

Causality Bounds

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“Causal Effective Field Theories”

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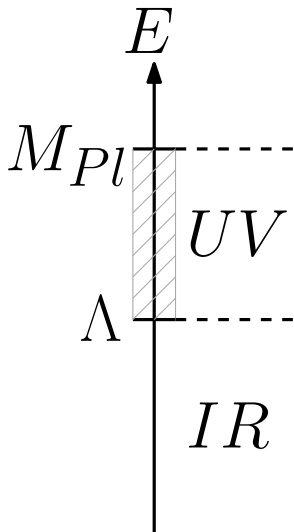
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EFTs: a bottom-up approach

EFT=Effective Field Theory

Effective description of low-energy physical phenomena

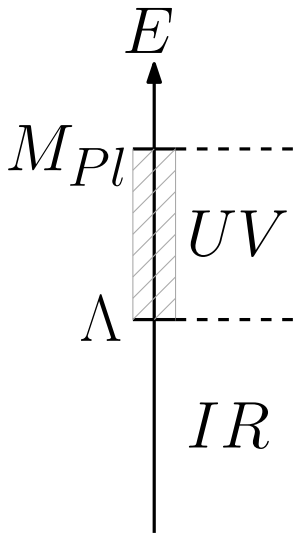
⇒ Not the whole picture but enough at low energies



- Infinite number of operators but only a few are relevant at low energy to a given order in the EFT expansion
- Operators constructed on symmetry principles
- **No need** to know the UV completion to compute IR observables

But is the low-energy EFT causal?

Causality of the EFT



- Symmetry arguments in the IR are **not sufficient** to ensure causality
- Causality needs to be imposed (by hand)

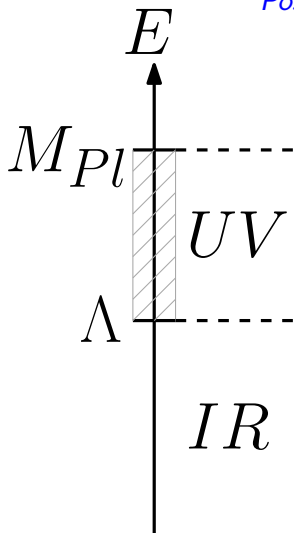


We get constraints on the Wilson coefficients of the low-energy EFT

Imposing IR causality: *Positivity bounds*

Well-studied and **successful** method:

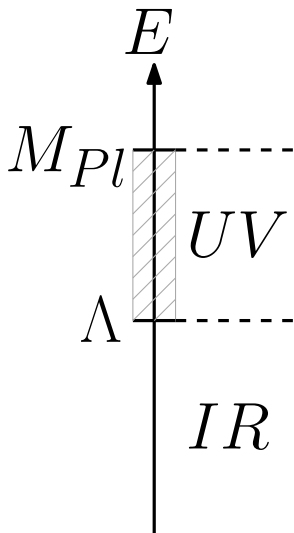
Positivity bounds \Leftrightarrow UV Causality \Leftrightarrow Analyticity



- Compute $2 \rightarrow 2$ scattering amplitudes
- Create positive bounded functions of these amplitudes
- Positivity \Rightarrow Bounds on the Wilson coefficients
- Extremise these bounds

But they are complicated and challenging to extend to (massless) gravitational theories and to arbitrary-curved backgrounds.

How to avoid these issues?



Issue: Many assumptions in the UV

Solution: Impose causality directly at the level of the IR theory (low energy)

Tool: Time delay

Criterion: No *resolvable* time advance

Semi-classical (WKB) approximation

Time delay = the delay of a scattered wave relative to a freely propagating wave.

Resolvable = measurable within the low energy EFT

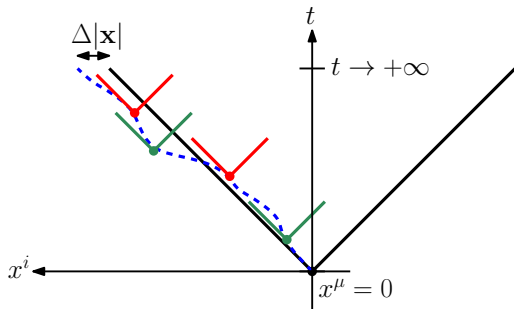
Causality vs Subluminality

For a generic EFT in Minkowski, the speed of propagation is modified by higher-derivative operators $c_s \neq 1$.

