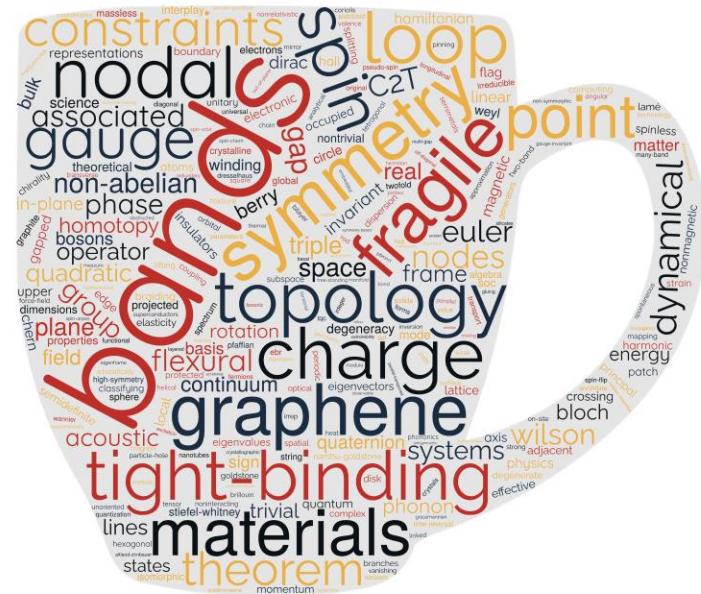


Multi-gap topology & non-abelian braiding in k -space

Outline

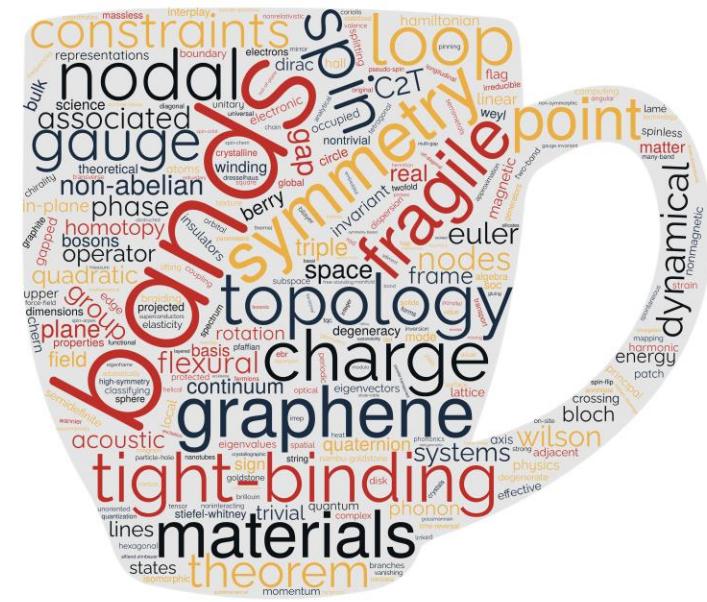
GFL25@CAM.AC.UK

- General comments on topology
 - Homotopy theory
 - Generalities
 - Anyons
 - Band structures
 - Weyl points
 - Multi-gap topology
 - Formality
 - Applications
 - Conclusion & Outlook



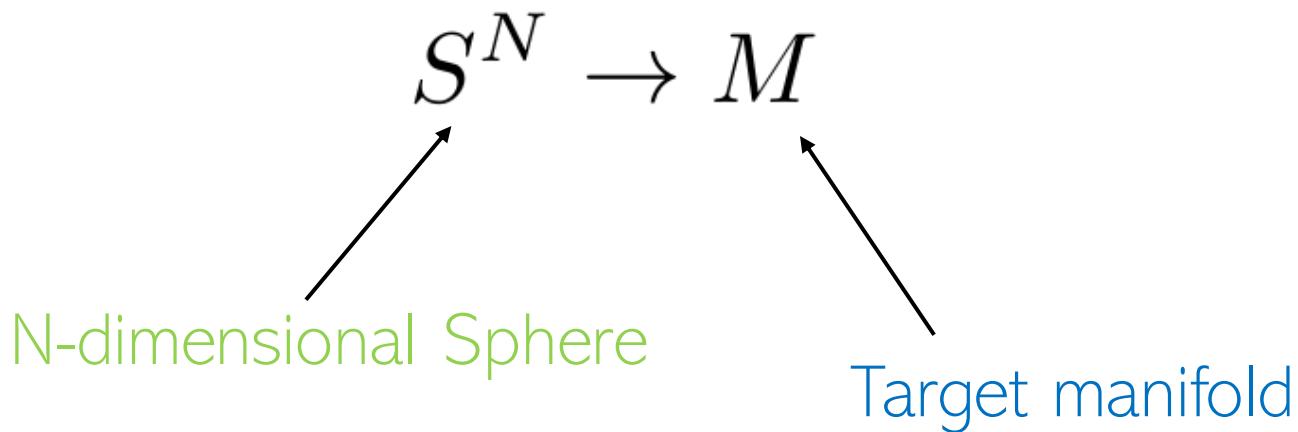
Topology?

- Study *global* properties of theories
 - Robust experimental quantities
 - Method today: Homotopy theory



