may arise in case the line element is not representable, as has been premised, by the square root of a differential expression of the second degree. Now, however, the empirical notions on which spatial measurements are based appear to lose their validity when applied to the indefinitely small, namely the concept of a rigid body and that of a light-ray; accordingly it is entirely conceivable that in the indefinitely small, the spatial relations are not in accord with the postulates of geometry(...) The question of the validity of the postulates of geometry in the indefinitely small is involved in the question concerning the ultimate basis of relations in space (...)" (Riemann, "On the Hypotheses Which Lie at the Foundation of Geometry" 1854 [8])

The breakdown of Reimannian paradigm in small scales

"It is true that this proposed physical interpretation of geometry breaks down when applied immediately to spaces of sub-molecular order of magnitude. (...) Success alone can decide as to the justification of such an attempt, which postulates physical reality for the fundamental principles of Riemann's geometry outside of the domain of their physical definitions. It might possibly turn out that this extrapolation has no better warrant than the extrapolation of the idea of temperature to parts of a body of molecular order of magnitude."(Einstein, "Geometry and experience" [3])

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1. First, we need to determine a conceptual geometric framework for quantum scale.
2. Then find a practical geometry that corresponds to it. (work in progress).

Supremacy of spectral over spatial quantities in quantum realm

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- Hint toward $\rightarrow$ Change of Paradigm of Geometry from spatial to spectral geometry

Heisenberg's Seminal Paper

Figure: Werner Heisenberg (1901-1976)

QUANTUM-THEORETICAL RE-INTERPRETATION OF KINEMATIC AND MECHANICAL RELATIONS

W. HEISENBERG

The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable.

Figure: Heisenberg Seminal 1925 paper [5]

From Bohr to Heisenberg

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- Bohr: Keep the link between spatial (electron trajectory) and spectral quantities (frequency, intensity etc.. of radiated light)
- Heisenberg (1925): Break the link between spatial and spectral quantities: Only spectral quantities are fundamental.

From Classical to Quantum Frequencies

Classical Theory of the atom:

- Fourier Series Representation for a periodic trajectory of the electron:

$$x(t) = \sum_{n=1}^{\infty} a_n \cos(n\omega t)$$

- Classical theory connects: *Kinematical* properties (spatial) of the electron $\leftrightarrow$ *Spectral* properties of light (frequency, intensity etc.)
- Kinematical (spatial) frequencies **coincide** with spectral frequencies.
- Spectral lines predicted to be equidistant:

$$\omega_n = n\omega$$

Classical Spectral Lines

Quantum Frequencies and New Mechanics

- Observed atomic spectral lines do not match classical predictions (not equidistant).

- Quantum Frequencies: Follow the Rydberg-Ritz combination rule:

$$\omega_{nk} + \omega_{km} = \omega_{nm}$$

- Heisenberg proposed a new "quantum kinematics" (while keeping classical dynamics).
- Redefined position as a spectral quantity, not a spatial one.

Heisenberg Spatial to Spectral Quantities

Spatial Concept: Trajectory

Spectral Concept: Set of Transition Components

$$x(n,t) = \sum_\alpha a_\alpha(n) e^{i\alpha\omega(n)t} \longrightarrow x \to \left\{ a(n, n-\alpha) e^{i\omega(n,n-\alpha)t} \right\}$$

Figure: Transition from Spatial to Spectral Concepts

Matrix Mechanics and Spectral Geometry

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- Heisenberg kinematical elements: Shift from spatial to spectral elements.
- Spectral quantities became the primary entities.
- There is no possible link between spatial and spectral quantities (as was the case in classical theory)

Change of Paradigm

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- There are no spatial elements in quantum mechanics, only spectral elements $\rightarrow$ Change the paradigm of geometry from a spatial geometry to a spectral geometry.

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- **Goal**: Recover geometry by analyzing vibrations/waves in the space.
- Helmholtz equation: $\Delta_M u = k^2 u$. Δ_M is the Laplace-Beltrami operator on the manifold (M, g), u is a scalar field representing the wave, and k is the wave number.

