

Coloured ~~Condensed~~ matter physics

Today's talk

Dark matter
Dark energy



Zhao Zhang
25.10.2023



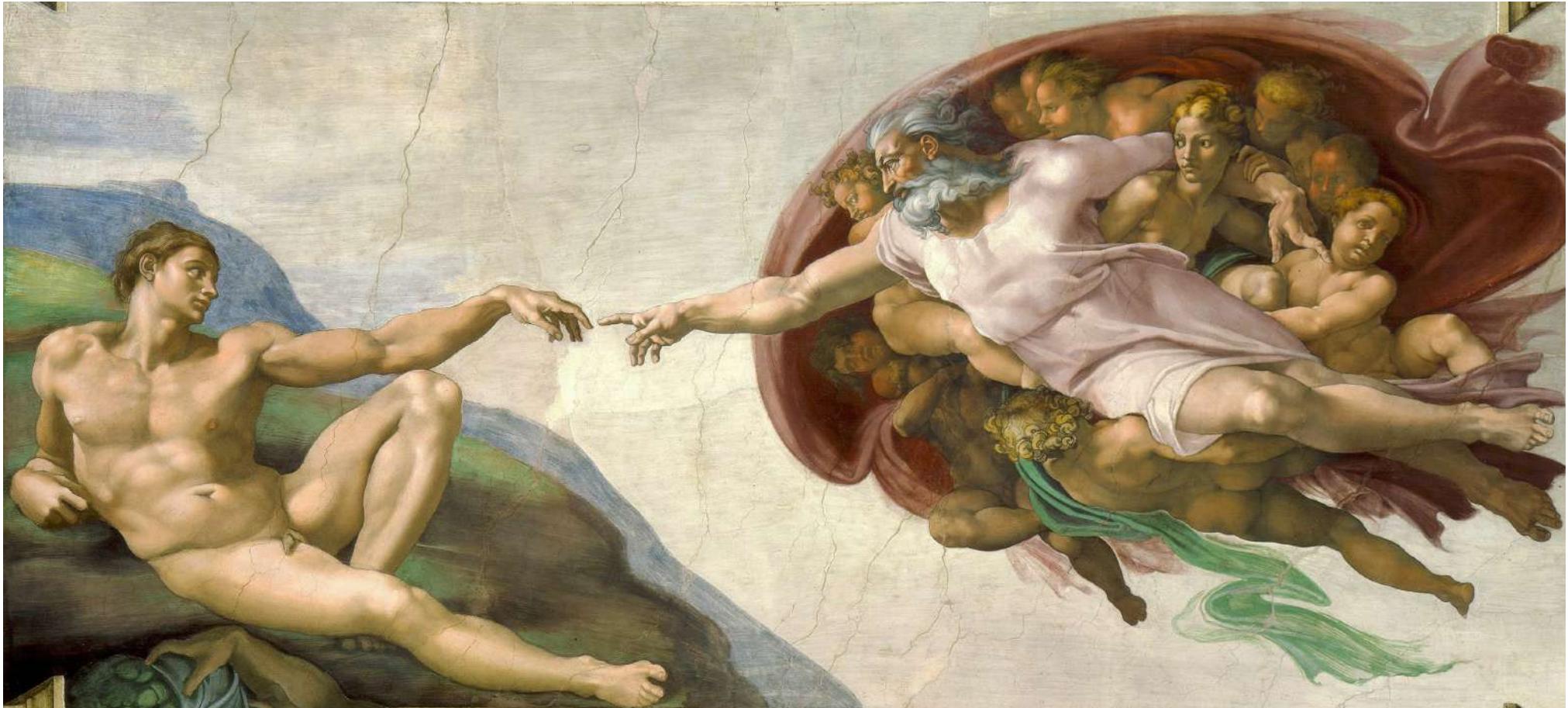
UiO :



Please excuse Wednesday. She's allergic to color.

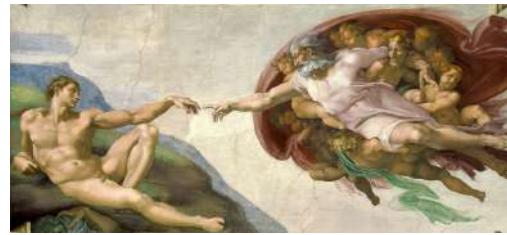
Hierarchy of theoretical research

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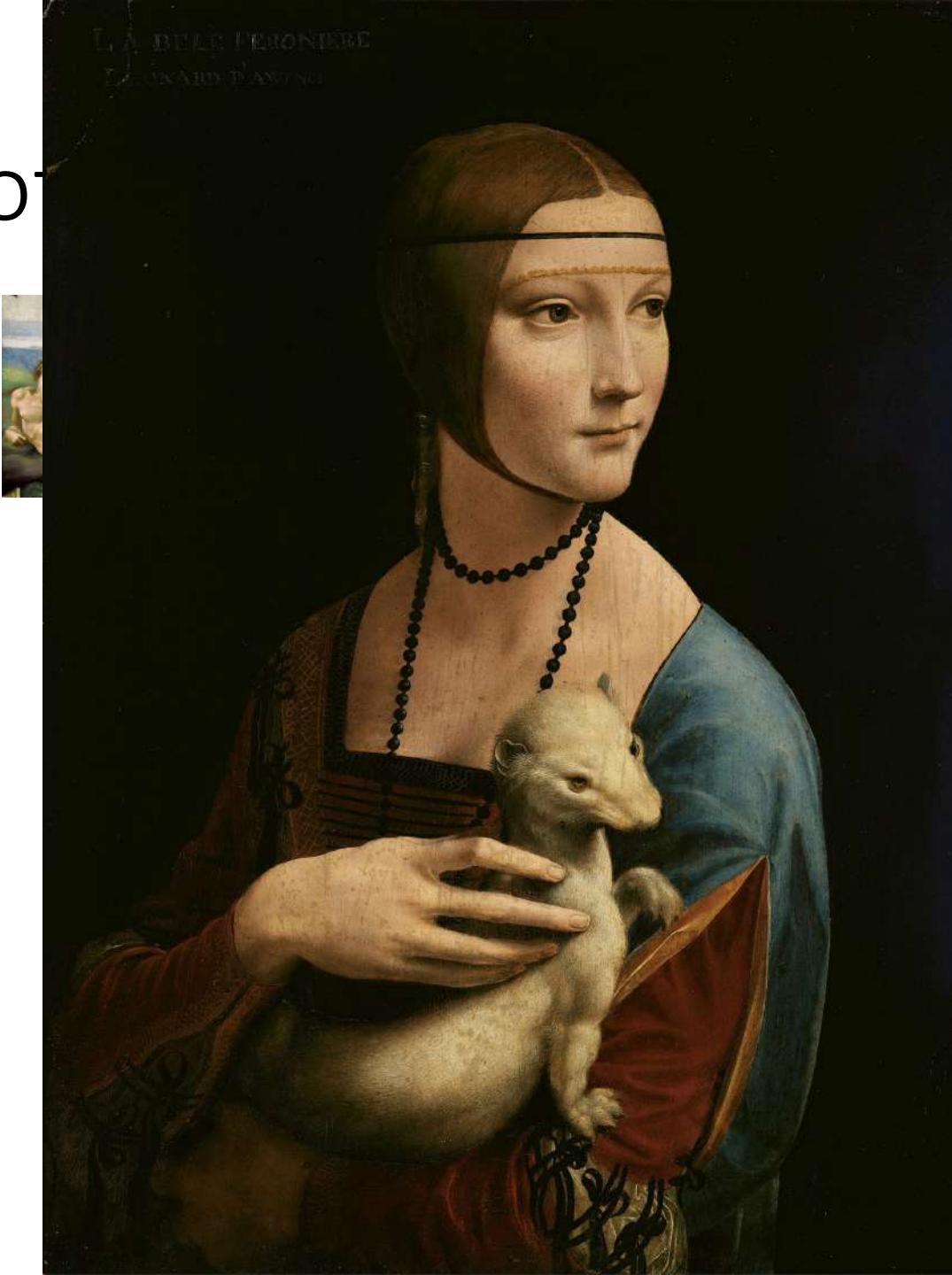
Hierarchy of theoretical research

1. History/religion/allegory



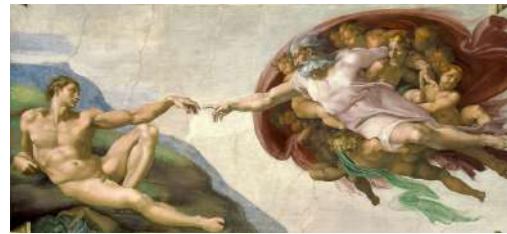
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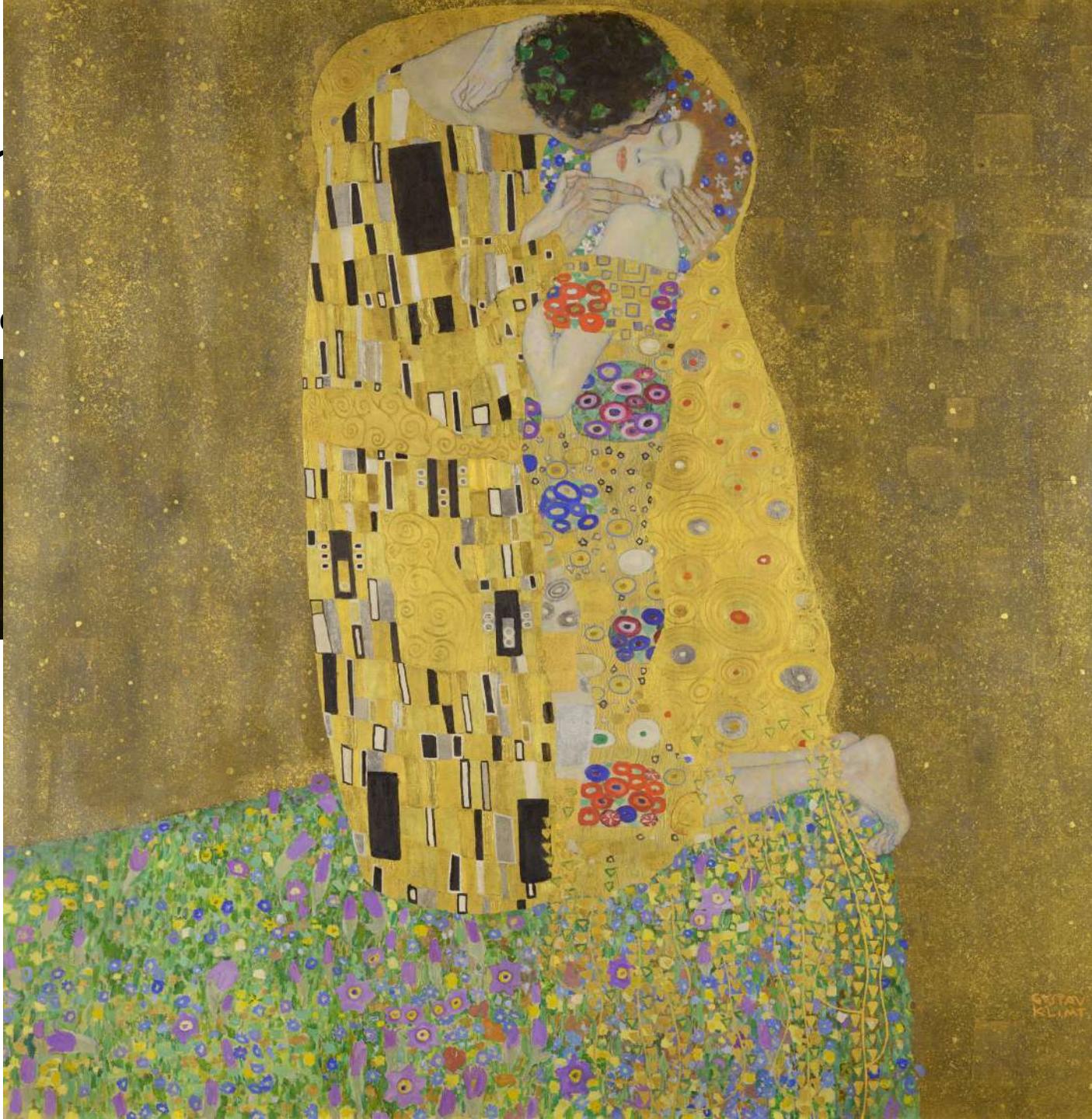