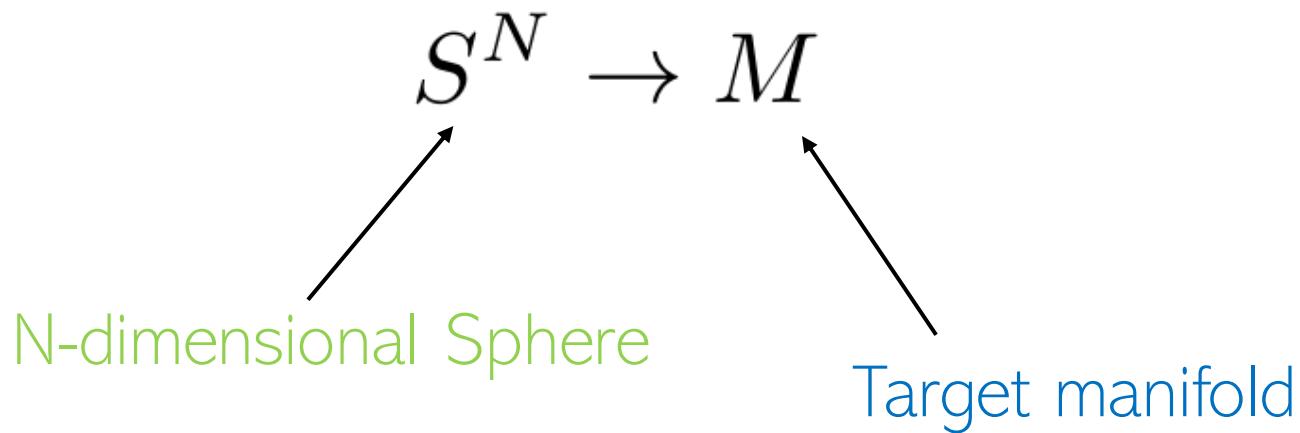
Homotopy theory

- Study *mappings of paths*



Homotopy theory

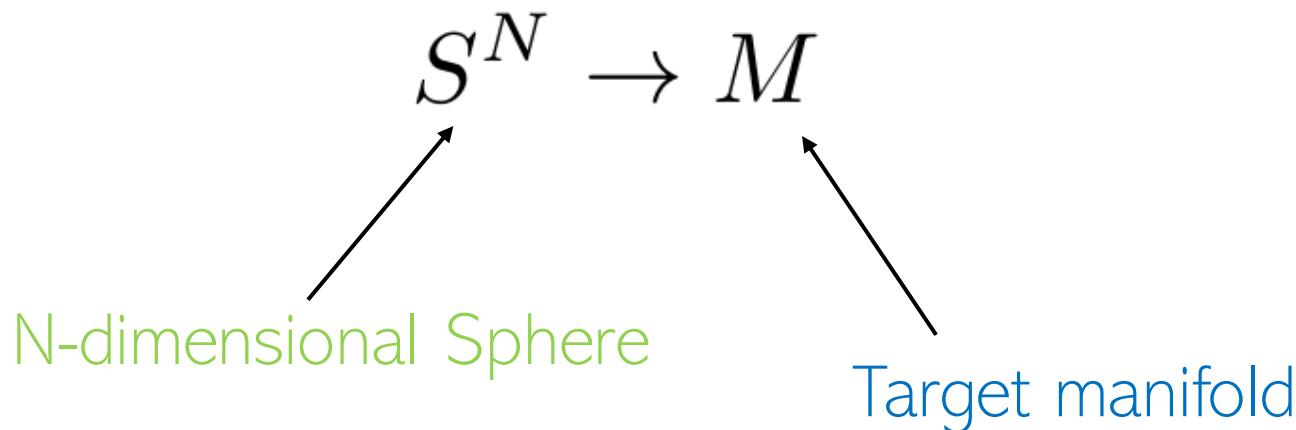
- Study *mappings of paths*



- N-th homotopy group: $\pi_N(M)$

Homotopy theory

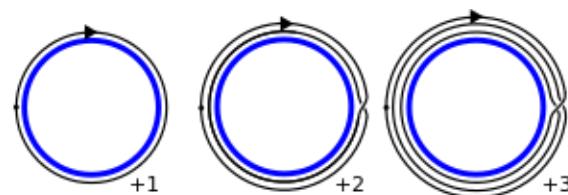
- Study *mappings of paths*



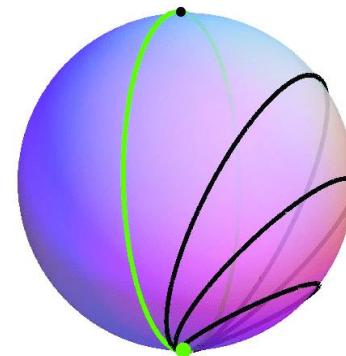
- N -th homotopy group: $\pi_N(M)$
 - Natural group structure: Composition of paths
 - Efficient algorithms exist to compute these groups

Homotopy examples

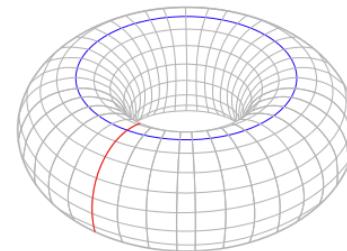
$$\pi_1(S^1) = \mathbb{Z}$$



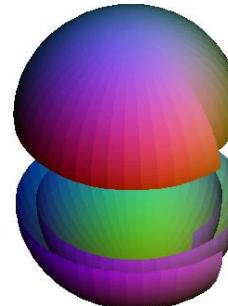
$$\pi_1(S^2) = \emptyset$$



$$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$$



$$\pi_2(S^2) = \mathbb{Z}$$



Homotopy in physics

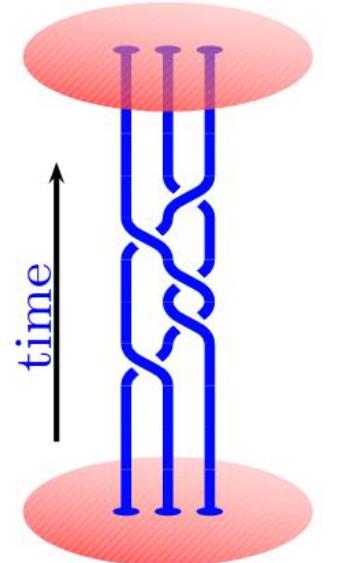
- Identical particles: Configuration space

$$M_d^N = \frac{(\mathbb{R}^d)^N - \Delta}{S_N}$$

Particle number
Spatial dimension

Avoid overlapping particles
Divide by permutations of coordinates

Braid group
Symmetric group



S. Simon Topological Quantum:
Lecture Notes (2021)

$$\pi_1(M_2^N) = B_N$$

$$\pi_1(M_3^N) = S_N$$

Homotopy in physics

$$\pi_1(M_2^N) = B_N \quad \longleftarrow \quad \text{Braid group}$$

- Abelian representations:
 - Anyons
 - Non-abelian representations:
 - Non-abelian anyons
-
- Quantum computing
- Abelian representations:
 - Boson/Fermions
 - Non-abelian representations:
 - Parastatistics
- Not too relevant for point particles

Central question

Can we find some non-abelian first-homotopy groups
to realize braiding?



Adrien Bouhon
NORDITA/Cambridge



Robert-Jan Slager
Harvard/Cambridge



Tomáš Bzdušek
Zürich

Homotopic ideas in condensed matter

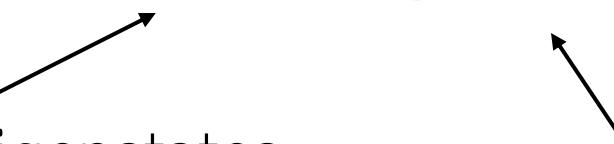
- Periodic media – Bloch Hamiltonians defined on Brillouin zone:

$$H(\mathbf{k}) \quad \mathbf{k} \in T^d$$

- For topology: Non-interacting, eigenstates form frame

$$|\psi_i(\mathbf{k})\rangle_{i \in \{1\dots N\}} \in \frac{U(N)}{[U(1)]^N}$$

Bloch eigenstates Ordered by energy



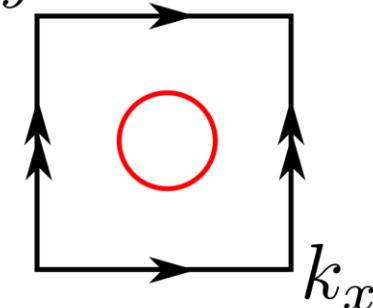
Homotopic ideas in condensed matter

$$H(\mathbf{k}) \quad \mathbf{k} \in T^d$$

- Study mappings: $T^d \rightarrow U(N)/[U(1)^N]$
- On patches in Brillouin zone: $S^{D < d} \rightarrow U(N)/[U(1)^N]$
- Impose gap conditions (equivalence relations):

$$T^d \rightarrow \frac{U(N)}{U(N-M) \times U(M)} \simeq \text{Gr}_{M,N}^{\mathbb{C}}$$