\Rightarrow Should we impose $c_s(x^\mu, \omega) \leq 1$?

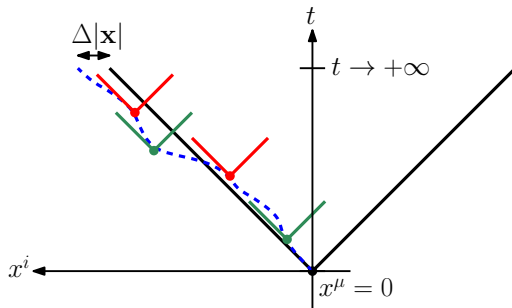
The answer is **NO**, this is too restrictive.

Subluminality \Rightarrow Causality *but* Causality \nRightarrow Subluminality



Allowed to propagate outside the forward Minkowski lightcone *locally* as long as this violation occurs in small regions of space.

Causality is ensured as long as the would-be violation is not *resolvable*



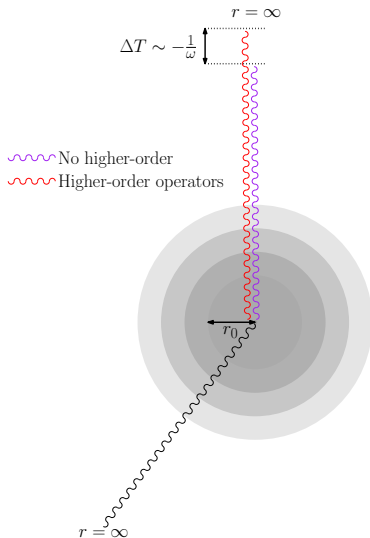
Is the distance $\Delta|\mathbf{x}|$ travelled *outside* of the forward Minkowski lightcone *resolvable*?

Nature of the probe: wave of momentum k and wavelength λ

Low-energy observer is probing with $k \ll \Lambda$ or $\lambda \gg 1/\Lambda$.

\Rightarrow Resolvable hence means $k\Delta|\mathbf{x}| \gtrsim 1$ or $\Delta|\mathbf{x}| \gtrsim \lambda$.

Causality criterion



So how shall we define resolvability?

Maximal allowed time advance given by **purely classical EFT validity** \Leftrightarrow

Resolvability

\Rightarrow Should we impose $\Delta T > -1/\Lambda?$
(where Λ is the cut-off)

\Rightarrow No, not Lorentz-invariant...

Causality violation criterion

$$\omega \Delta T_\ell \simeq -\mathcal{O}(1).$$

Causality constraint

$$\omega \Delta T_\ell > -1.$$

Low energy EFT

For simplicity's sake, we choose to study a **scalar** field theory with

- Shift symmetry $\phi \rightarrow \phi + c$
- Up to quartic operators in ϕ ($2 \rightarrow 2$ scattering)
- Up to dimension-12 operators

Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g_8}{\Lambda^4}(\partial\phi)^4 + \frac{g_{10}}{\Lambda^6}(\partial\phi)^2 [(\phi_{,\mu\nu})^2 - (\Box\phi)^2] \\ + \frac{g_{12}}{\Lambda^8}((\phi_{,\mu\nu})^2)^2 - g_{\text{matter}}\phi J,$$

with

- m : mass of the field ϕ
- Λ : cut-off of the low energy EFT
- g_{matter} : coupling strength to external matter
- J : arbitrary external source

Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g_8}{\Lambda^4}(\partial\phi)^4 + \frac{g_{10}}{\Lambda^6}(\partial\phi)^2 [(\phi_{,\mu\nu})^2 - (\Box\phi)^2] \\ + \frac{g_{12}}{\Lambda^8}((\phi_{,\mu\nu})^2)^2 - g_{\text{matter}}\phi J.$$