2. Potrait

Hier

1. History/re

2. Potrait



Hierarchy of theoretical research

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2. Potrait



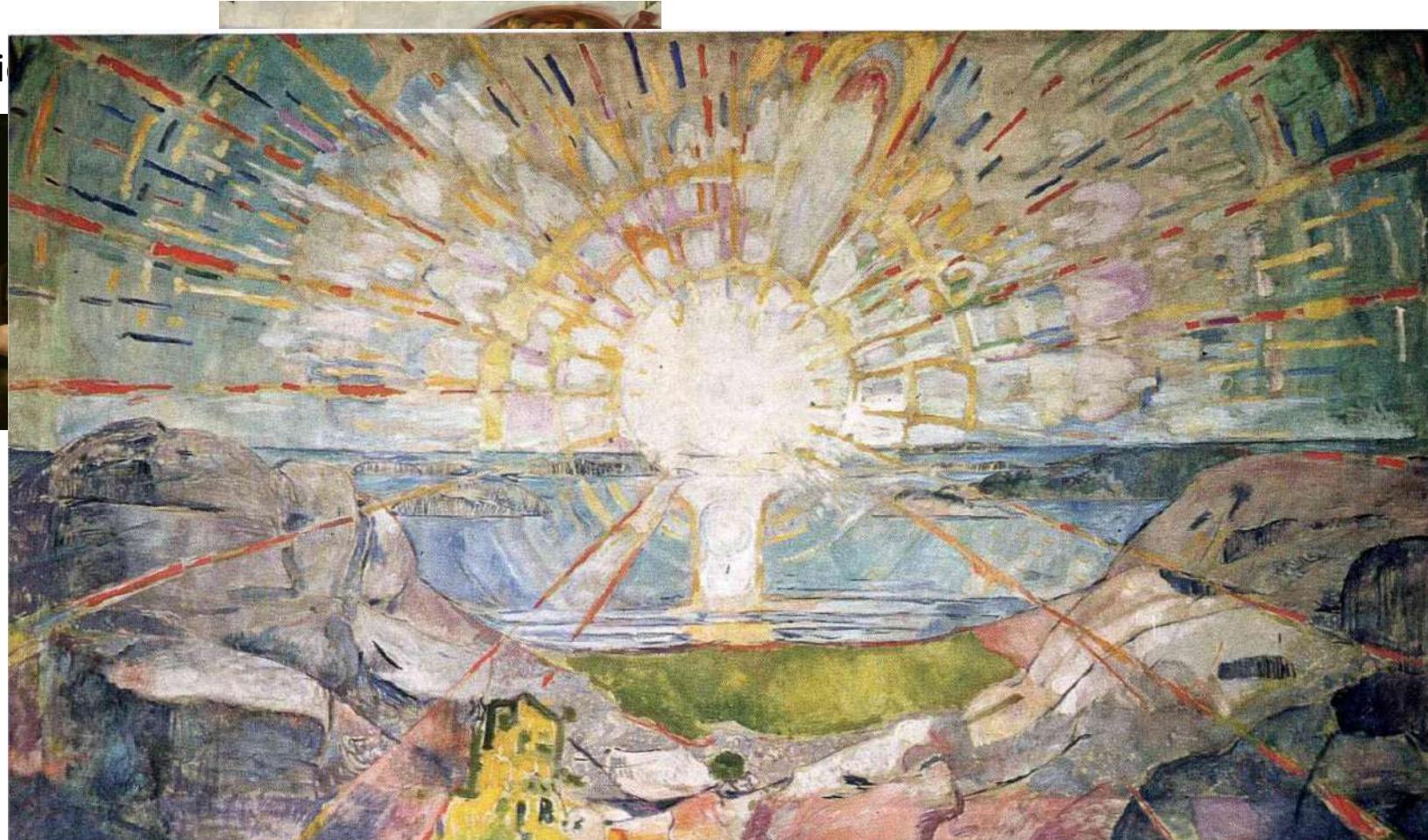
3. Genre

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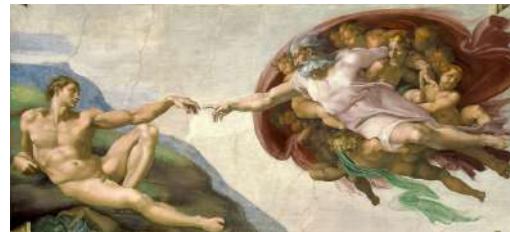
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Hierarchy of theoretical research

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4. Landscape

Hierarchy of genres

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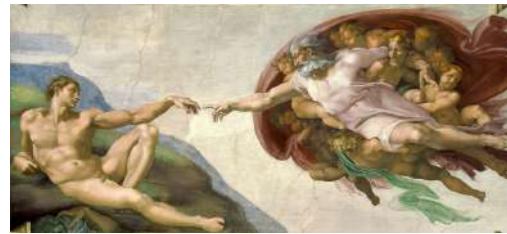


4. Landscape

5. Still life

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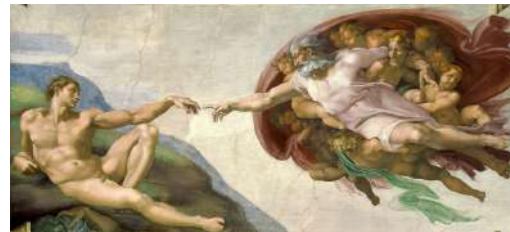


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1. Discovery of new a law or equation
e.g.: KPZ equation

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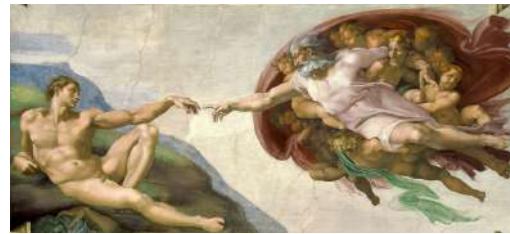
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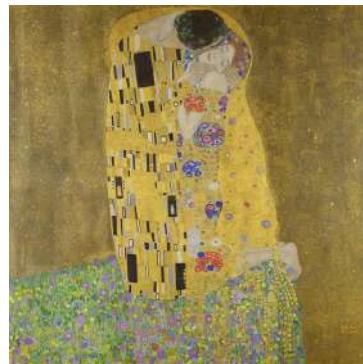
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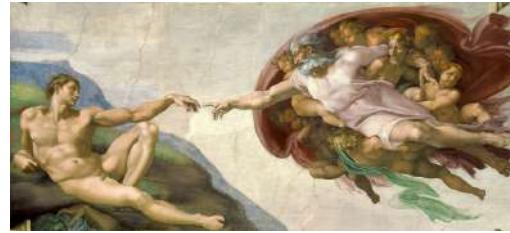
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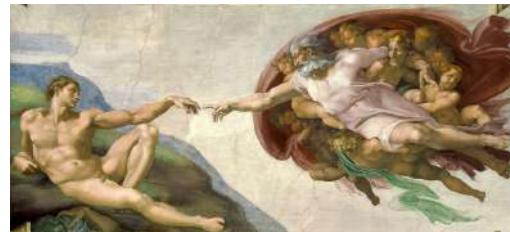
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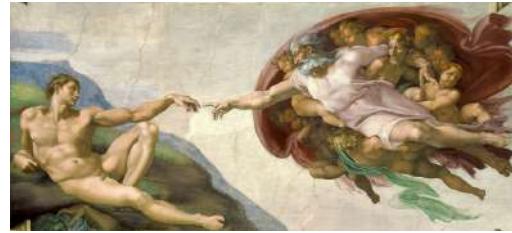
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5. Generalization of known model
e.g.: parafermion

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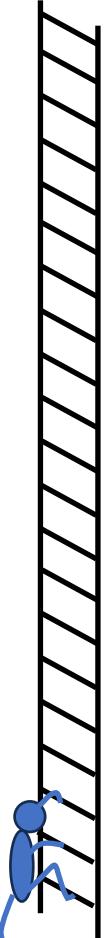
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Few- to many-body physics

High energy theory

Condensed matter theory

Few- to many-body physics

High energy theory

- Spacetime and gauge symmetry

Condensed matter theory

Few- to many-body physics

High energy theory

- Spacetime and gauge symmetry

Condensed matter theory

- Lattice and local Hilbert space

Few- to many-body physics

High energy theory

- Spacetime and gauge symmetry
- Diagonalize Lagrangian

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- vev., particle content, Feynman rules

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- Lattice and local Hilbert space
- Diagonalize Hamiltonian
- Ground state, quasiparticles, S-matrix

Few- to many-body physics

High energy theory

- Spacetime and gauge symmetry
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- vev., particle content, Feynman rules
- Scattering amplitudes

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Few- to many-body physics

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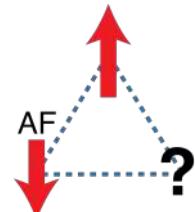
Condensed matter theory

- Lattice and local Hilbert space
- Diagonalize Hamiltonian
- Ground state, quasiparticles, S-matrix
- Correlation functions
- Strongly correlated

Challenges

- Frustration

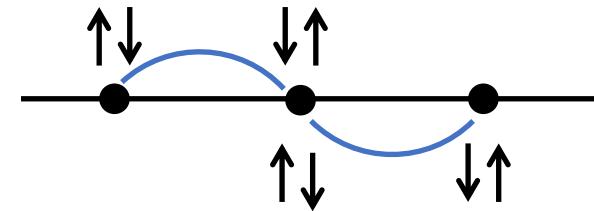
- Geometric
 - Quantum



- Heisenberg Hamiltonian $h_{i,i+1} = \vec{S}_i \cdot \vec{S}_j$

lowest energy eigenstate $|\uparrow_i \downarrow_{i+1}\rangle - |\downarrow_i \uparrow_{i+1}\rangle$

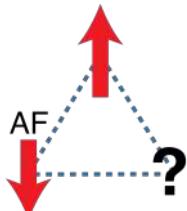
- Fluctuation – mean field picture breakdown



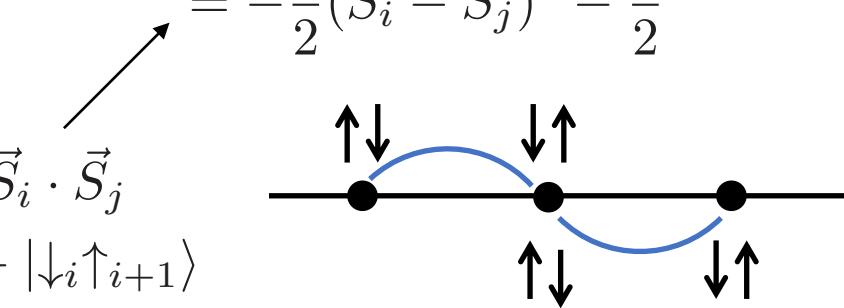
$$\langle \sigma_x \sigma_y \rangle \neq \langle \sigma_x \rangle \langle \sigma_y \rangle$$

Challenges

- Frustration
 - Geometric
 - Quantum



$$= -\frac{1}{2}(\vec{S}_i - \vec{S}_j)^2 - \frac{3}{2}$$

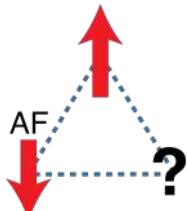


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Challenges

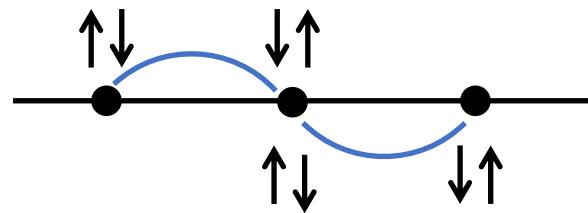
- Frustration
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$$= -\frac{1}{2}(\vec{S}_i - \vec{S}_j)^2 - \frac{3}{2}$$

\nearrow

$h_{i,i+1} = \vec{S}_i \cdot \vec{S}_j$
lowest energy eigenstate $|\uparrow_i \downarrow_{i+1}\rangle - |\downarrow_i \uparrow_{i+1}\rangle$