- Impose symmetries: $U(N) \rightarrow O(N)$



$$E_F - \frac{\equiv}{\equiv} \}_{M}^{N-M}$$

Example: Weyl points

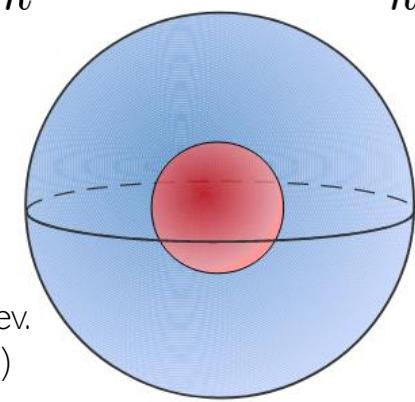
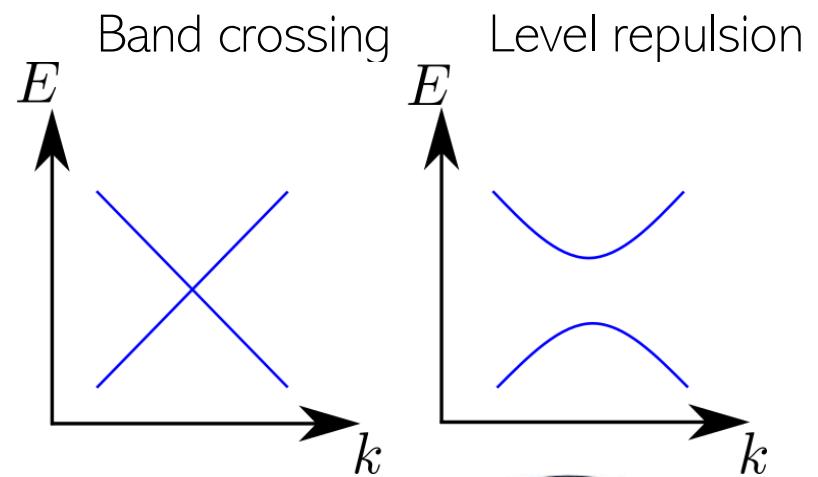
$$H(\mathbf{k}) \quad \mathbf{k} \in T^d$$

$$S^{D-d} \rightarrow U(N)/[U(1)^N]$$

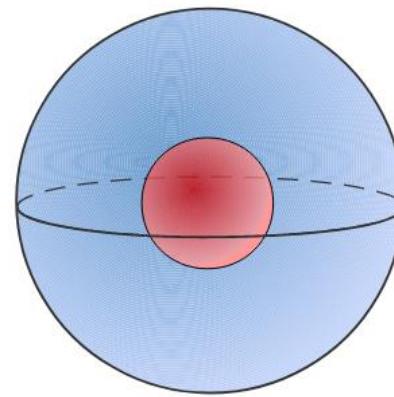
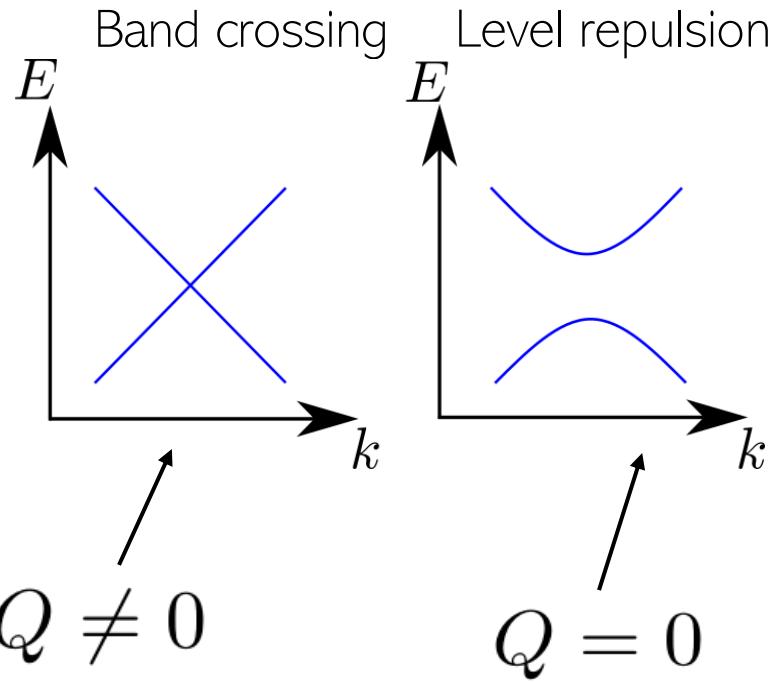
Specialize to two-band subspace in 3D:

$$S^2 \rightarrow \frac{U(2)}{U(1) \times U(1)}$$

$$\pi_2 \left[\frac{U(2)}{U(1) \times U(1)} \right] = \mathbb{Z}$$



Example: Weyl points

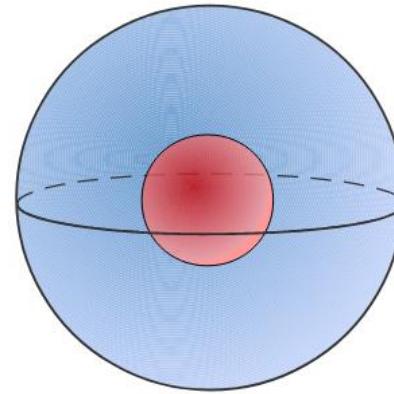
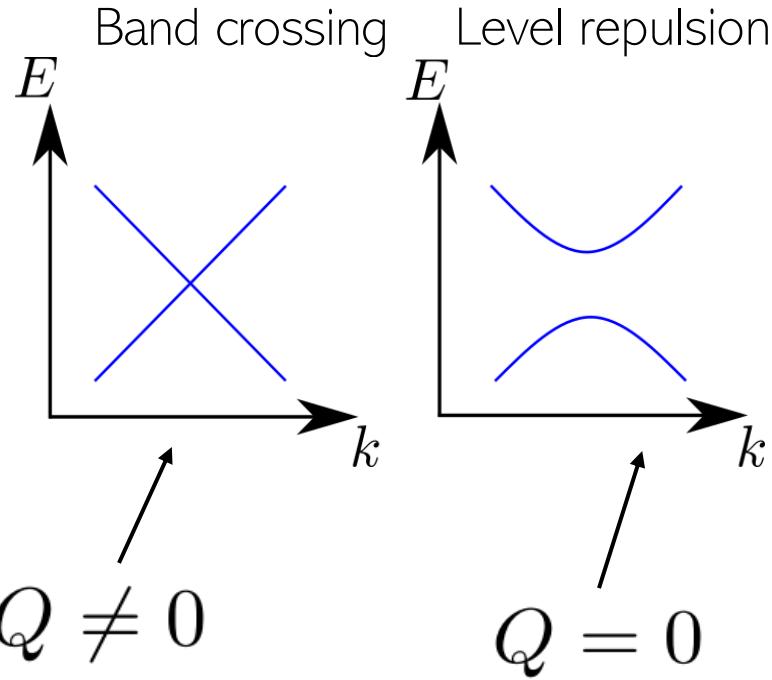


Bzdušek et al., Phys. Rev. B 96, 155105 (2017)

- Well defined charges
- Well-defined dispersion
- Charge conservation (Nielsen-Ninomya)

$$\pi_2 \left[\frac{U(2)}{U(1) \times U(1)} \right] = \mathbb{Z}$$

Example: Weyl points



Bzdušek et al., Phys. Rev. B 96, 155105 (2017)

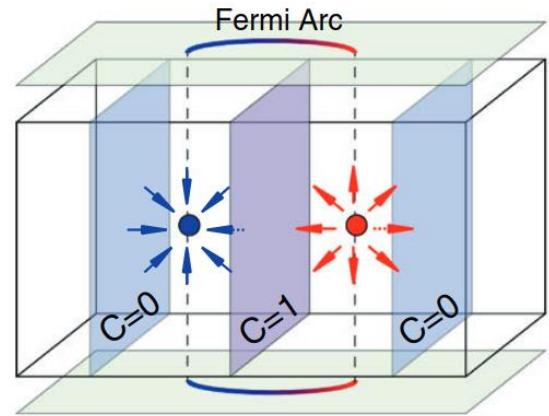
$$\pi_2 \left[\frac{U(2)}{U(1) \times U(1)} \right] = \mathbb{Z}$$

- Well defined charges
- Well-defined dispersion
- Charge conservation (Nielsen-Ninomya)

Particles in k -space!