The small mass m is introduced because *Positivity bounds* require a mass gap. This breaks the shift-symmetry but does not induce any further symmetry-breaking operator at the quantum level. The previously derived *Positivity bounds* give

$$g_8 > 0, \quad g_{12} > 0, \quad g_{10} < 2g_8, \quad g_{12} < 4g_8,$$

$$-\frac{16}{3}\sqrt{g_8 g_{12}} < \underbrace{g_{10} < \sqrt{g_8 g_{12}}}_{\text{full crossing symmetry}}.$$

Hierarchy between scales of variation

WKB approximation states that

$$\lambda_{\text{pert}} \ll \lambda_{\text{bkg}} ,$$

where we consider perturbations $\psi = \phi - \bar{\phi}$ over an arbitrary background $\bar{\phi}$. In this regime, we have causality violation if

$$\omega \Delta T \sim \left(\frac{\lambda_{\text{bkg}}}{\lambda_{\text{pert}}} \right) \left[\int_{X \subset \mathbb{R}^{3+1}} (1 - c_s(\lambda_{\text{pert}})) + \mathcal{O} \left(\frac{\lambda_{\text{pert}}}{\lambda_{\text{bkg}}} \right) \right] \lesssim -1 .$$

- Subluminality $c_s < 1$ implies causality
- The first term (...) is large, but the integrand is small within the regime of validity of the EFT. It can even be negative and not necessarily lead to violations of causality.

Method to derive *Causality bounds*

Now that the theoretical grounds have been (hopefully) clarified, there is a **simple recipe** to derive *Causality bounds*

- 1 Choose the symmetry of the background
- 2 Choose the functional form of the background profile and compute the equations of motion of the perturbation while ensuring validity of the EFT and WKB approximation
- 3 Compute the time-delay and extremise the *Causality bounds* ($\omega \Delta T_\ell > -1$) by re-computing points 2 and 3
- 4 Compare with *Positivity bounds* (if they exist!)

Background symmetry

Choose the symmetry of the background profile $\bar{\phi}$

e.g. **Homogeneous**: $\bar{\phi} = \bar{\phi}(t)$

- Could be interesting for some problems
- Slightly too trivial in our case (reproduces some basic inequalities)

or **Spherically-symmetric**: $\bar{\phi} = \bar{\phi}(R) = \bar{\Phi}_0 f(r/r_0)$ where

- $\bar{\Phi}_0$: scale of the background ($[\bar{\Phi}_0] = +1$)
- r_0 : typical scale of variation of the background ($[r_0] = -1$)
- f : dimensionless function such that $f \sim \mathcal{O}(1)$
- R : dimensionless radial coordinate
- **Perturbation**: $\psi = \phi - \bar{\phi}$
 - Azimuthal symmetry \Rightarrow neglect φ -dependence
 - Expansion in partial waves: $\psi = \sum_{\ell} e^{i\omega t} Y_{\ell}(\theta) \delta\rho_{\ell}(R)$
 - $Y_{\ell}(\theta)$: spherical harmonics
 - $\delta\rho_{\ell}(R)$: radial perturbation

Regime of validity of the EFT

The advantage of turning to **dimensionless quantities** is that it makes the expansion parameters more tractable

- First, we have $\partial^n f / \partial R^n \sim \mathcal{O}(1)$

- Second, one can show that it is sufficient to require

$$\epsilon_1 \equiv \frac{\bar{\Phi}_0}{r_0 \Lambda^2} \ll 1, \quad \epsilon_2 \equiv \frac{1}{r_0 \Lambda} \ll 1, \quad \epsilon_\Omega \equiv \Omega \epsilon_2 \ll 1.$$

to ensure higher-order terms remain negligible.

- Finally, we want to truncate at $\mathcal{O}(\epsilon_1^4, \epsilon_1^2 \epsilon_2^2)$ so we additionally require

$$\epsilon_1^2 \ll \epsilon_2, \quad \epsilon_2^2 \ll \epsilon_1.$$

At the level of the phase shift/time delay, we have the following scaling

$$g_8 : \mathcal{O}(\epsilon_1^2), \quad g_{10} : \mathcal{O}(\epsilon_1^2 \epsilon_2^2), \quad g_{12} : \mathcal{O}(\epsilon_1^2 \epsilon_2^2 \Omega^2).$$

Equation of motion

In the **spherically-symmetric** case, we write down the linear equation of motion (eom) for the **perturbation** $\delta\rho_\ell(R)$.