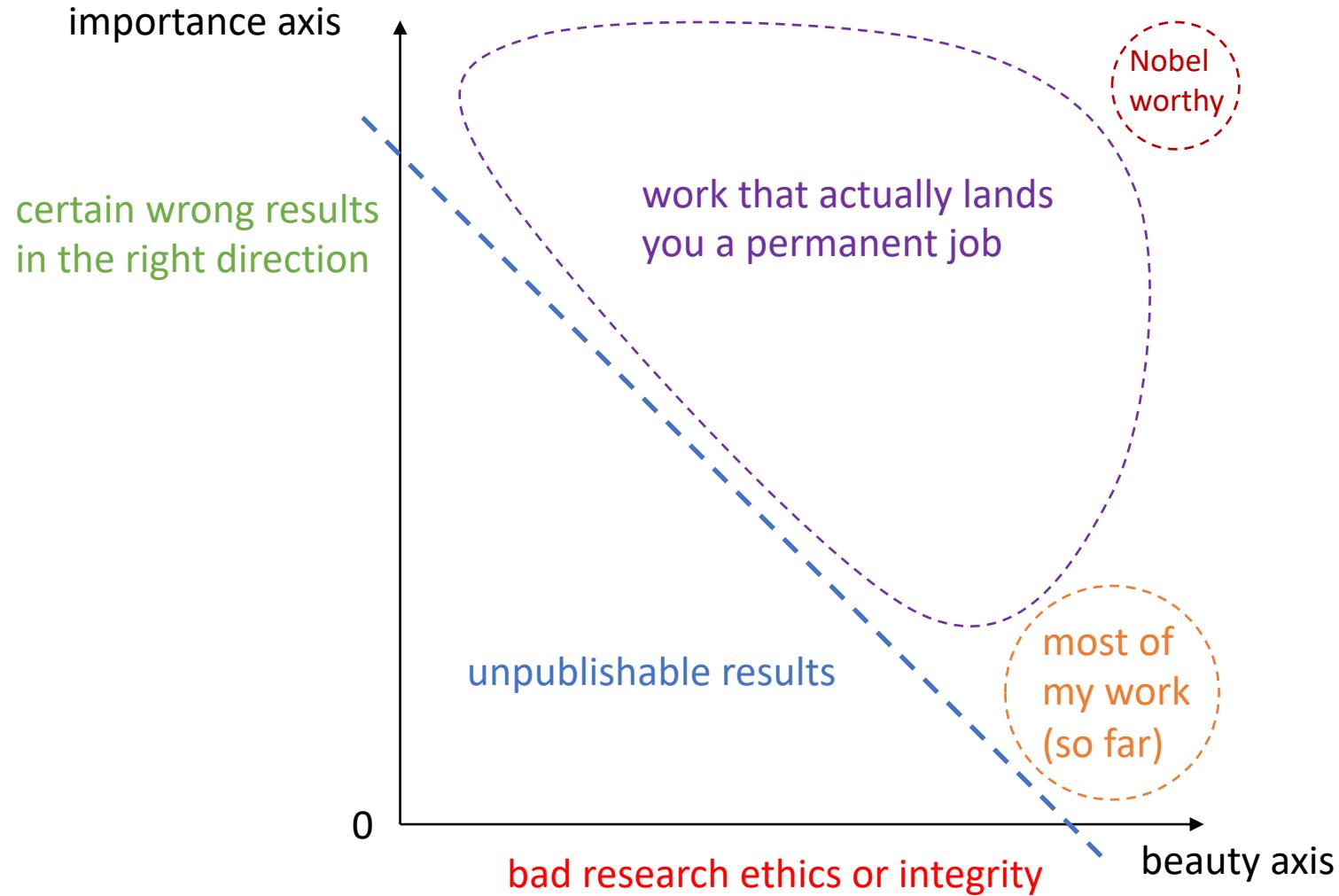
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- Fluctuation – mean field picture breakdown

On the bright side:

- Low threshold – combinatorics, linear algebra, quantum mechanics, Mathematica, Adobe Illustrator
- Low risk of being scouped!

The beauty-importance diagonal



Outline

- (incomplete) survey of models with colour d.o.f.
 - $SU(N)$ spin chain
 - Simplex matryoshka
 - Beyond area-law ground states via color
- Colored loop models and sub-area-law entanglement
 - Hilbert space fragmentation
 - Tower of exact excited states
 - Topological entanglement entropy
 - Non-intersecting bicolor loop models
 - Balasubramanian–Lake–Choi model
- Summary and outlook

Colored spin chain

- Heisenberg chain $H = \sum_{i=1}^L P_{i,i+1} \rightarrow H = \sum_{i=1}^L [X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}]$ XXZ chain
- Multi-component generalization

$$H = \sum_{i=1}^L P_{i,i+1} \rightarrow H + \sum_{i=1}^L \left[\sqrt{4\Delta^2 - 1} \sum_{a,b=1}^s \Delta_{ab} e_i^{(aa)} e_{i+1}^{(bb)} \right], \quad \Delta_{ab} = \text{sgn}(a-b) + \frac{2}{s}(b-a)$$

Sutherland, Phys. Rev. B, 75'

Babelon, de Vega, Viallet, Nucl. Phys. B, 82'

- Partial integrability

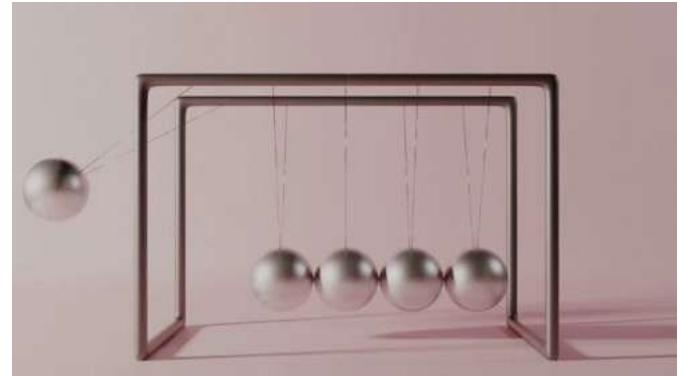
$$H = \sum_{i=1}^L [P_{i,i+1} + 2\Delta C_{i,i+1}]$$

Kiwata, Akutsu, Sato, J Phys. Soc. Japan, 94', 95', 96'

$$\begin{aligned} P_{i,i+1} &= \sum_{a,b=1}^s e_i^{(ab)} \otimes e_{i+1}^{(ba)}, & P_{i,i+1}(v_i \otimes v_{i+1}) &= v_{i+1} \otimes v_i \\ C_{i,i+1} &= \sum_{a=1}^s e_i^{(aa)} \otimes e_{i+1}^{(aa)} & (e^{(ab)})_{cd} &= \delta_c^a \delta_d^b \end{aligned}$$

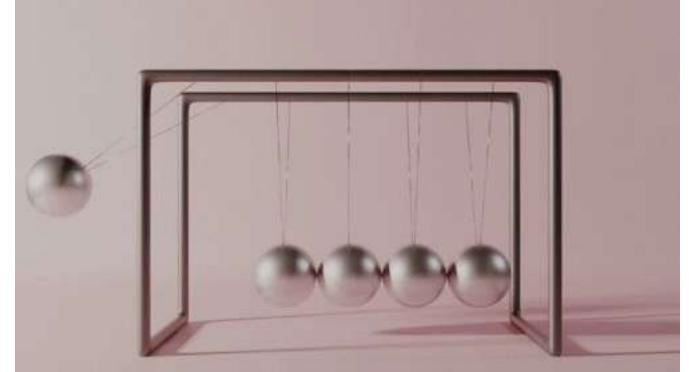
“ $\partial\partial$ ” integrability breaking

$$|v\rangle = \sum_{Q \in \mathfrak{S}_n} \sum_{1 \leq x_{Q1} < \dots < x_{Qn} \leq L} \psi_Q(\mathbf{x}) |\mathbf{x}\rangle,$$



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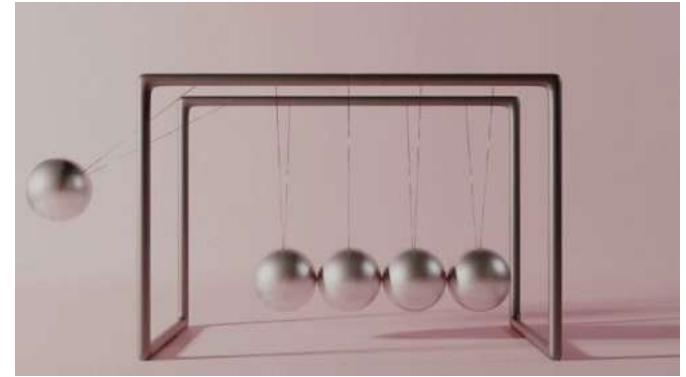
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$A_{Q,P}$ are $n! \times n!$ matrices with column vectors ξ_P

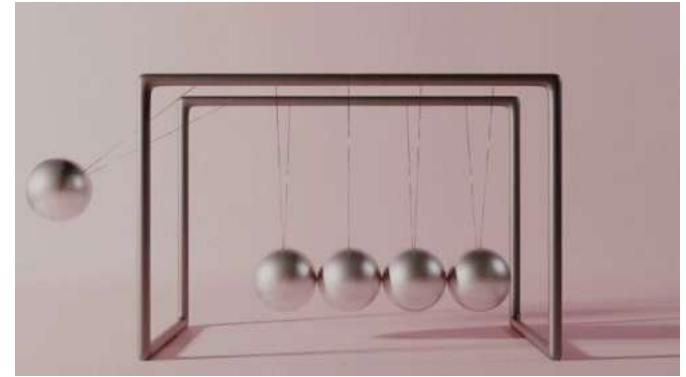


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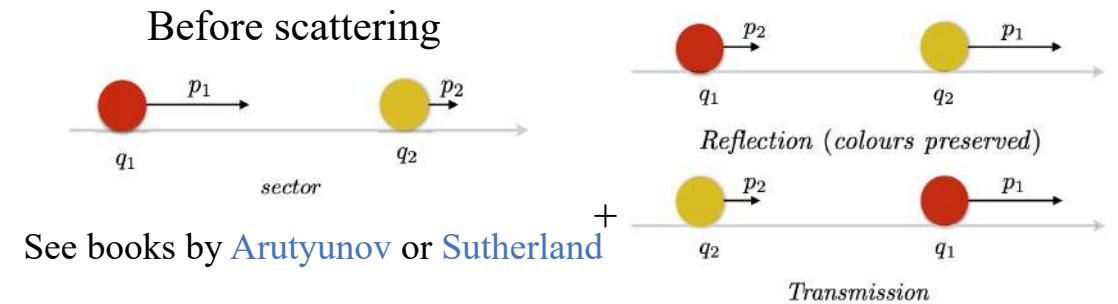
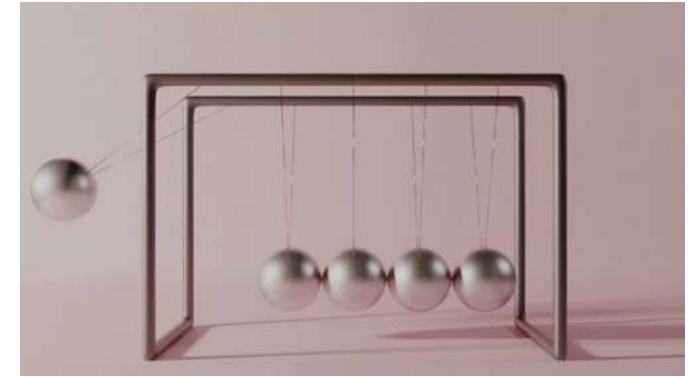
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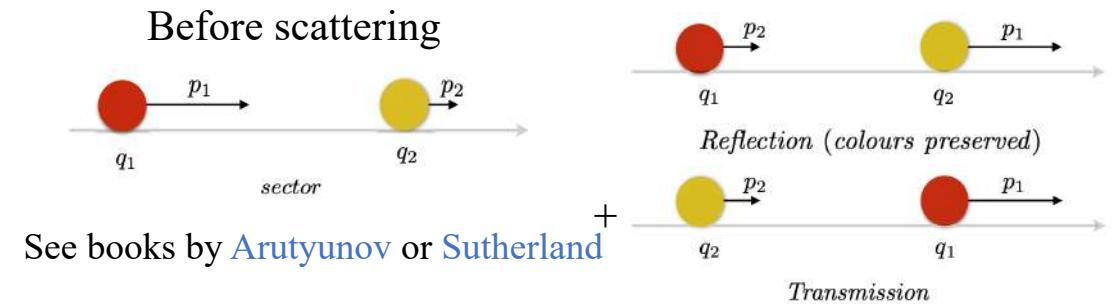
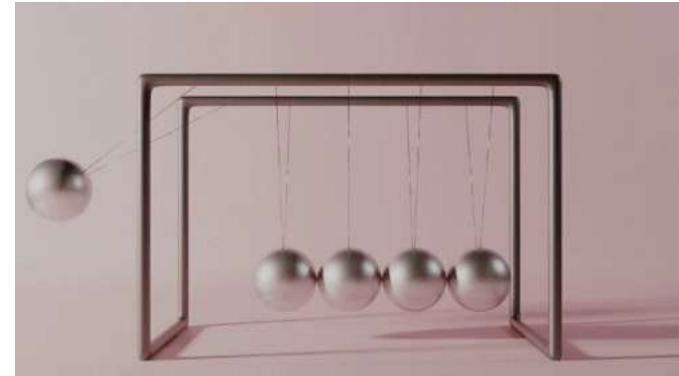
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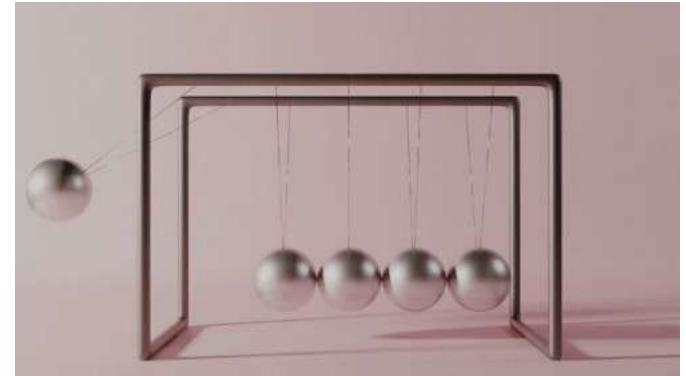
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YBE:
$$Y_{jk}^{ab}(\Delta) Y_{ik}^{bc}(\Delta) Y_{ij}^{ab}(\Delta) = Y_{ij}^{bc}(\Delta) Y_{ik}^{ab}(\Delta) Y_{jk}^{bc}(\Delta)$$