Disk

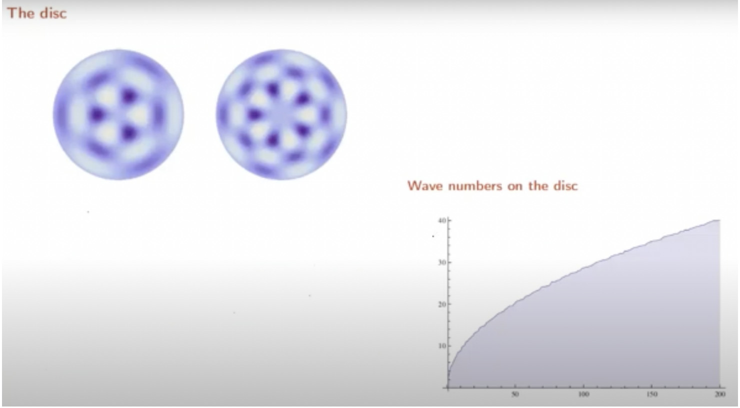

Figure: Spectral data from the disc [9]

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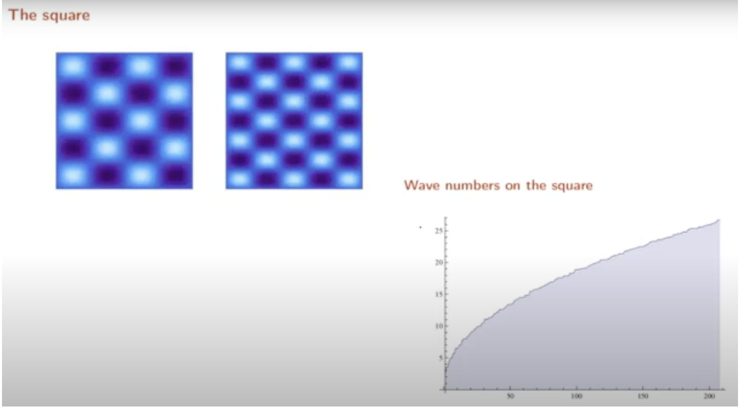

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Retrieving geometrical properties from the spectrum

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- By analyzing the spectrum, we can deduce if the shape is a disc or a square.
- We can do much more than that. Let's talk a bit about the birth of spectral geometry.

Weyl's law and quantum mechanics

In late October 1910, H. A. Lorentz visited Göttingen to present a series of lectures. His series of five talks was titled "Old and new problems of physics." In the concluding remarks of his fourth lecture, Lorentz introduced a mathematical problem inspired by Jeans's radiation theory:

> *In conclusion, there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present (...) to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure, and is simply proportional to its volume. For many shapes for which calculations can be carried out, this theorem has been verified. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures, like membranes, air masses, etc., should also hold. (Lorentz "Old and new problems of physics" [7])*

Weyl's law : history

Hilbert was not very optimistic to see a solution in his lifetime. But Hermann Weyl, his bright student, settled this conjecture of Lorentz and announced a proof within a year (in 1911).

Weyl's Law

- **Weyl's Law [10]:** We can also gain information about geometrical properties by analyzing the spectrum of Laplace operator:

  $$N(k) \sim \frac{\mathrm{Vol}(M)}{(2\pi)^n} k^n,$$

  where $N(k)$ is the number of eigenvalues of the Laplace operator Δ_M that are less than or equal to k, n is the dimension of the manifold, and $\mathrm{Vol}(M)$ is the volume of the manifold.

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- Spectral geometry: Framework where geometrical properties are encoded in a spectral manner.
- What would replace the line element ds in spectral geometry?
- **Alain Connes** [2]: Dirac operator can be seen as an adequate replacement of the line element: $ds = D^{-1}$.

Dirac Operator and Spectral Geometry

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- Examples: **Topological invariants**: Certain global properties of the space (topological properties) can be encoded in the spectrum of the Dirac operator.
  Local geometry: Dirac operator is connected with the curvature of space through the Atiyah-Singer index theorem.

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- In contrast to Einstein's practical geometry, which utilized rods and clocks to probe geometry encoded in line element ds, spectral geometry uses the Dirac operator as a the equivalent of a line element, which is inherently linked to the fundamental interactions within a space.
- Interestingly, Riemann had a similar idea:

Riemann

"While in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either therefore the reality which underlies space must be discrete, or we must seek the foundation of its metric relations outside it, in binding forces which act upon it. This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate. [8]"

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- Non-commutativity cannot be incorporated in classical manifold picture.
- Mathematical speculation: Let's try to extend the duality (**Alain Connes**):
  "Geometric Space $\iff$ Commutative Algebra".

