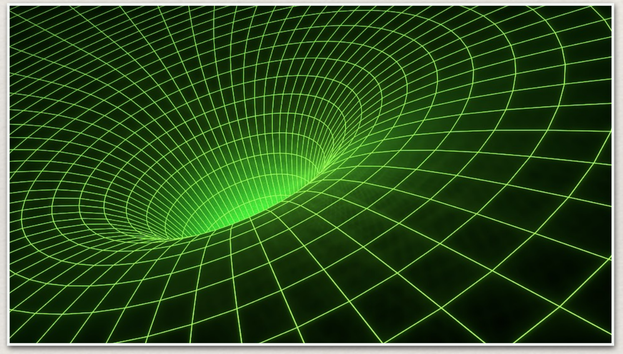


University of Oslo, December 2025

Black Hole Dynamics from Scattering Amplitudes

Andreas Helset
CERN

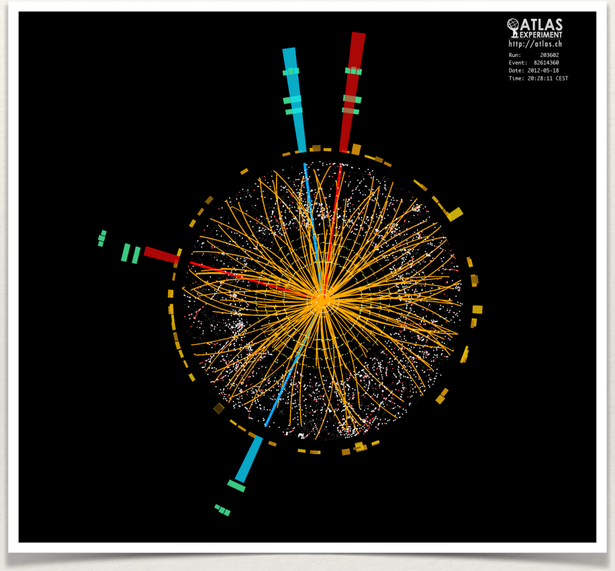
Goals of this talk



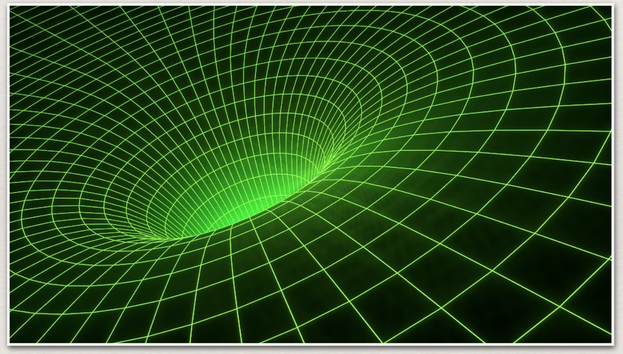
1. Black holes?
   What, how, why, …

2. Scattering amplitudes?
   What they are and how to use them.

3. How are these things connected?



Einstein's general theory of relativity



Central player: space-time metric

$$g_{\mu\nu}$$

The gravitational action is

$$S_{\text{Einstein}-\text{Hilbert}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

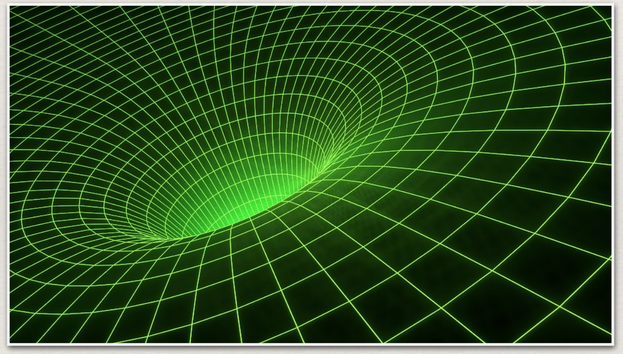
Einstein's equation (in vacuum)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

Solving Einstein's equations – black holes

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$



Schwarzschild metric (1916)