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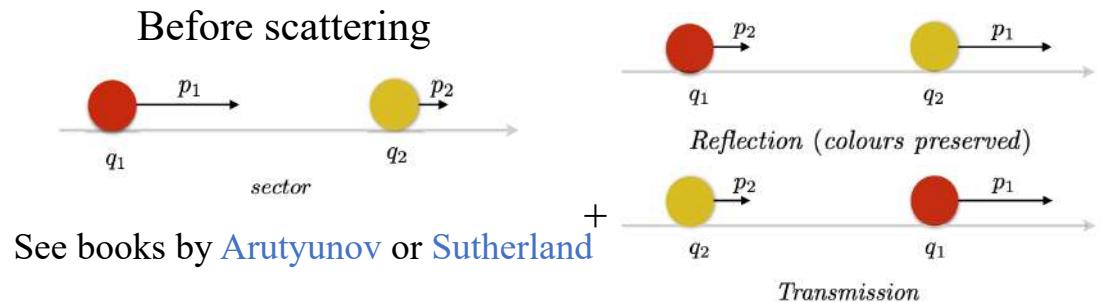
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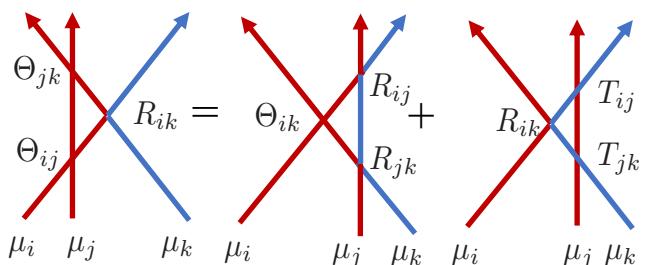
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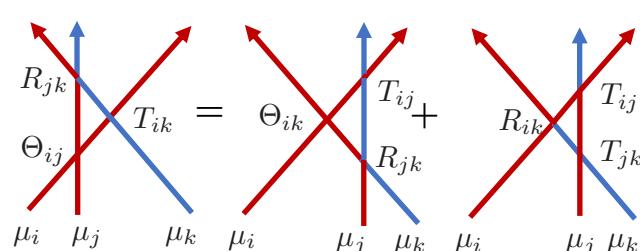
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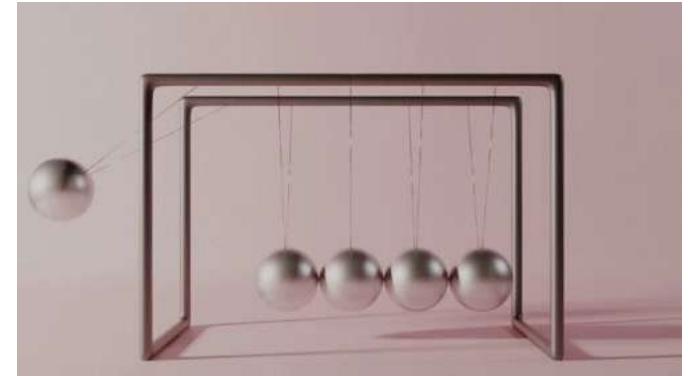
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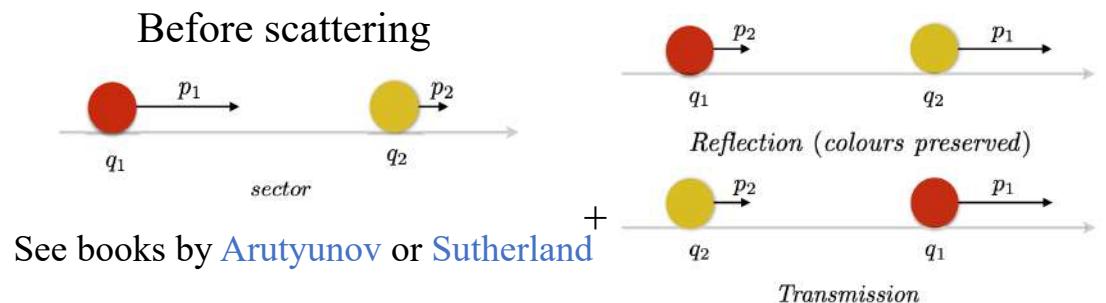
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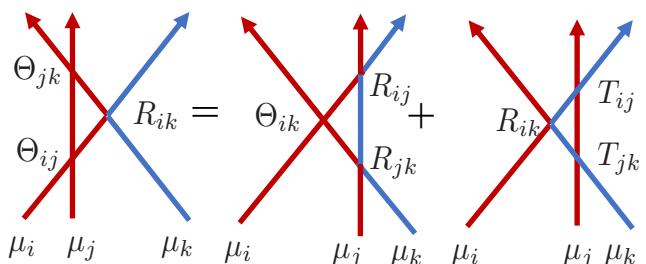
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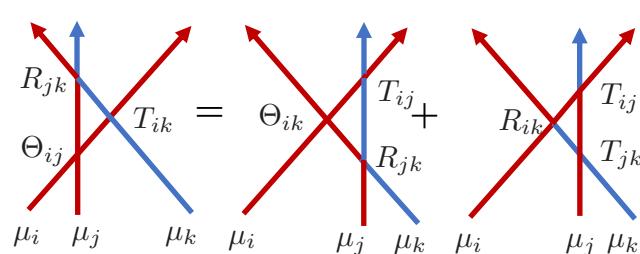
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$$R_{jk} T_{ik} \Theta_{ij} = T_{ij} \Theta_{ik} R_{jk} + R_{ij} R_{ik} T_{jk}$$

- $c_a \neq c_b \neq c_c$ ✓
- $c_a = c_b = c_c$ ✓
- $c_a = c_b \neq c_c$ ✗

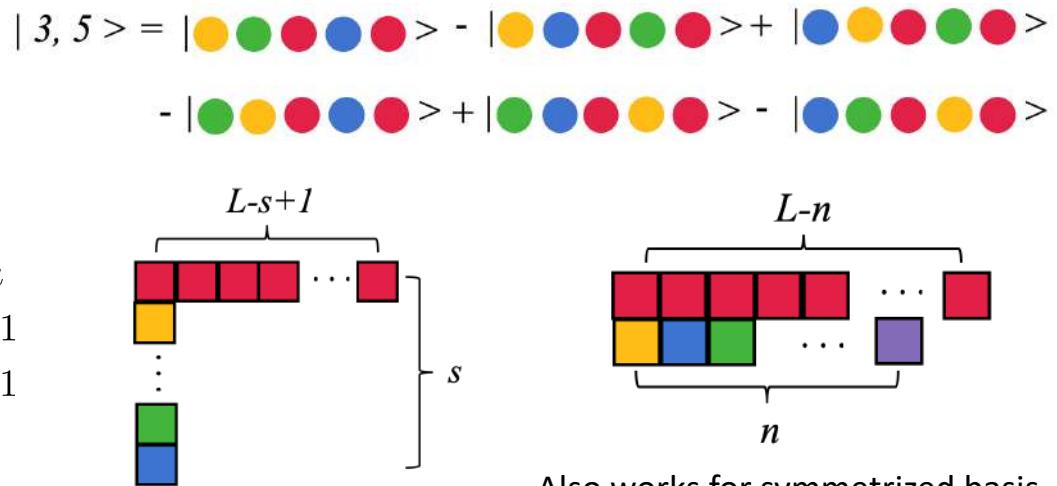
Bethe states with anti-symmetrized basis

Zhang, Mussardo, PRB, '22

- Reduced frustration

$$|i_1, i_2, \dots, i_n\rangle = \sum_{\sigma \in \mathfrak{S}_{L-n}} \text{sgn}(\sigma) |i_1, i_2, \dots, i_n; \sigma(c_2 c_3 \dots c_{L-n})\rangle$$

$$P_{j,j+1}|i_1, \dots, i_n\rangle = \begin{cases} -|i_1, \dots, i_n\rangle, & j \neq i_a, i_a - 1, \forall a \\ |i_1, \dots, i_{a-1}, i_a + 1, i_{a+1}, \dots, i_n\rangle, & j = i_a \neq i_{a+1} - 1 \\ |i_1, \dots, i_n\rangle, & j = i_a = i_{a+1} - 1 \end{cases}$$



Also works for symmetrized basis

- Finite energy density of excited states
- Decomposition of entanglement entropy

$$\epsilon_v - \epsilon_{\text{GS}} = \epsilon_{\text{XXZ}}$$

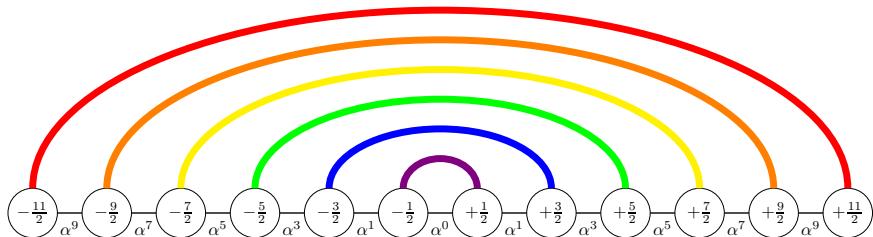
$$S_A - S_A^{\text{XXZ}} \sim (2l - n) \log(2l - n) - \sum_{n_A=n-l}^l p_A^{\text{XXZ}}(n_A) ((l - n_A) \log(l - n_A) + (l - n + n_A) \log(l - n + n_A)).$$

- Slow thermalization
- Solve YBE as eq.'s of momenta \rightarrow Solitonic excited states: e.g.

$$\mu_j = \mu_i; \quad \mu_j = 2 - \frac{1}{\mu_i};$$

Rainbow chain and entanglement blossom

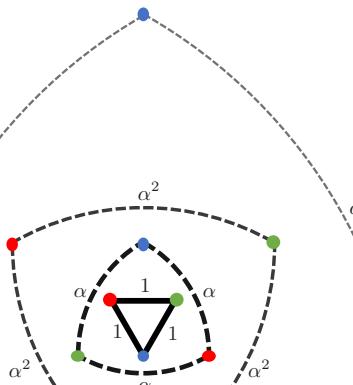
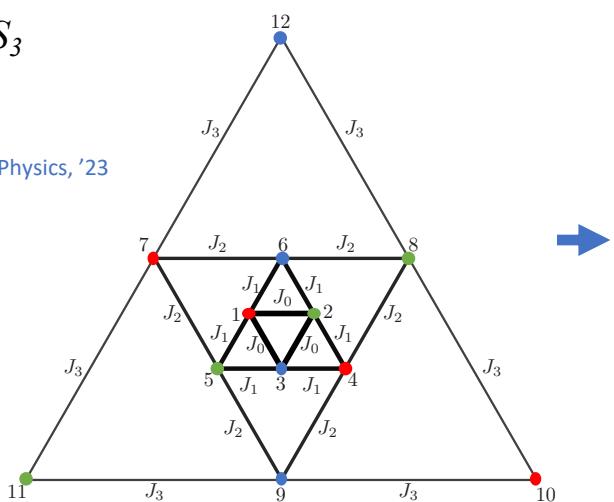
- Strong disorder RG → effective long range interaction