- Solve perturbatively order by order in the scale Λ
- Remove higher-order radial derivatives (> 2) using lower-order eom

\Rightarrow Get a second-order equation eom for the perturbation

$$\delta\rho_\ell''(R) + A\delta\rho_\ell'(R) + B\delta\rho_\ell(R) = 0.$$

- Field-redefine the perturbation to remove the friction term

$$\chi_\ell''(R) + W_\ell \chi_\ell(R) = 0, \quad W_\ell = \frac{(\omega r_0)^2}{c_s^2(\omega^2, R, \ell)} \left(1 - \frac{V_{\text{eff}}(\omega^2, R, \ell)}{(\omega r_0)^2} \right),$$

where

- $c_s^2 = 1 + \mathcal{O}(g_i)$
- $V_{\text{eff}} = \frac{\ell(\ell+1)}{R^2} + \mathcal{O}(g_i)$

Regime of applicability of WKB

Equation of motion

$$\chi_\ell''(R) + (\omega r_0)^2 \hat{W}_\ell \chi_\ell(R) = 0.$$

n^{th} -order approximation to the exact solution χ_ℓ

$$\chi_\ell^{(n)}(R) \propto \text{Exp} \left[i(\omega r_0) \int_{R_t}^R \sum_{j=0}^n \delta_{WKB}^{(j)} dR \right], \text{ where } \hat{W}_\ell(R_t) = 0.$$

WKB series

$$\delta_{WKB}^{(0)} = \sqrt{\hat{W}_\ell}, \dots \text{ with } \delta_{WKB}^{(j)} \sim \mathcal{O} \left(\left(\frac{\lambda_{\text{pert}}}{\lambda_{\text{bkg}}} \right)^j \right) \sim \mathcal{O}((\omega r_0)^{-j}).$$

Now we can establish the validity of the WKB approximation

- Small error:

$$\Delta \chi_\ell^{(n)} = \frac{\chi_\ell - \chi_\ell^{(n)}}{\chi_\ell} \sim \int_{R_t}^R \delta_{WKB}^{(n+1)} dR \sim \mathcal{O}((\omega r_0)^{-(n+1)}) \ll 1$$

- Convergence: $\int_{R_t}^R \delta_{WKB}^{(n+1)} dR \ll \int_{R_t}^R \delta_{WKB}^{(n)} dR$

Here, one needs to specify the functional form of the background profile

$$\text{e.g. } f(R) = \sum_{n \geq 0}^{n_{\max}} a_{2n} R^{2n} e^{-R^2}.$$

Target

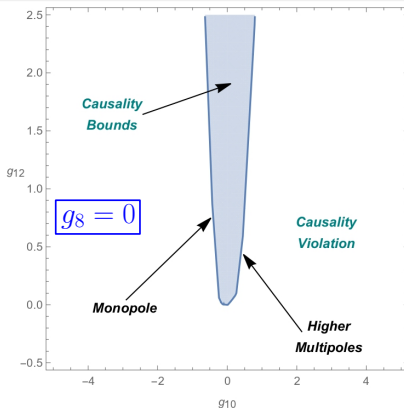
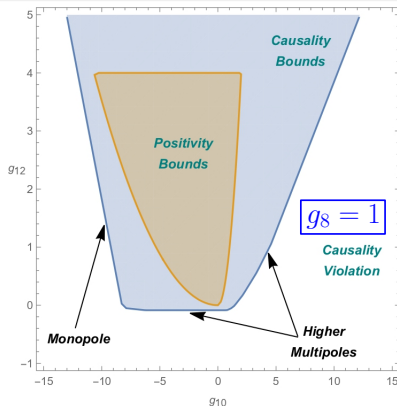
Solve $(\omega \Delta T_\ell) = -1$.