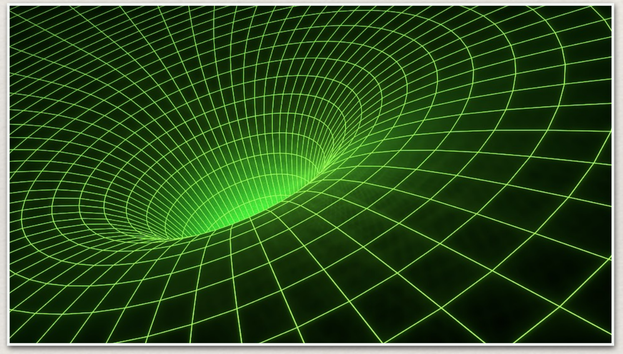
$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

Kerr metric (1963)

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma} \sin^2\theta\right) \sin^2\theta d\phi^2 - \frac{2 r_s r a \sin^2\theta}{\Sigma} dt d\phi$$

$$\Sigma = r^2 a^2 \cos^2\theta \qquad \Delta = r^2 - r_s r + a^2$$

Solving Einstein's equations – gravitational waves



Einstein's equations

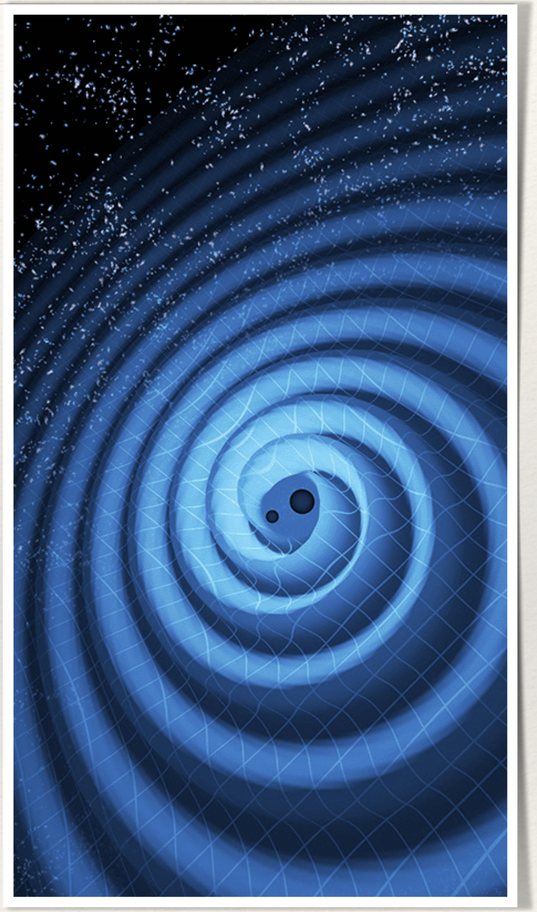
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Gravitational waves

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

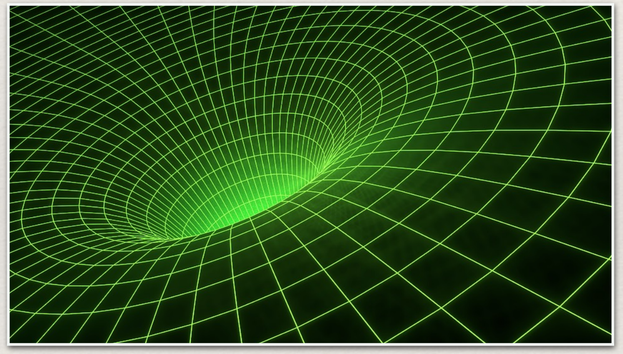
Wave equation

$$\Box \bar{h}_{\mu\nu} = 0$$



$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

Coupling matter to gravity



We can couple matter to gravity.