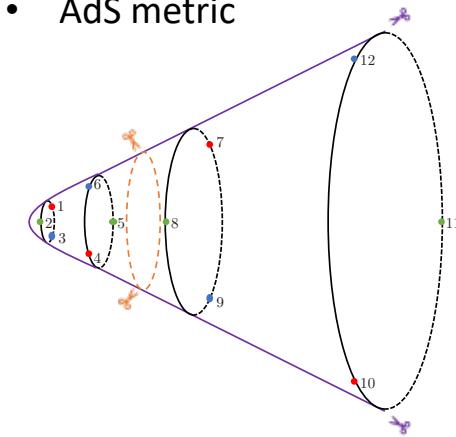
Vitagliano, Riera, Latorre, New J Phys '10
 Ramirez, Rodriguez-Laguna, Sierra, J Stat Mech, '14, '15
 Rodriguez-Laguna, et al, J Phys A, '17
 MacCormack, Liu, Nozaki, and Ryu, J Phys A, '19

- S_3

ZZ, Annals of Physics, '23



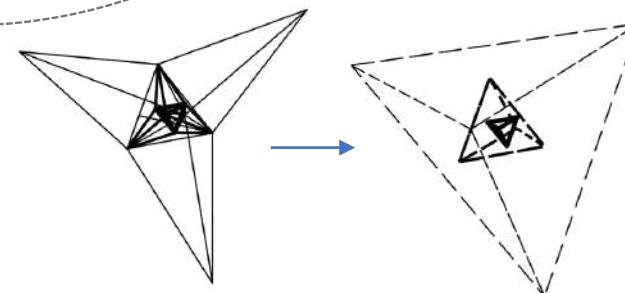
- AdS metric



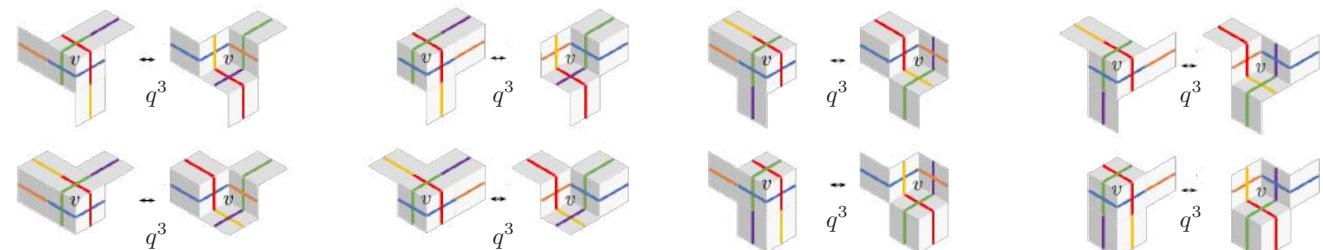
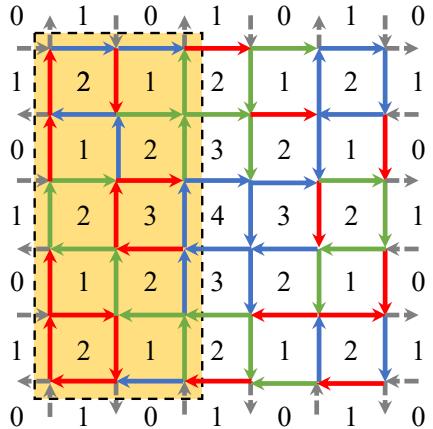
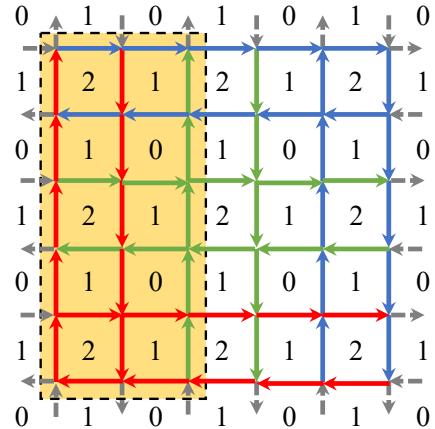
- S_k

- all-to-all Hamiltonian
 diagonalized by group algebra – Young operators

$$H_0^{(k)} = \sum_{\substack{i,j=1 \\ i \neq j}}^{k+1} \sum_{\substack{a,b=1 \\ a \neq b}}^{k+1} e_i^{ab} e_j^{ba}$$



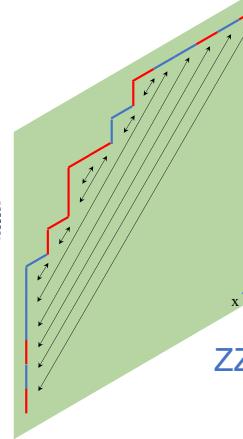
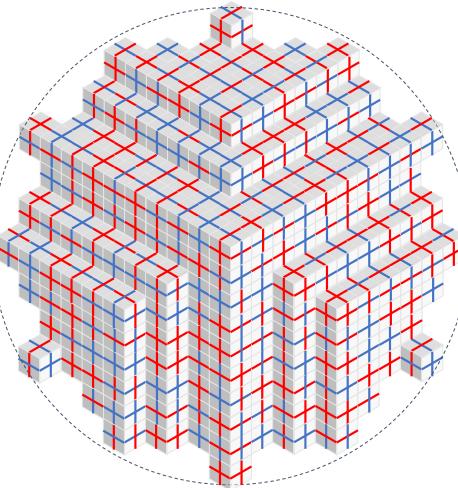
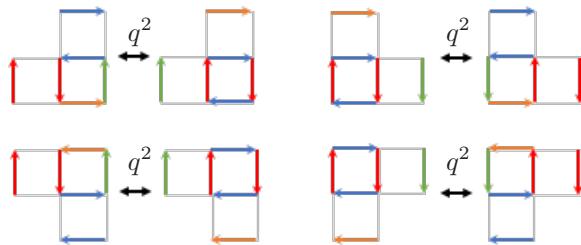
Colored vertex and dimer/tiling models



ZZ, I Klich, SciPost '23

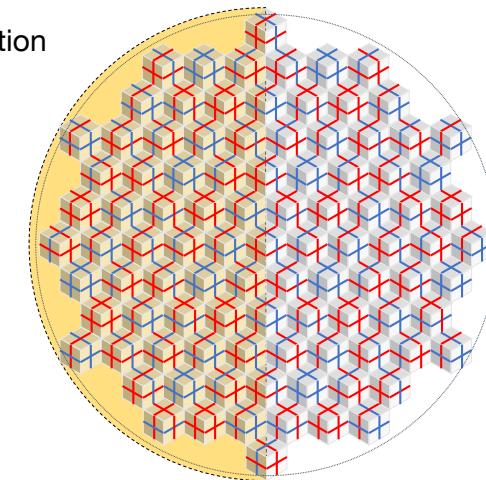
- 2-component colouring each with s colors $\rightarrow s^2$ colors of lozenge

Correlated Fredkin moves



lowest configuration
(rhombille tiling)

ZZ, I Klich, arXiv:2210.01098



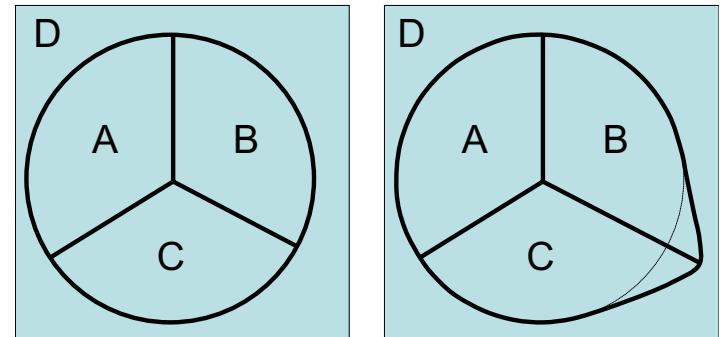
Topological entanglement entropy

$$S = \alpha L - \gamma + \dots$$

$$\gamma = \log \mathcal{D}$$

$$S_{\text{topo}} \equiv S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$

Kitaev, Preskill, PRL, '06

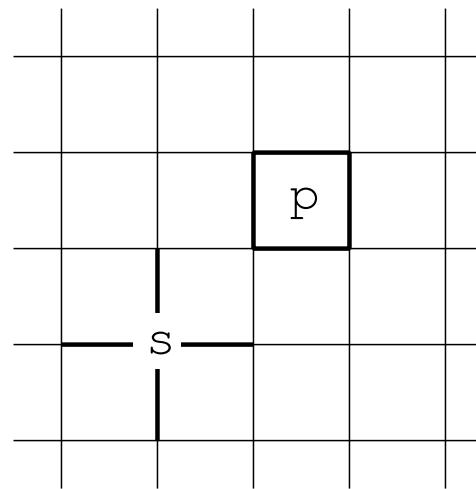


Example: Toric code

Kitaev, Annals of Physics, '03

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x \quad B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z$$

$$H_0 = - \sum_s A_s - \sum_p B_p$$



Sub-area law entanglement

- 2D conformal quantum critical points: $S = \alpha L - \beta \log L + \mathcal{O}(1)$
 - Universal: $\beta = \frac{c}{6}(\chi_A + \chi_B - \chi_{A \cup B})$ for smooth boundaries

for sharp corner with interior angle γ

$$\beta = \frac{c\gamma}{24\pi} \left(1 - \frac{\pi^2}{\gamma^2}\right) \log L$$

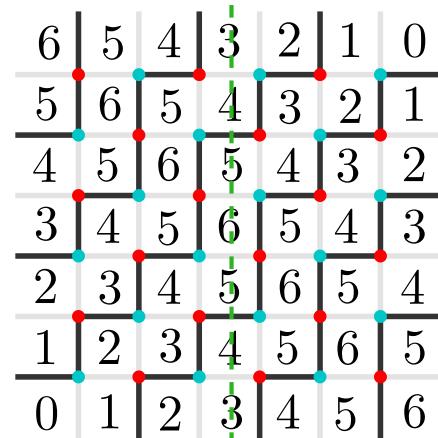
Fradkin, Moore, PRL, '06; Casini, Huerta, Nucl. Phys. B, '07; Hirata, Takayanagi, JHEP, '07; Stoudenmire et al., PRB, '14

- quantum dimer, vertex, and loop models

$$p = \frac{1}{\binom{L}{L/2}}$$

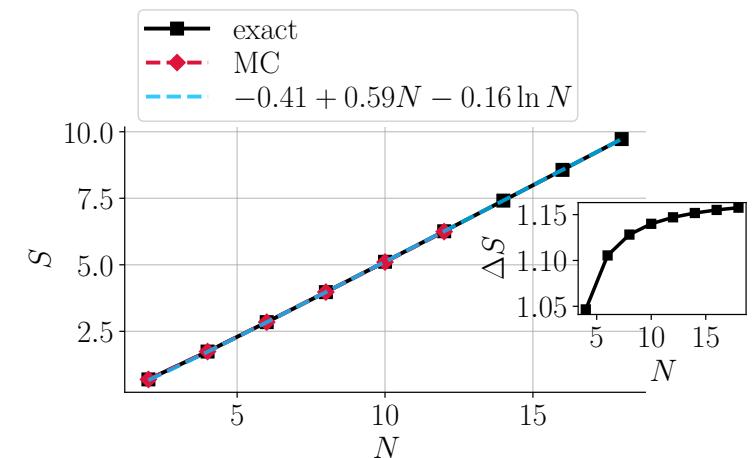
$$S \leq L \log 2 - \frac{1}{2} \log L + \frac{1}{2} \log \frac{2}{\pi}$$