\Rightarrow Gives boundary between causality-violating and allowed regions in parameter space

Constraints

- EFT regime of validity
- WKB regime of applicability

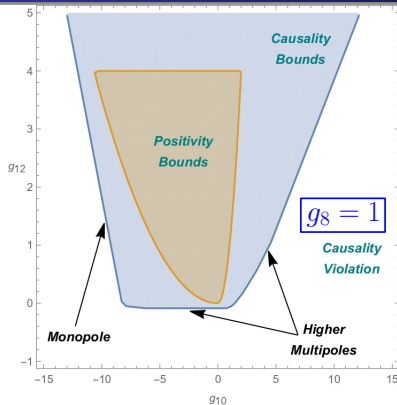
Causality bounds vs Positivity bounds



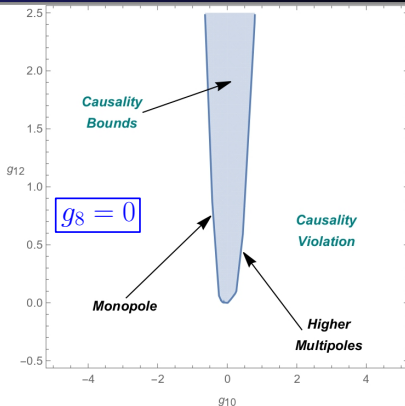
Lagrangian

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Causality bounds vs Positivity bounds



- $g_8 = 1 \Leftrightarrow$ Redefine Λ
- Compact *Causality bounds*
- Excellent agreement with *Positivity bounds*



- $g_8 = 0 \Leftrightarrow$ Impose galileon symmetry
- Ruling out quartic galileon (as a causal uncoupled low-energy EFT)

Conclusion

Take-home message

Requiring causality for low-energy EFTs places tight **compact** bounds on the Wilson coefficients (independently of UV completion)

Our intuition could have told us:

Causality $\Rightarrow c_s \lesssim 1 \Rightarrow$ One-sided bounds

But we get **two-sided** bounds and a **much richer** constraint structure than expected!

Comparison with *Positivity bounds*

Causality bounds not as constraining as *Positivity* ones (**just yet!**)
But results are **remarkably close** and we provide a **proof of principle** that the method works.

What's to come?

Possible improvements

- Less-symmetric background could lead to tighter bounds
- One could explore a wider range of functions
- A solid extremisation procedure is yet to be applied

Going beyond *Positivity bounds*

- Bound **higher-order operators** in the field ϕ^n with $n \geq 5$
(do *not* contribute to tree-level $2 \rightarrow 2$ *Positivity bounds*)
- Extend to any spin, e.g. vectors (ongoing work)
- Apply *Causality bounds* to **gravitational theories** and **curved backgrounds**
 \Rightarrow Cosmological and black hole gravitational bounds
(no S-matrix, broken Lorentz symmetry for *Positivity bounds*)
- Constrain potentials $V(\phi)$ which is useful for **inflation**
(not possible for *Positivity bounds*)

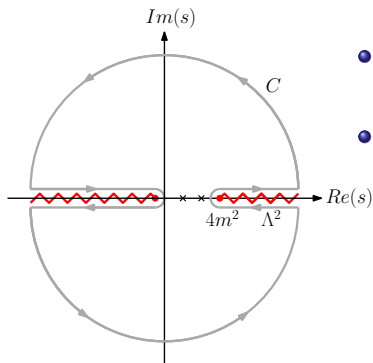
Thank you!

Thank you very much
for your attention!

Positivity bounds: How does it work?

General idea:

- Compute some integrals $\int_{4m^2}^{\infty}$
- Cut them into $\int_{4m^2}^{\Lambda^2}$ (IR) + $\int_{\Lambda^2}^{\infty}$ (UV)
- How to compute the UV part?
 - \Rightarrow Deform into an infinite contour in the complex s -plane
 - \Rightarrow Use **analyticity**
 - \Rightarrow Get relations between IR and UV quantities without even knowing the UV completion!

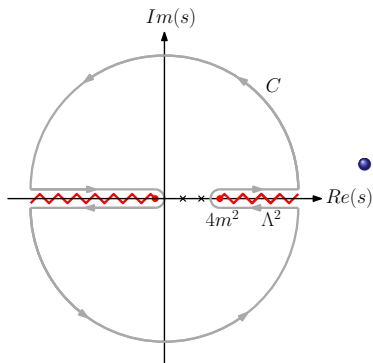


Very **successful** indeed, **but...**

Positivity bounds: What are the cons?