Relationship to high-energy physics

- Various dispersions possible (not constrained by Lorentz symmetries)
- Naturally realize chiral anomaly
- More generally: Crystals have been found to host axions, higher-spin particles, etc.



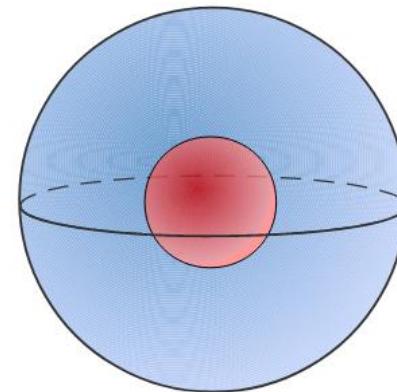
Ma et al., *Nature Communications*
Volume 12, Article number: 3994 (2021)

Braiding of Weyl points?

$$\pi_1 \left[\frac{U(2)}{U(1) \times U(1)} \right] = \emptyset$$



No non-trivial braiding



Bzdušek et al., Phys. Rev. B 96, 155105 (2017)

$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(2) \times \mathrm{O}(1)]} \right] = \mathbb{Z}_2$$

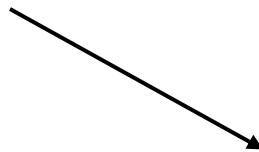
Quaternion group – Non-abelian!

$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)]} \right] = \mathbb{Q}$$



Multi-gap topology

$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(2) \times \mathrm{O}(1)]} \right] = \mathbb{Z}_2$$



2+1 bands, symmetry making eigenstates real



$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)]} \right] = \mathbb{Q}$$



1+1+1 bands (multi-gap), symmetry making eigenstates real

Reality condition

$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)]} \right] = \mathbb{Q}$$


1+1+1 bands (multi-gap), symmetry making eigenstates real

$$C_2 : \quad H(\mathbf{k}) = U_{C_2} H(C_2 \mathbf{k}) U_{C_2}^\dagger$$

$$\mathcal{I} : \quad H(\mathbf{k}) = U_{\mathcal{I}} H(-\mathbf{k}) U_{\mathcal{I}}^\dagger$$

$$\mathcal{T} : \quad H(\mathbf{k}) = U_{\mathcal{T}} H(-\mathbf{k})^* U_{\mathcal{T}}^\dagger$$

Spin even/odd

$$[C_2 \mathcal{T}]^2 = +1$$

$$[\mathcal{I} \mathcal{T}]^2 = \pm 1$$

Reality condition

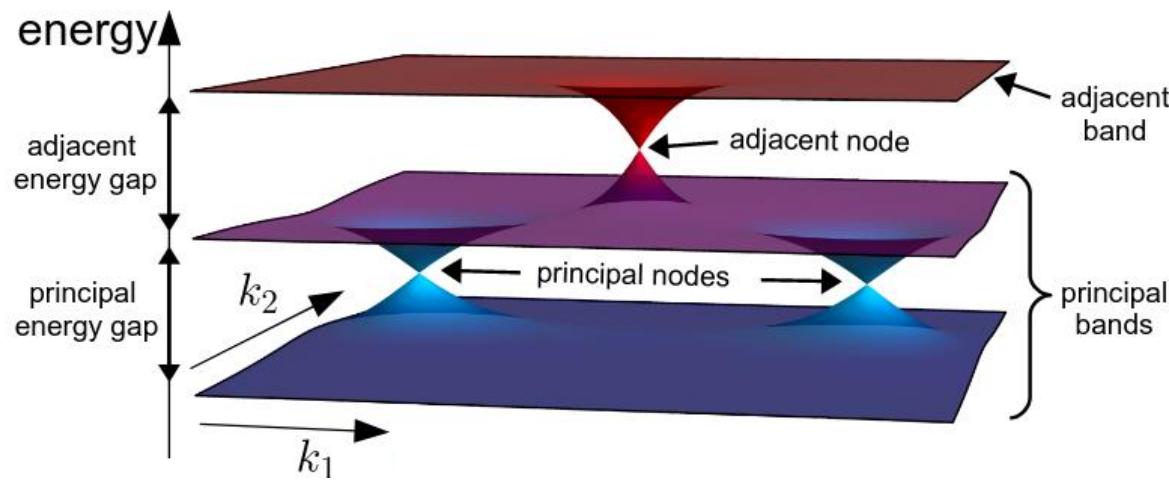
$$\pi_1 \left[\frac{\mathrm{SO}(3)}{S[\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)]} \right] = \mathbb{Q}$$


1+1+1 bands (multi-gap), symmetry making eigenstates real

Even spin and $\mathcal{IT} \longrightarrow H(\mathbf{k})$ real everywhere

Any spin and $C_2\mathcal{T} \longrightarrow H(\mathbf{k})$ real in plane

Braiding Weyl points



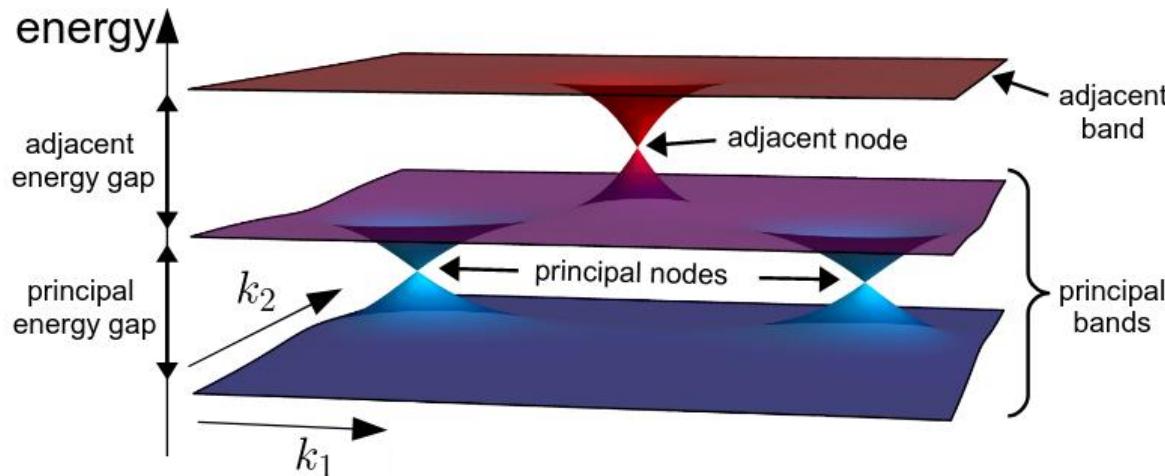
Bouhon et al. *Nature Physics* volume 16, pages 1137–1143 (2020)



Multi-gap states in $C_2\mathcal{T}$ -invariant plane in k -space

Characterized by Quaternion group!