Zhang, Røising, J Phys. A, '23



L

R

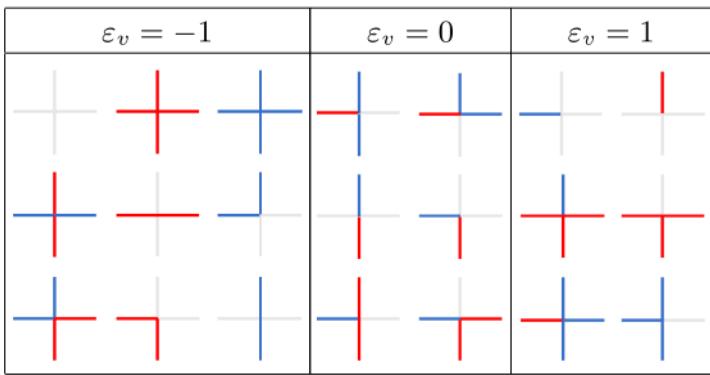


Bicolor toric code

- S_3 invariant Hamiltonian

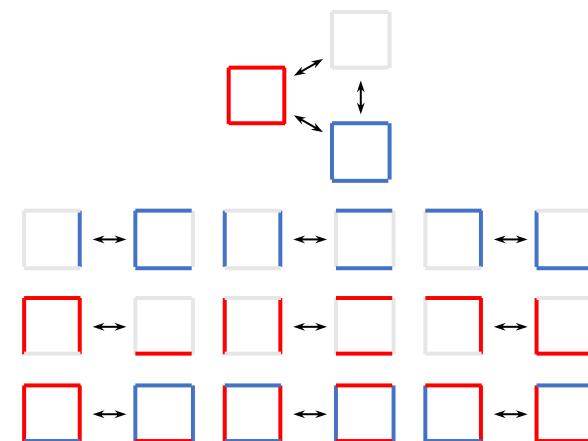
- Vertex: $h_v = -\sum_{a=1}^3 A_v^{(a)} + \hat{\Delta}_v$

$$A_v^{(a)} = \prod_{j \in +_v} Z_j^{(a)}, \quad \text{for } a = 1, 2, 3,$$



- Plaquette: $h_f = -\sum_{a=1}^3 B_f^{(a)} - \hat{N}_f$

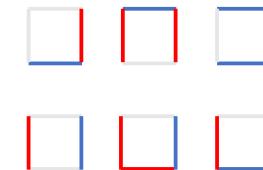
$$B_f^{(a)} = \prod_{j \in \square_f} X_j^{(a)},$$



$$X^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$



Piet Mondrian artwork displayed upside down for 75 years

⌚ 28 October 2022

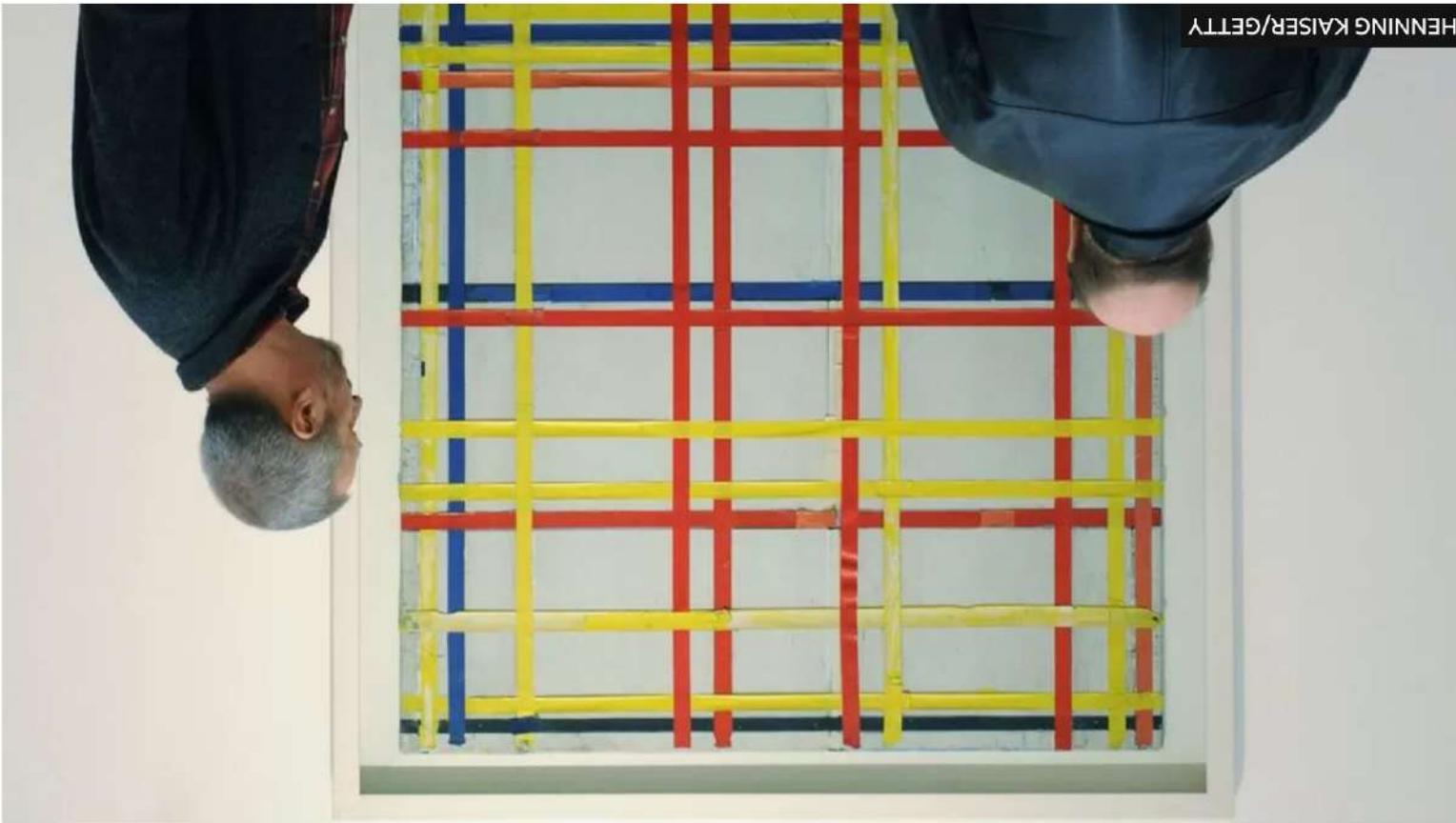


"Does that look right to you?"

By Paul Glynn

"Does that look right to you?"

HENNING KAISER/GETTY



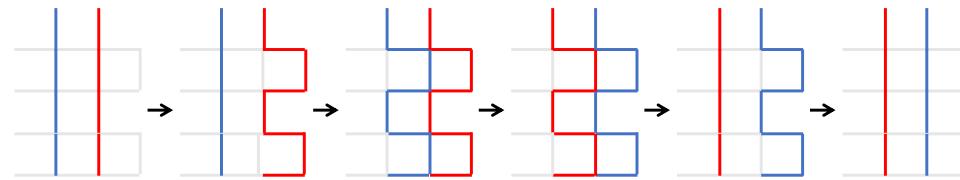
© 28 October 2022

Piet Mondrian artwork displayed
upside down for 75 years

Ground state degeneracy

$4^2=16$ topological sectors: 4 for each direction

$$\{|\emptyset\rangle, |r\rangle, |b\rangle, |rb\rangle\}$$

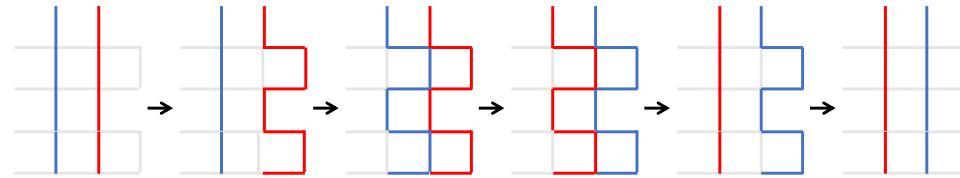


$$|GS_{\emptyset_x r_y}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\emptyset_x r_y}}} \left(\left| \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & \text{---} & & \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & \text{---} & & \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & \text{---} & & \\ \hline \end{array} \right\rangle + \dots \right)$$

Ground state degeneracy

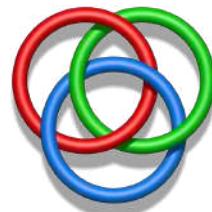
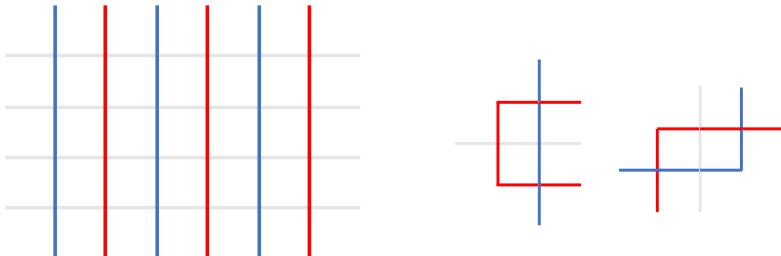
$4^2=16$ topological sectors: 4 for each direction

$$\{|\emptyset\rangle, |r\rangle, |b\rangle, |rb\rangle\}$$



$$|\text{GS}_{\emptyset_x r_y}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\emptyset_x r_y}}} \left(\left| \begin{array}{|c|c|} \hline & | \\ \hline & \square \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|} \hline & | \\ \hline & | \\ \hline & \square \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|} \hline & | \\ \hline & | \\ \hline & | \\ \hline & \square \\ \hline \end{array} \right\rangle + \dots \right)$$

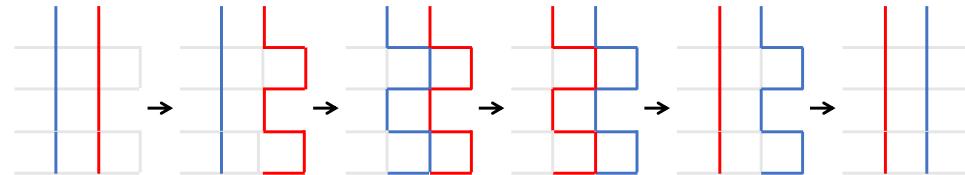
But what about these two?



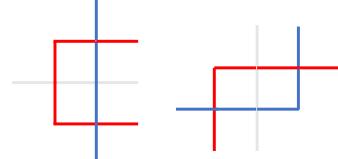
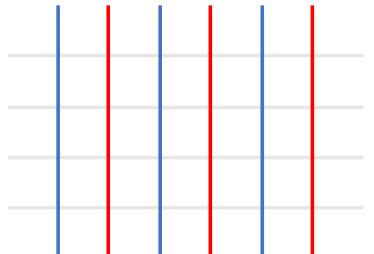
Ground state degeneracy

$4^2=16$ topological sectors: 4 for each direction

$$\{|\emptyset\rangle, |r\rangle, |b\rangle, |rb\rangle\}$$



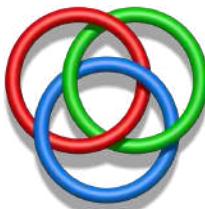
But what about these two?



$$h_{\langle f, f' \rangle} = - \sum_{a=1}^3 C_{\langle f, f' \rangle}^{(a)} - \hat{N}_{\langle f, f' \rangle}$$

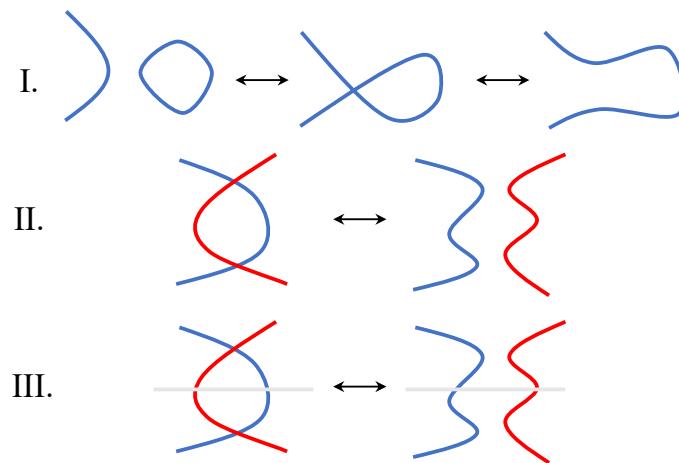
$$H_{\text{int}} = \sum_v h_v + \sum_f h_f + \sum_{\langle f, f' \rangle} h_{\langle f, f' \rangle}$$

$$C_{<f,f'>}^{(a)} = \prod_{j \in (f \cup f') \setminus (f \cap f')} X_j^{(a)}, \quad \text{for } a = 1, 2, 3.$$