$$S_{\text{matter}} = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$

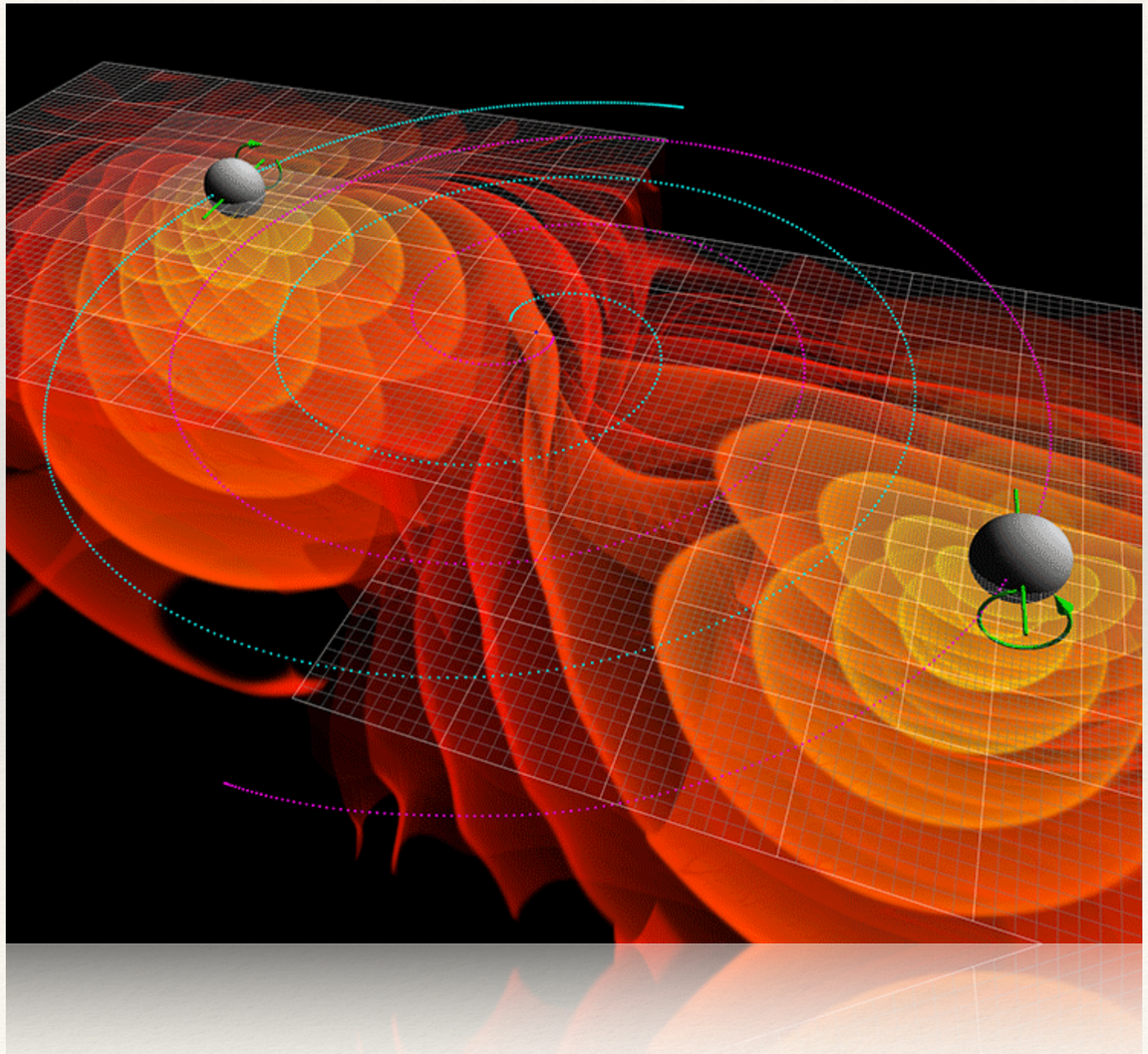
Black holes as laboratories

Black holes and gravitational waves are predicted by Einstein's theory of gravity.

Gravitational waves are ripples in space-time and black-hole binaries cause them.

Unique test of our theories of Nature in extreme environments.

Numerical simulation of two black holes in orbit.