Quaternion group



Bouhon et al. *Nature Physics* volume 16, pages 1137–1143 (2020)

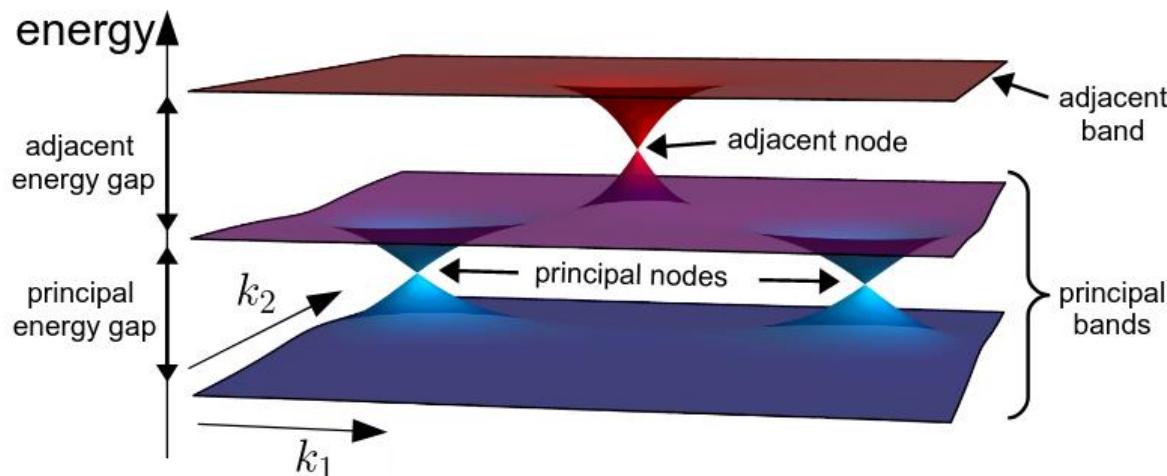
Physically:

- i: Node in first gap
- j: Node in both gaps
- k: Node in second gap
- -1: Double node

Braiding between nodes in different gaps!

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Quaternion group



Bouhon et al. *Nature Physics* volume 16, pages 1137–1143 (2020)

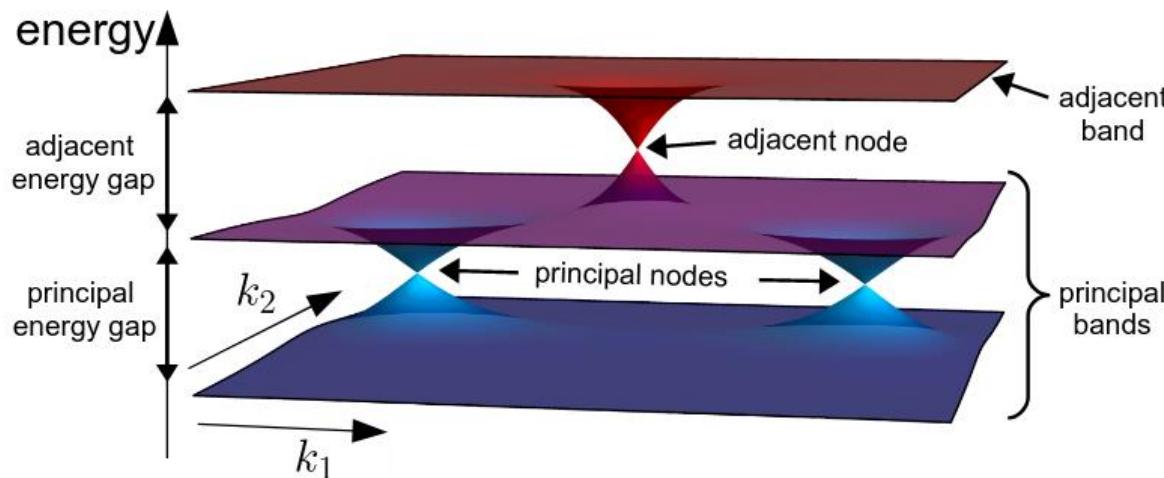
$$\mathbb{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$ij = k \neq ji = -k$$

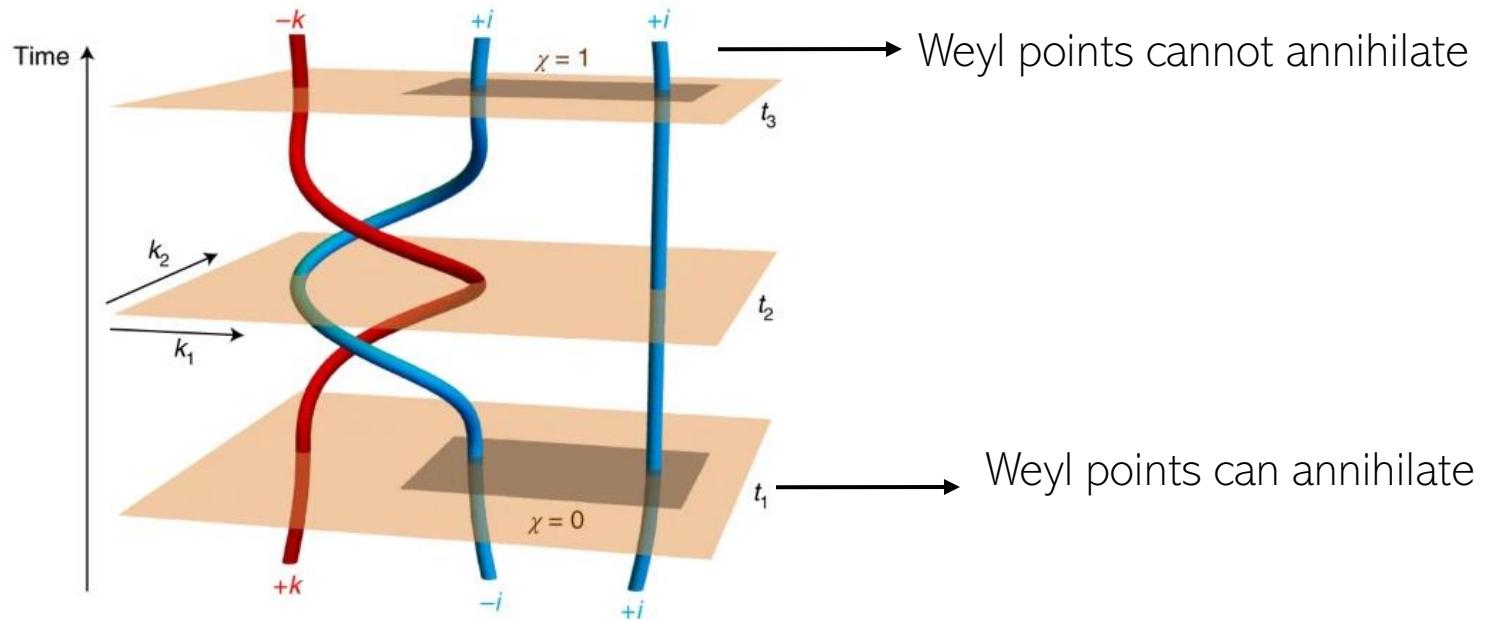
Non-abelian braiding!