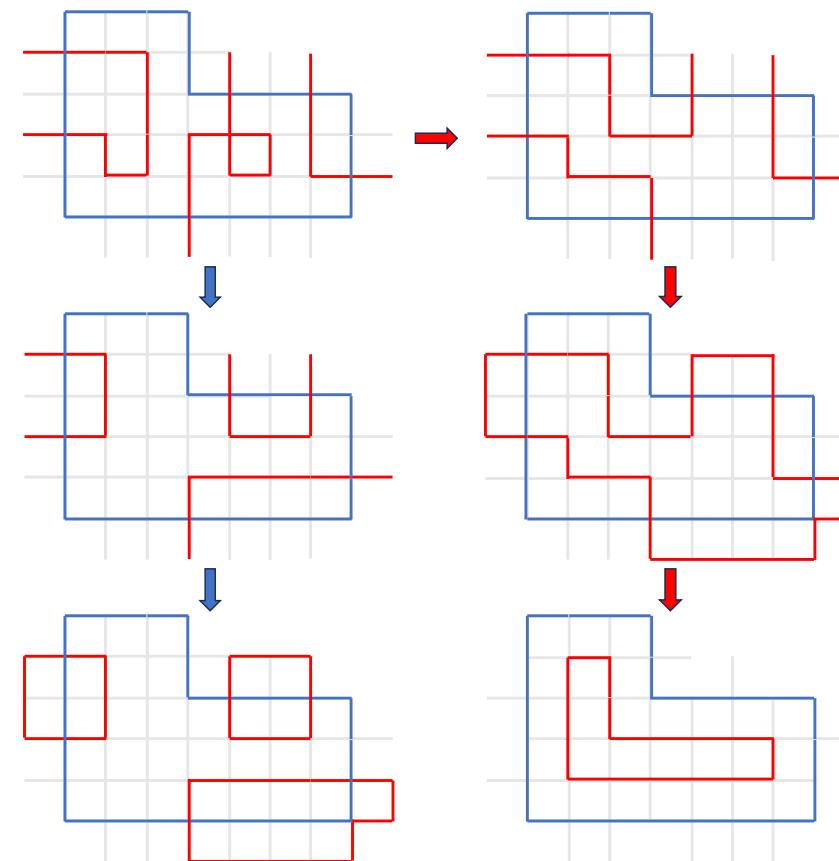
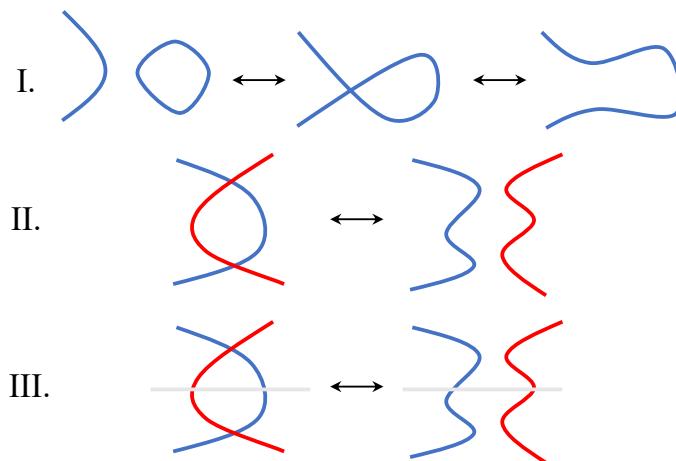
Hilbert space fragmentation

Similarity to Reidemeister moves



Hilbert space fragmentation

Similarity to Reidemeister moves

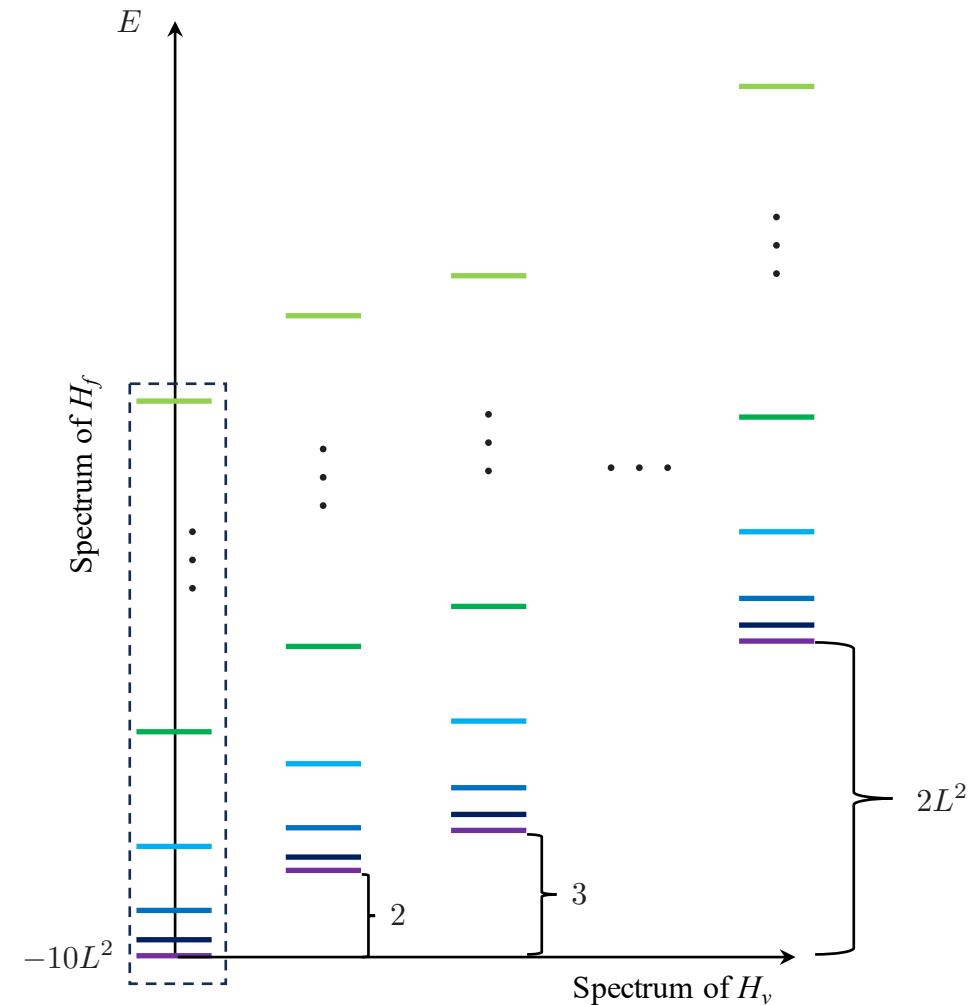
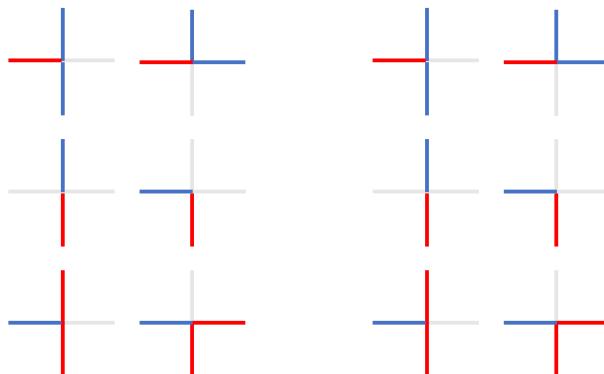


Exact excited states

$$[B_f^{(a)}, B_f^{(b)}] \neq 0 \quad \text{for } a \neq b$$

But $[A_v^{(a)}, B_f^{(b)}] = 0, \quad [A_v^{(a)}, C_{<f,f'>}^{(b)}] = 0,$

So $[H_v, H_f] = 0$

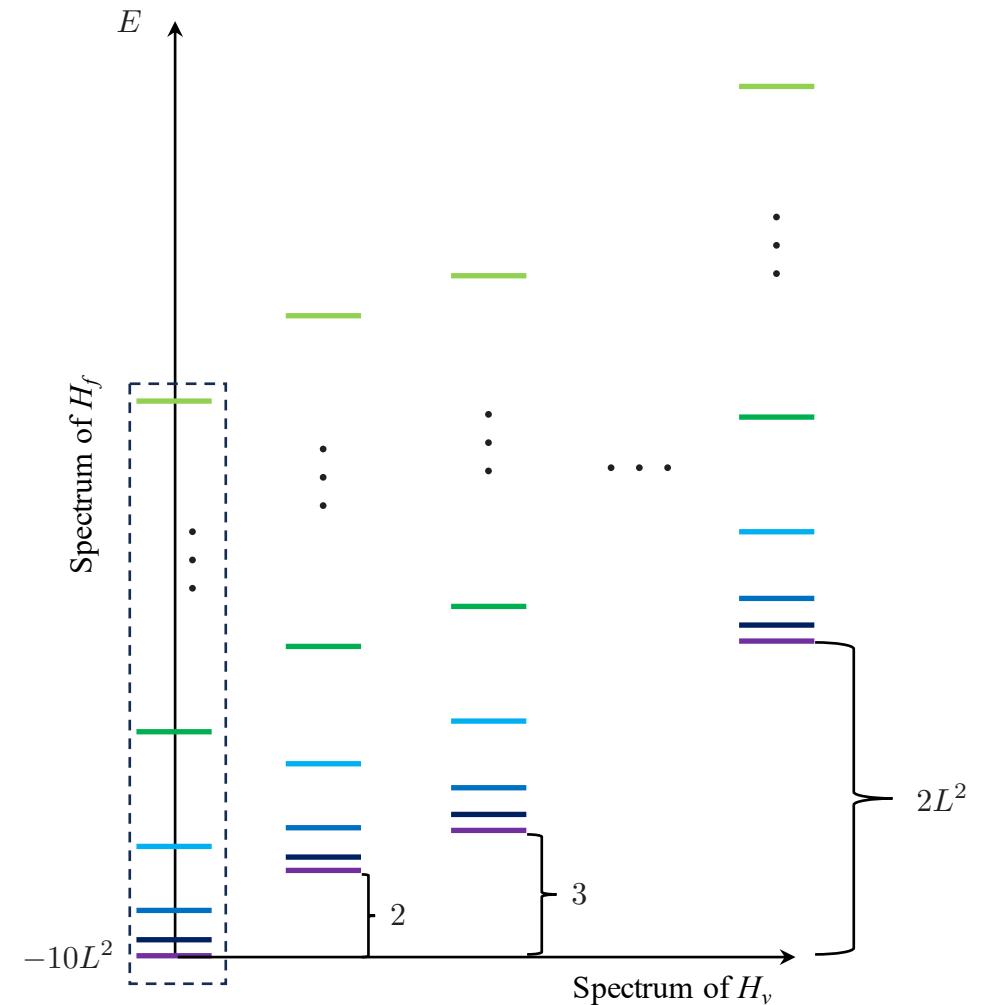
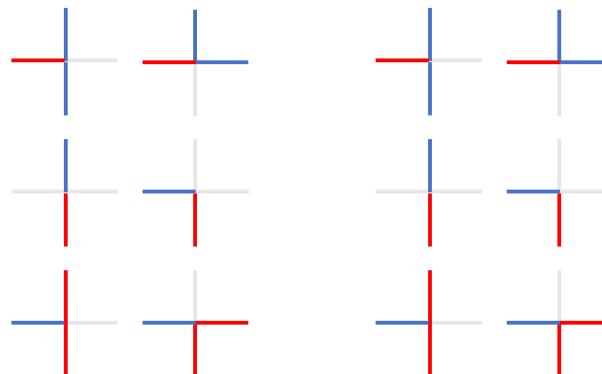


Exact excited states

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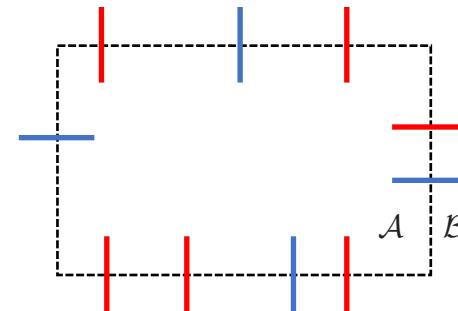
So $[H_v, H_f] = 0$