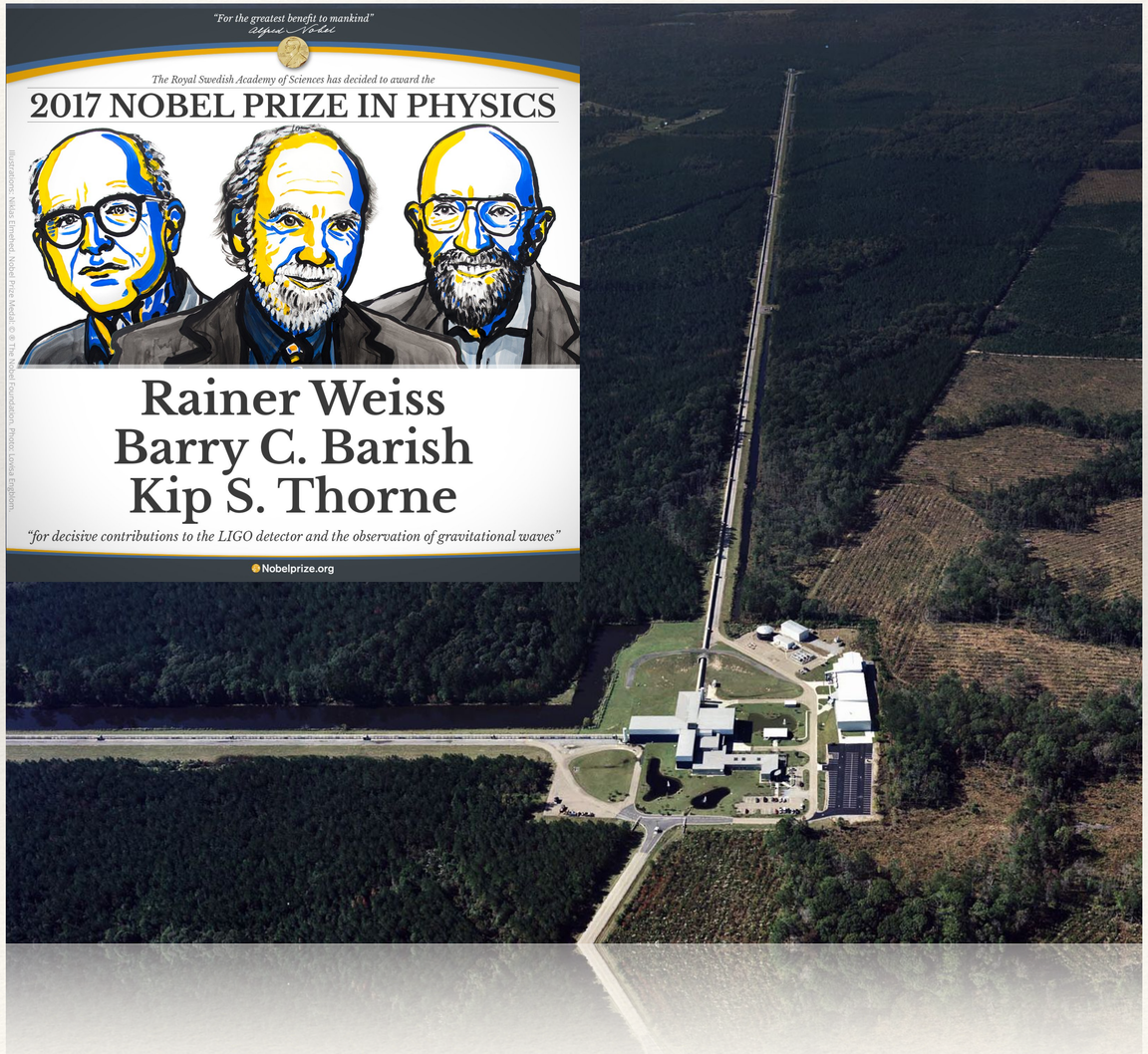
Observation of gravitational waves

First detected by LIGO. Now we have entered a precision era for gravitational waves.

Direct test of Einstein's theory of gravity.

New window to study black holes and the universe.

Next-generation LISA will detect signals from extreme-mass-ratio binaries.