	1	i	j	k
1	1	i	j	k
i	i	-1	k	- j
j	j	- k	-1	i
k	k	j	- i	-1

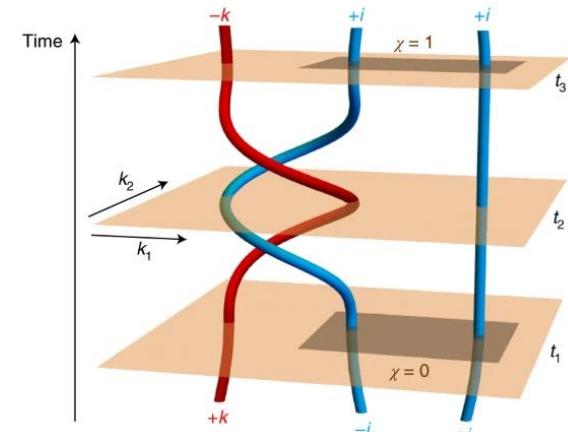
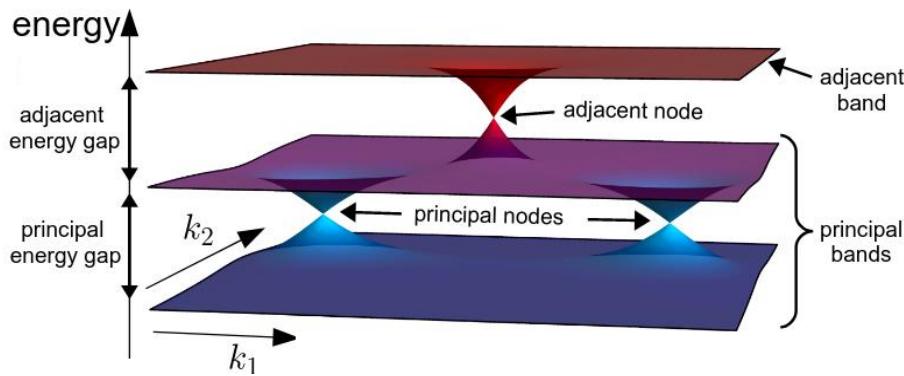
Braiding Weyl points



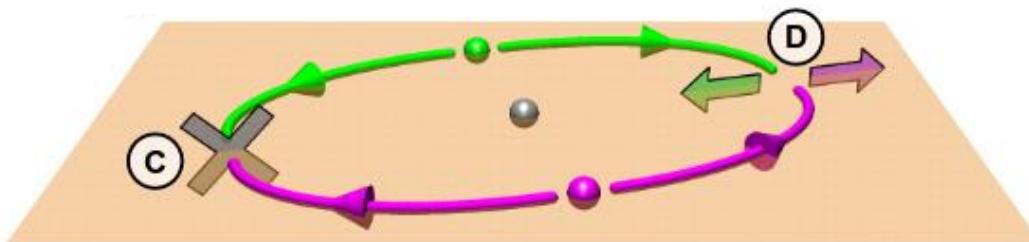
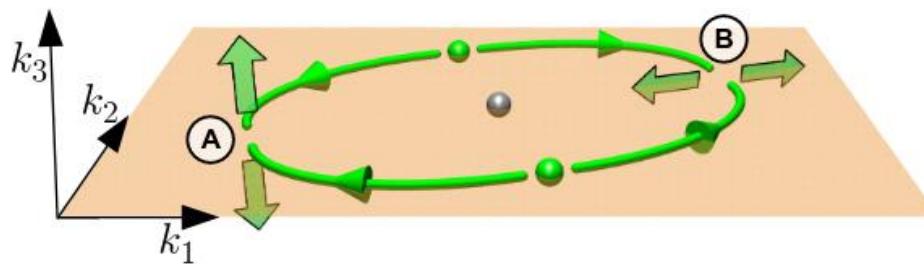
Bouhon et al. *Nature Physics* volume 16, pages 1137–1143 (2020)



Braiding Weyl points



Bouhon et al. *Nature Physics* volume 16, pages 1137–1143 (2020)

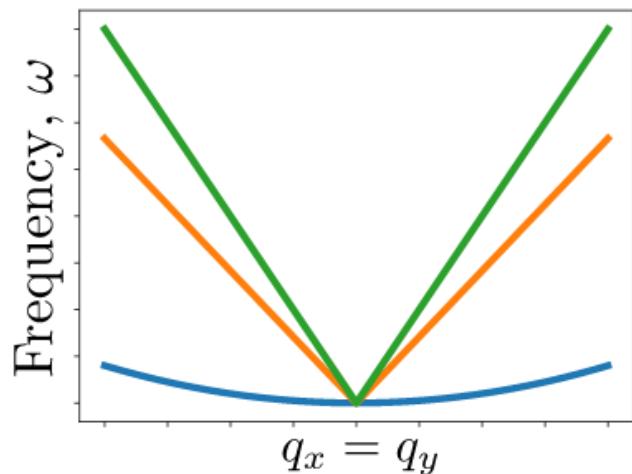


Application: Acoustic phonons in 2D

Vibrations of atoms (phonons): $D(\mathbf{q})v(\mathbf{q}) = \omega^2(\mathbf{q})v(\mathbf{q})$

Dynamical matrix

- Positive semi-definite
 - Local inversion symmetry



Phonon eigenstates

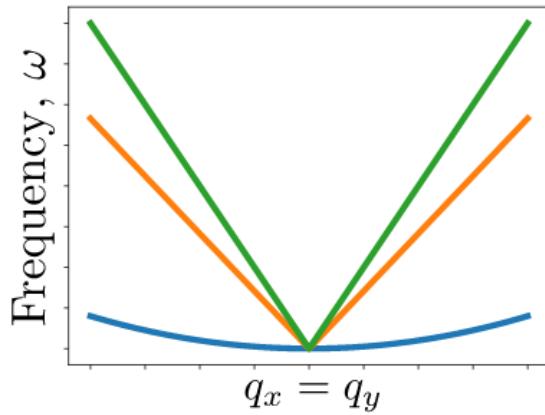
- Spin-0
- Automatically time-reversal invariant

Low-q dispersion in 2D

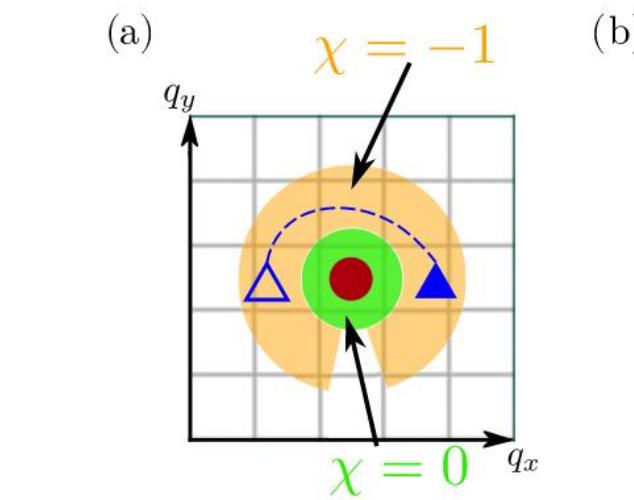
- Acoustic phonons (Goldstone bosons)
- 2 linear modes & 1 quadratic mode (flexural)

Charge?