Upper bound on entanglement entropy

$$|\text{GS}\rangle = \sum_{\sigma \in \{\phi, r, b\}^{\otimes 2l}} \sqrt{p_\sigma} |\text{GS}_A(\sigma)\rangle \otimes |\text{GS}_B(\sigma)\rangle$$

$$N_i = \sum_{i,j=0 \mod 2} \binom{2l}{i-j-2l-i-j} = \frac{3^{2l} + 3}{4}$$



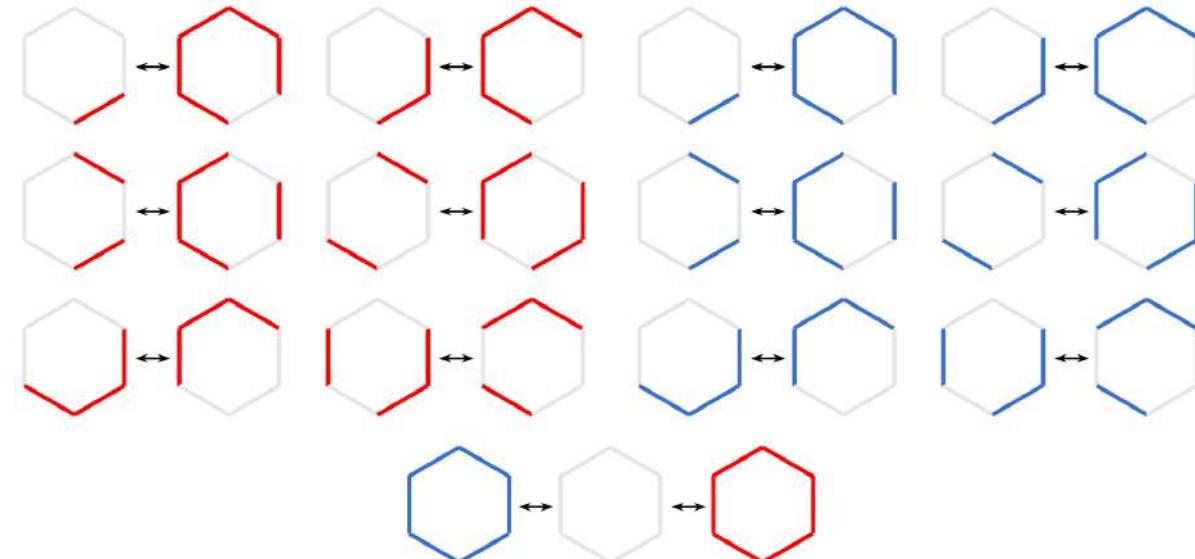
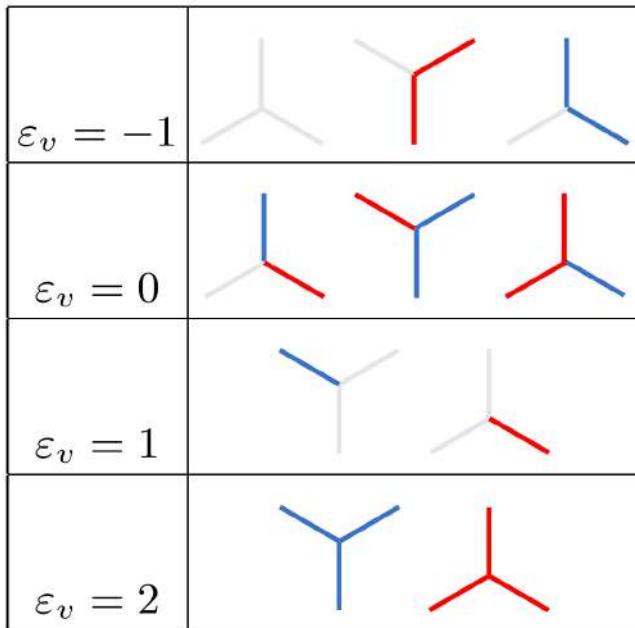
$$(1 + s_r + s_b)^{2l} = \sum_{i+j=0}^{2l} \binom{2l}{i-j-2l-i-j} s_r^i s_b^j$$

$$p_\sigma = \frac{1}{N_i} \quad \rightarrow \quad S_i = - \sum p_\sigma \log p_\sigma = 2l \log 3 - \log 4 + \epsilon$$

Agrees with quantum dimension

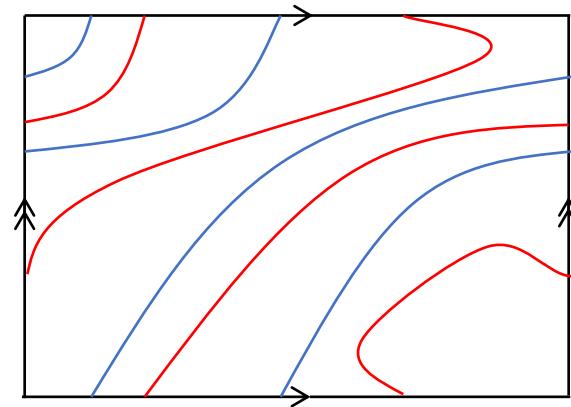
Non-intersecting bicolor loop model

- Honeycomb lattice



Topological sectors

$\{ |\emptyset_x, \emptyset_y\rangle,$
 $|\emptyset_x\rangle \otimes \{ |r_y\rangle, |b_y\rangle, |rb_y\rangle, |rbrb_y\rangle, \dots \},$
 $\{ |r_x\rangle, |b_x\rangle, |rb_x\rangle, |rbrb_x\rangle, \dots \} \otimes |\emptyset_y\rangle,$
 $\{ |r_x, r_x\rangle, |b_x, b_x\rangle, |rb_x, rb_y\rangle, |rbrb_x, rbrb_y\rangle, \dots \} \},$



Upper bound on entanglement entropy

Transfer matrix method

$$T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} |\phi\rangle \\ |r\rangle \\ |b\rangle \\ |rb\rangle \\ |br\rangle \end{array}$$

$$N_n = \text{tr } T^{2l} = (\sqrt{3} + 1)^{2l} + 2^{2l} + 1 + (1 - \sqrt{3})^{2l}$$

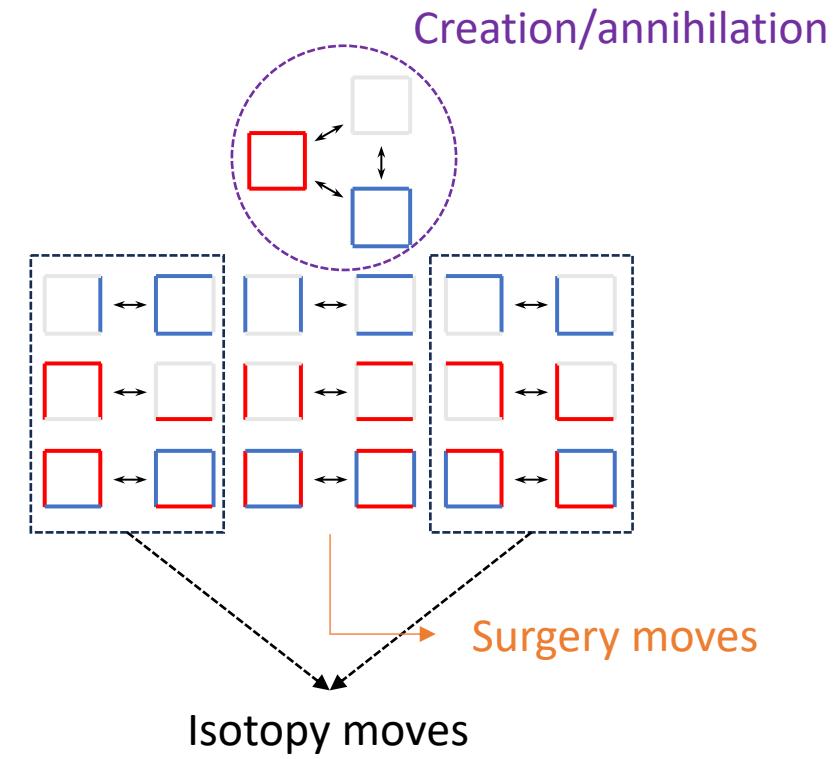
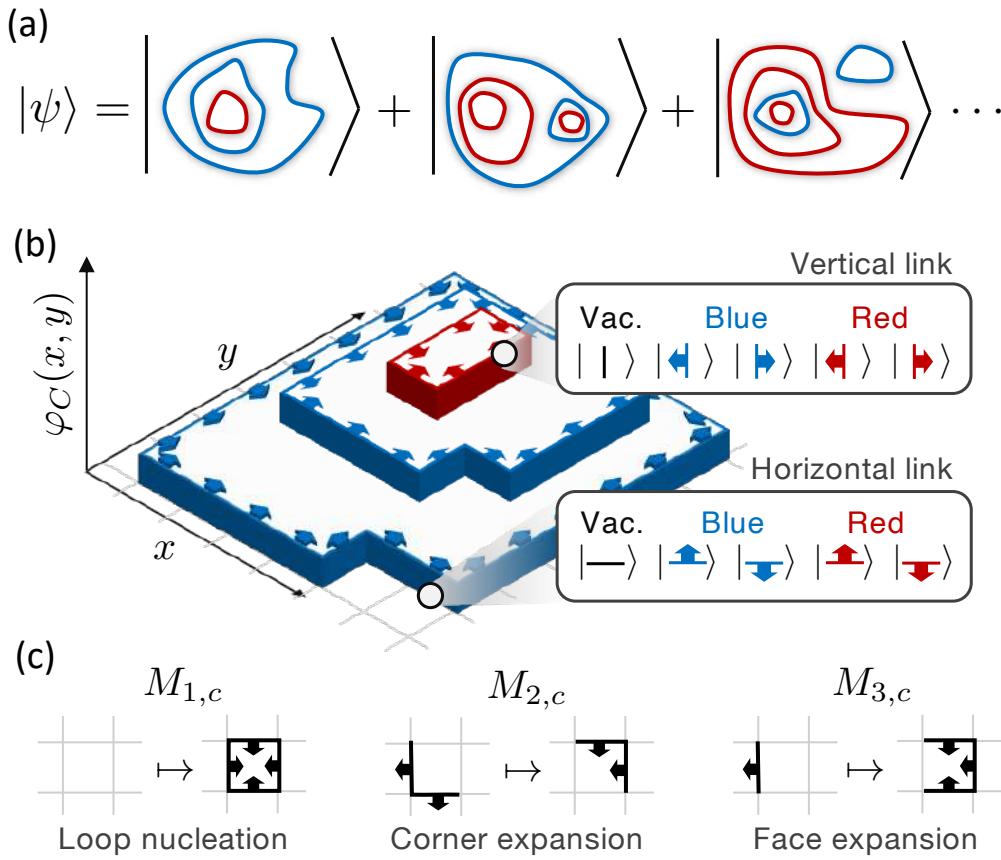
$$p_\sigma = \frac{1}{N_n}$$



$$S_n = 2l \log(1 + \sqrt{3}) + \epsilon$$

Balasubramanian--Lake--Choi model

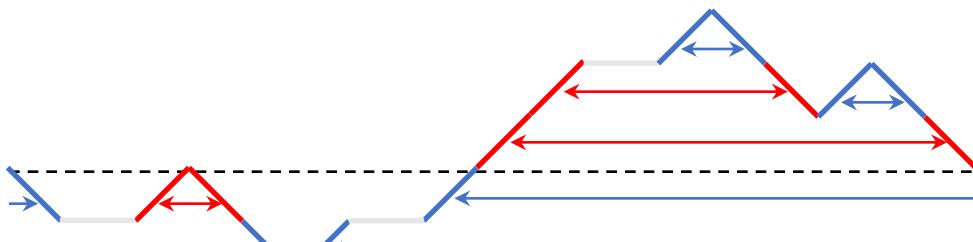
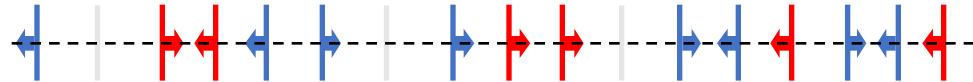
arXiv: 2305.07028



**Only creation/annihilation + isotopy moves,
No surgery moves!**

Sub-leading logarithmic contribution

Bijective mapping to colored random walks



Emergent height function

$$N_{\text{BLC}} = \sum_{n=0}^l \binom{l}{n} \binom{2l}{2l-2n} s^n$$

Saddle-point approximation

$$\begin{aligned} N_{\text{BLC}} &\approx \frac{1}{4\pi\sqrt{sl}} \left(\frac{2\sqrt{s}}{\sigma}\right)^{2l+\frac{3}{2}} \int_{-\infty}^{\infty} dx \exp\left\{-\frac{\left((2\sqrt{s}+1)x\right)^2}{2\sqrt{sl}}\right\} \\ &\approx \frac{(2\sqrt{s}+1)^{2l+\frac{1}{2}}}{2\sqrt{2\pi l}} \end{aligned}$$

$$S_{\text{BLC}} = 2l \log(2\sqrt{s} + 1) - \frac{1}{2} \log(2l) - \frac{1}{2} \log \frac{4\pi}{2\sqrt{s} + 1}$$

Summary and Outlook

- Long range entanglement in colored loop models can be reflected in the (upper bound on) entanglement entropy in 3 different ways
- Coloring toric code breaks exact sovablility, results in a kinetically constrained model with towers of exact excited states
 1. Spectrum generating algebra for the tower states?
 2. Spectral gap and loop correlation
 3. Topological invariant to classify Krylov subspace
 4. Generalization of anyonic excitations
 5. Fault tolerant quantum computation of qtrits



Thank you for your attention!