Very **successful** indeed, **but...**

- Need to make a number of **assumptions** on the UV completion: unitarity, locality, causality, Poincaré symmetry, and especially (full) crossing symmetry



- Challenging to extend to (massless) gravitational theories and to arbitrary-curved backgrounds
 - Broken Lorentz symmetry
 - Lack of an S-matrix
- \Rightarrow Analyticity is hard to generalise and hence dispersion relations are not straightforward to build

Scattering time delay

If a state with energy ω is scattered in an event described by an S-matrix, the time delay reads

$$\Delta T = -i \langle in | \hat{S}^\dagger \frac{\partial}{\partial \omega} S | in \rangle = 2 \frac{\partial \delta}{\partial \omega},$$

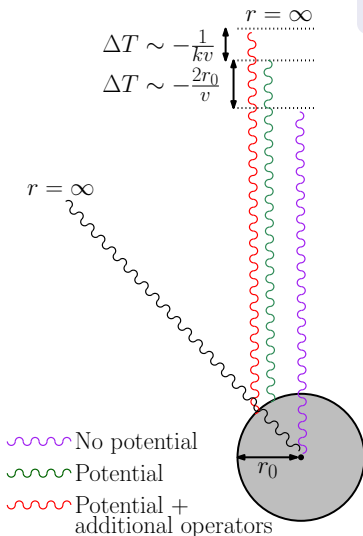
where δ is the eigenvalue $S|in\rangle = e^{2i\delta}|in\rangle$ and will later be identified with the *phase-shift*.

Spherically-symmetric backgrounds: the S-matrix diagonalises in multipoles ℓ and one can define the time-delay for each multipole

$$\Delta T_\ell = 2 \left. \frac{\partial \delta_\ell}{\partial \omega} \right|_\ell.$$

Note that the well-known eikonal approximation is done at large ℓ and fixed impact parameter $b = (\ell + 1/2)\omega^{-1}$.

Causality criterion



So how shall we define resolvability?

- Example: Monopole ($\ell = 0$) with speed v and momentum k scattering in a spherically-symmetric potential vanishing for $r > r_0$ can experience a time advance due to
 - Uncertainty principle $\Delta T \sim \omega^{-1}$ where $\omega \sim \mathcal{O}(kv)$
 - Wave scattering at the boundary $r = r_0$

$$\Delta T_{\ell=0} \geq \underbrace{-\frac{2r_0}{v}}_{\text{Potential}} \underbrace{-\frac{1}{kv}}_{\text{Uncertainty}}$$

Note on the first-order relation to the phase-shift

When no higher-order derivative operators are included

- $\frac{\partial c_s}{\partial \omega} = 0$
- $\delta^{EFT} \propto \omega$
 \Rightarrow Causality violation for $\delta^{EFT} < -1$

However, when including such operators,

- c_s becomes ω -dependent
- δ^{EFT} becomes non-linear $\delta^{EFT} = \sum_{n=n_{min}}^{n_{max}} a_n \omega^{2n+1}$ (where n_{min} can be negative)
 \Rightarrow There is no clear causality violation criterion in terms of the phase-shift
 \Rightarrow Go back to using $\omega \Delta T < -1$

- **Monopole ($\ell = 0$):** $R_t = 0$ and one can compute arbitrarily high WKB corrections (here we need up to $\delta_{WKB}^{(4)}$)
 \Rightarrow Can achieve small corrections for $(\omega r_0) \sim 3$
- **Higher multipoles ($\ell > 0$):**
 - Langer trick to better capture the low- ℓ regime
 - $R_t > 0$: finite value
 - $\chi_\ell^{(0)}$ can be computed
 - Corrections are **hard** to compute but we can ensure their smallness by taking a large WKB suppression: $(\omega r_0) \gg 1$

What we found

- $\ell = 0$: lower bounds (\sim triple crossing symmetry in the UV for *Positivity bounds*)
- $\ell > 0$: higher bounds ($\sim s \leftrightarrow u$ dispersion relations in the UV for *Positivity bounds*)

Comparison with *Positivity bounds*

Causality bounds not as constraining as *Positivity* ones (just yet!)
But results are remarkably close and we provide a proof of principle that the method works.