Application: Acoustic phonons in 2D



Artificially split degeneracies

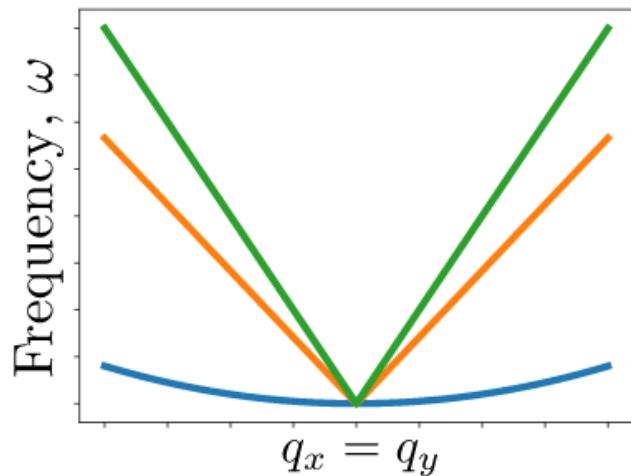


Lange et al., Phys. Rev. B 105, 064301 (2022)

Nodes characterized by Quaternions

Application: Acoustic phonons in

2D

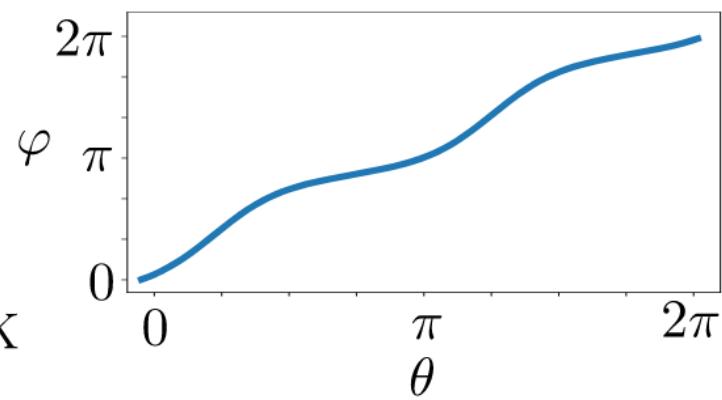
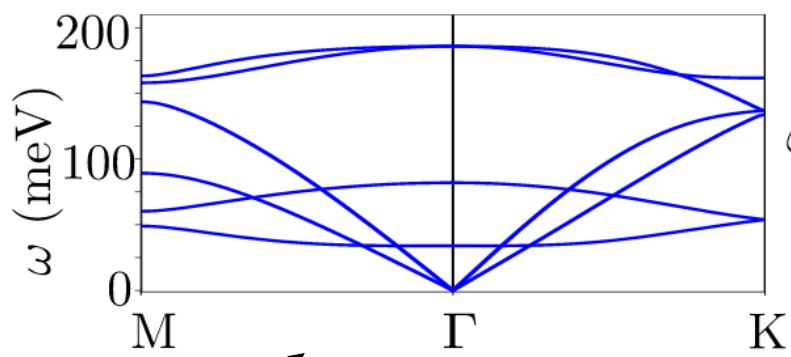
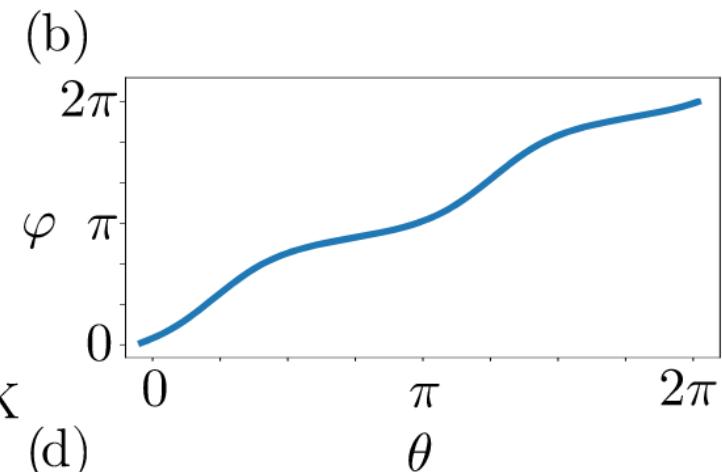
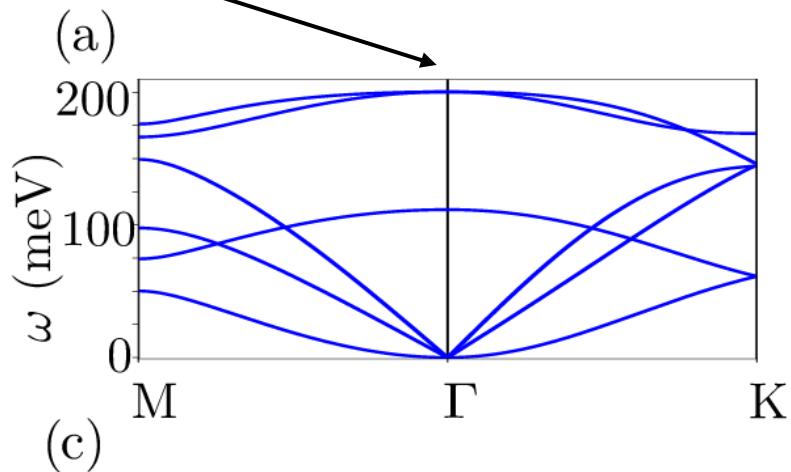


Lange et al., Phys. Rev. B 105, 064301 (2022)

Caveat: If system has \mathcal{I} and \mathcal{T} separately, possible charges at triple-point are ± 1