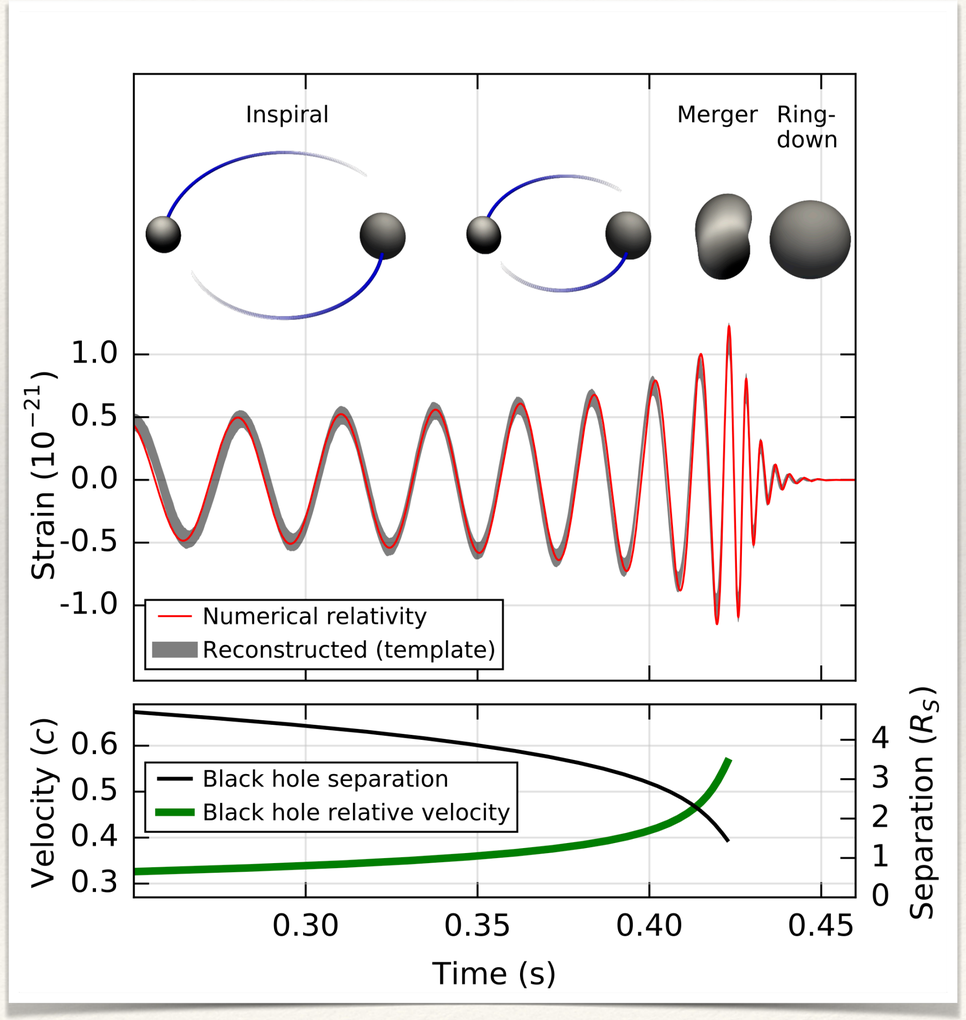
Predicting the gravitational-wave signal

Need high-precision predictions for the gravitational-wave signal.

Three phases:

1. Inspiral
2. Merger
3. Ringdown

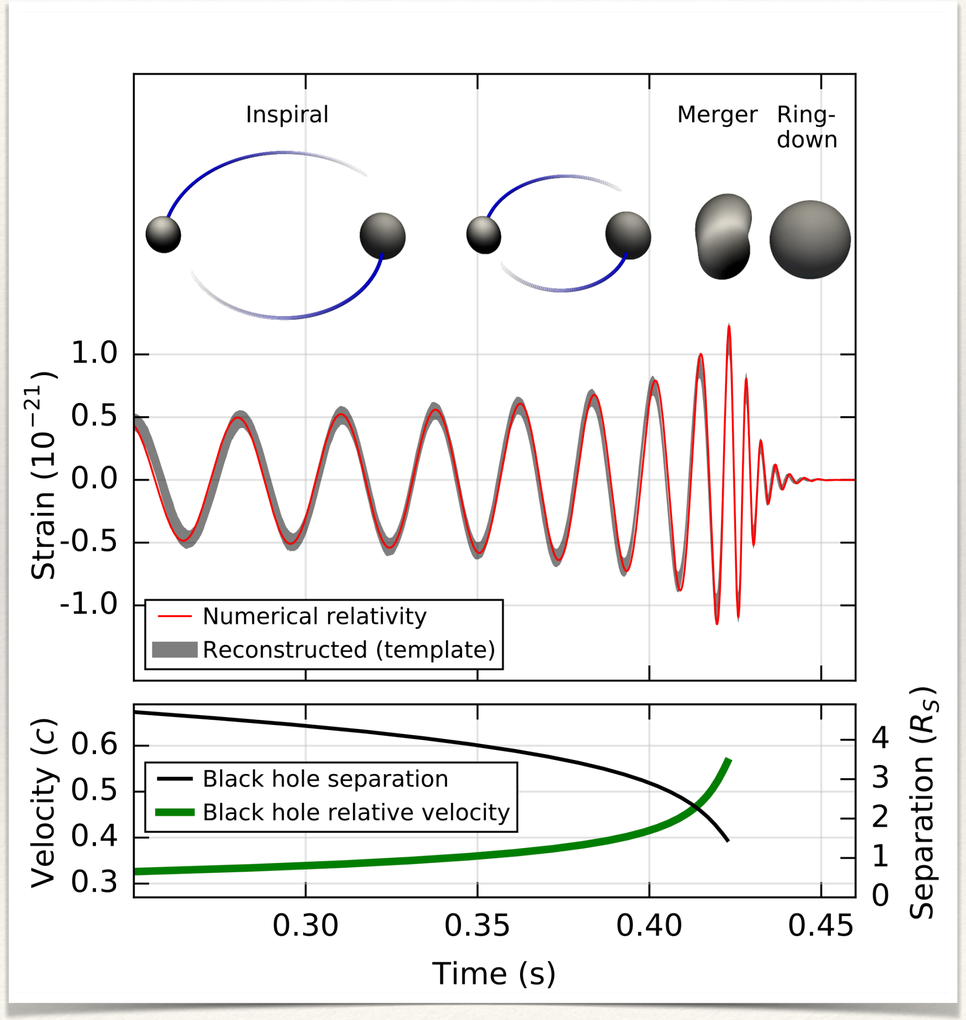
Analytical methods useful for inspire phase.