The interplay between algebra and geometry

"Algebra is but written geometry and geometry is but figured algebra." - **Sophie Germain(1776-1831)**

$$\begin{array}{c}\text{Manifold } M \\ \big\downarrow \\ (\textbf{Commutative}) \text{ Algebra of Functions } C^{\infty}(M) \\ \big\downarrow \\ \text{All Geometric Objects: Vector Fields, Riemannian metric, etc.}\end{array}$$

All the differential geometric properties of a manifold M are encoded in the algebra C(M), the commutative algebra of the infinitely differentiable functions on M. As soon a one has the algebra C(M), the manifold M itself becomes superfluous.

Non-Commutative case

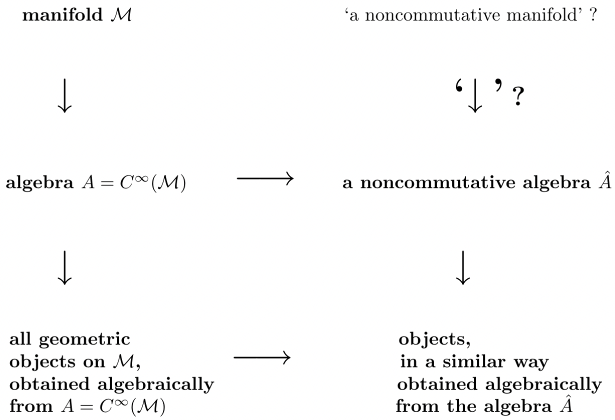

Figure: Extending the duality Geometry - Algebra to non-commutative case [1]

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  3. The Dirac operator $\mathcal{D}$ encodes the spectral information of geometry.

Noncommutative Geometry: Measuring Distances

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- Physically: $\mathcal{A}$ represents the algebra of physical observables, and the Dirac operator $\mathcal{D}$ encodes geometry through its spectrum.
  $\|[f, D]\| \leq 1$ can be interpreted as a condition on energy of the physical observables f.

Some text on spectral distance

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- It is built on physical observables of the quantum system itself, making it more suitable for describing physics at the quantum scale from a practical geometry point of view.

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- It is built on physical observables of the quantum system itself, making it more suitable for describing physics at the quantum scale from a practical geometry point of view.
- The spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ provides a unified framework to discuss geometry and physics in the quantum realm beyond the Riemanian paradigm, while ensuring geometrical concepts has an empirical equivalent within the quantum realm (i.e. compatible with a practical view of geometry)

Comparison of Riemannian and Spectral Non-commutative Geometry

Riemannian Geometry	**Spectral Non-commutative Geometry**
Space: (X, ds^2)	Spectral triple: (A, H, D)
Line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$	Line element: $ds = D^{-1}$
Distance: $d(a, b) = \inf \int_\gamma \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$	Distance: $d(a, b) = \sup_{f \in A} \{ \lvert f(x) - f(y) \rvert ; \lVert [D, f] \rVert \leq 1 \}$

Einstein's Practical Geometry vs. Spectral Practical Geometry

Einstein's Practical Geometry	**Spectral Practical Geometry**
Uses rigid rods and clocks to probe geometry that's encoded through the line element *ds*.	Use fundamental physical interaction to probe geometry that is encoded in the spectrum of the Dirac operator. (Work in progress to make a clearer sense of that)
Riemannian distance valid only on arc-connected spaces and relies on the line element *ds* and commutative coordinate functions.	Spectral distance works in any space and is defined using any physical observable (including quantum non-commutative observables).

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1. Construct concrete physical examples for the application of algebraic distance both in classical and quantum scales.
2. Construct a consistant "Practical spectral geometry", by deeply understanding the physical meaning of considering Dirac operator as line element $ds = D^{-1}$ (Relating it to fundamental interactions of nature encoded in D.)
3. Understand how the probabilistic framework of quantum theory could emerge from this spectral framework of geometry.

Finally

"I want to know God's thoughts; the rest are details."
- **Albert Einstein**

- Final words: I deeply believe that the lack of clear understanding of what is going on in the quantum realm comes from keeping an *a priori way* of thinking about the world as embedded in Riemanian space, and thus using a practical geometry that is not valid in the quantum realm. I think we need to adapt our geometrical framework to comprehend the quantum realm. Spectral geometry gives an interesting new paradigm of geometry that could guide us toward knowing god's thoughts in the quantum scale.

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