Application: Acoustic phonons in Graphene

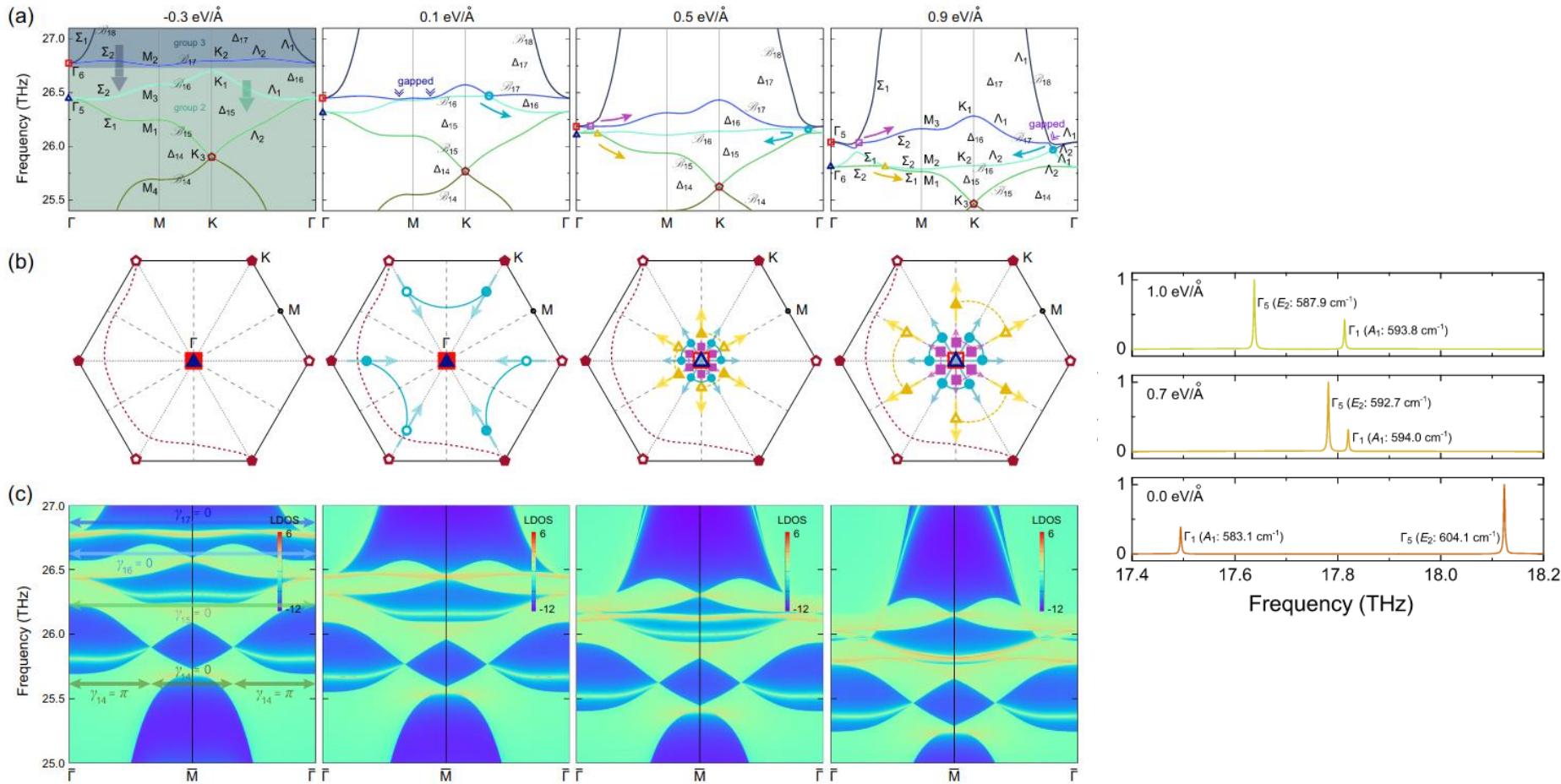
Free standing



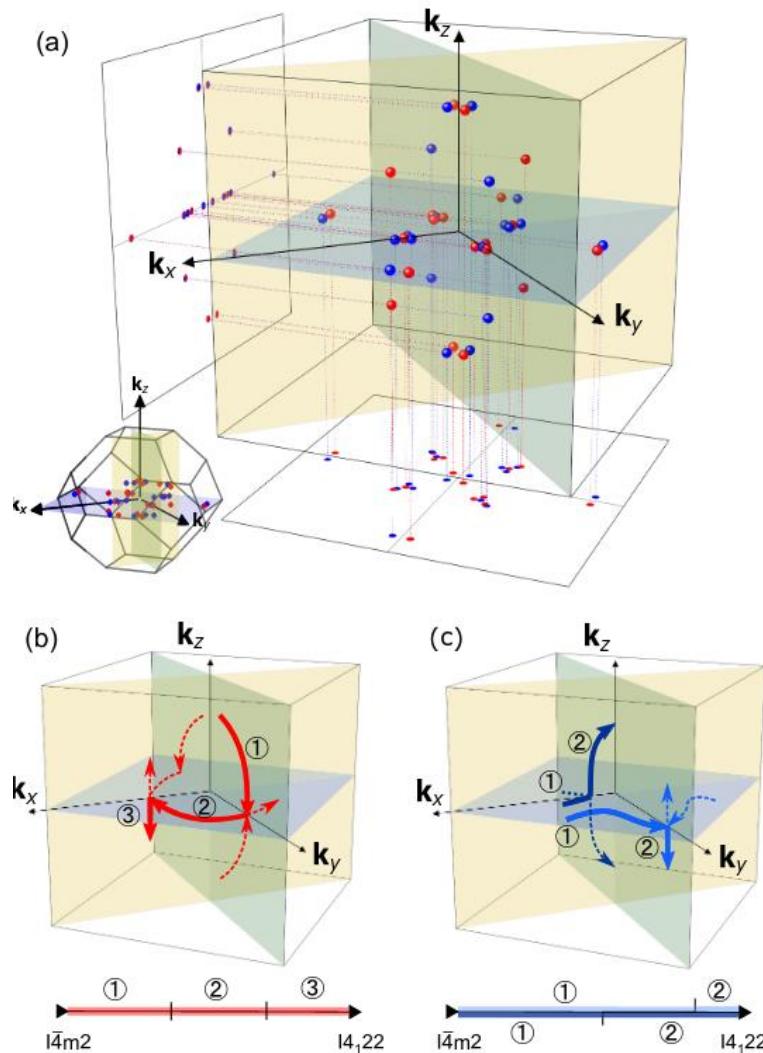
On substrate

Lange et al., Phys. Rev. B 105, 064301 (2022)

Application: Phonons manipulated by electric field in silicates



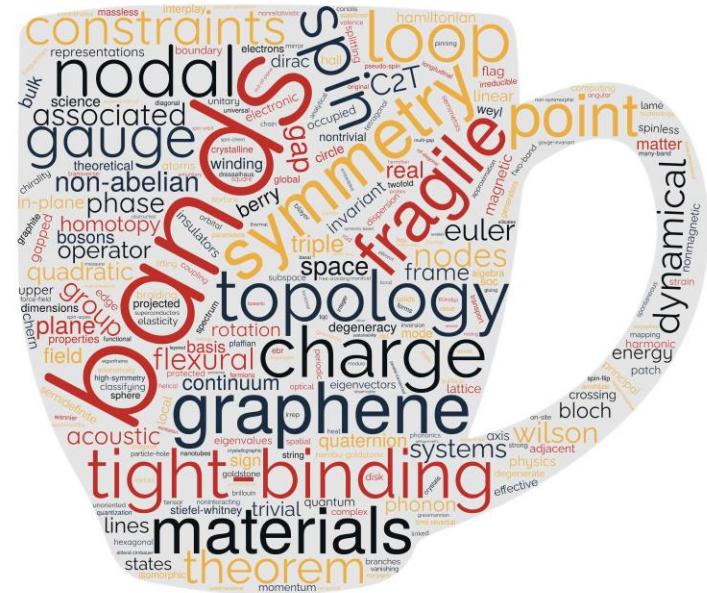
Application: Electron phase transitions manipulated by temperature in $\text{Cd}_2\text{Re}_2\text{O}_7$



Chen et al., Phys. Rev. B 105,
L081117 (2022)

Conclusion/Outlook

- Many topological phases in band structures
 - Weyl points feature interesting physics
 - Multi-gap topology can interact with Weyl points
 - Exists in phonons and electrons
 - Lots of exciting things to come!



TCM

Acknowledgements



UNIVERSITY OF
CAMBRIDGE
Cavendish Laboratory



Gunnar Lange
Cambridge



Adrien Bouhon
NORDITA/Cambridge



Robert-Jan Slager
Harvard/Cambridge



Peter Orth
Iowa State



Thais Trevisan
Iowa State /Berkley



Bartomeu Monserrat
Cambridge



Bo Peng
Cambridge



Siyu Chen
Cambridge



Dominik Hamara
Cambridge



Chiara Ciccarelli
Cambridge



GFL25@CAM.AC.UK