Predicting the gravitational-wave signal

Theoretical tools:

- Numerical relativity
- Self-force theory
- Non-relativistic General Relativity (NRGR)
- Effective One Body (EOB)
- Scattering amplitudes



Predicting the gravitational-wave signal

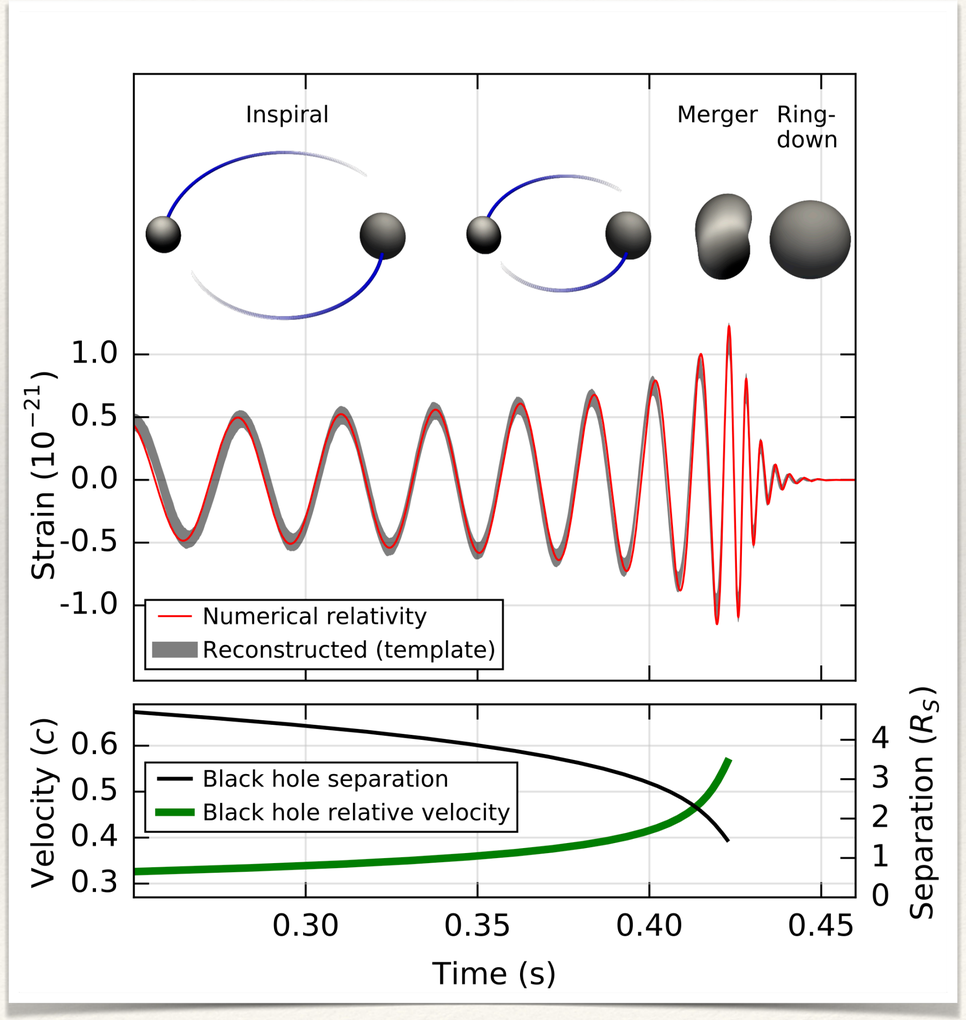
Precision calculations

G	Gv^2	Gv^4	Gv^6
	G^2	G^2v^2	G^2v^4
		G^3	G^3v^2
			G^4

post-Minkowskian

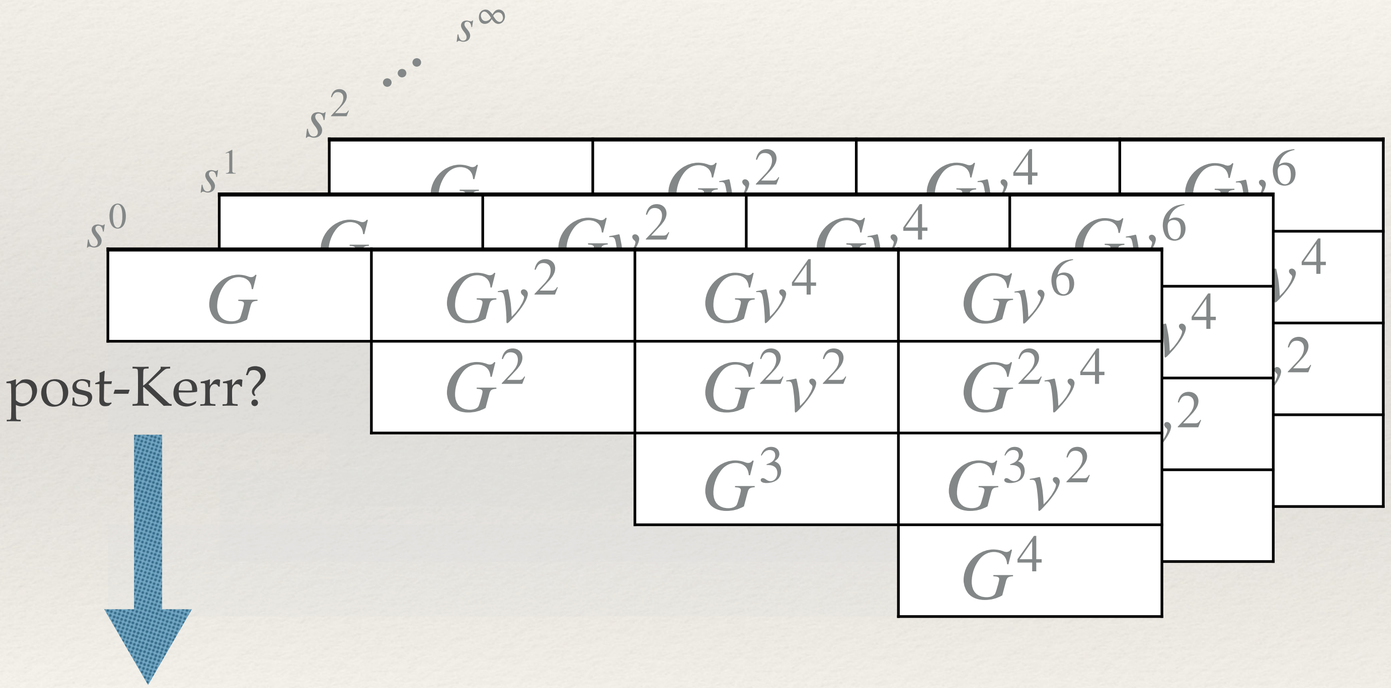
post-Newtonain

State of the art $\mathcal{O}(G^5)$ [Driesse et.al., PRL '24] [Bern et.al., '25]



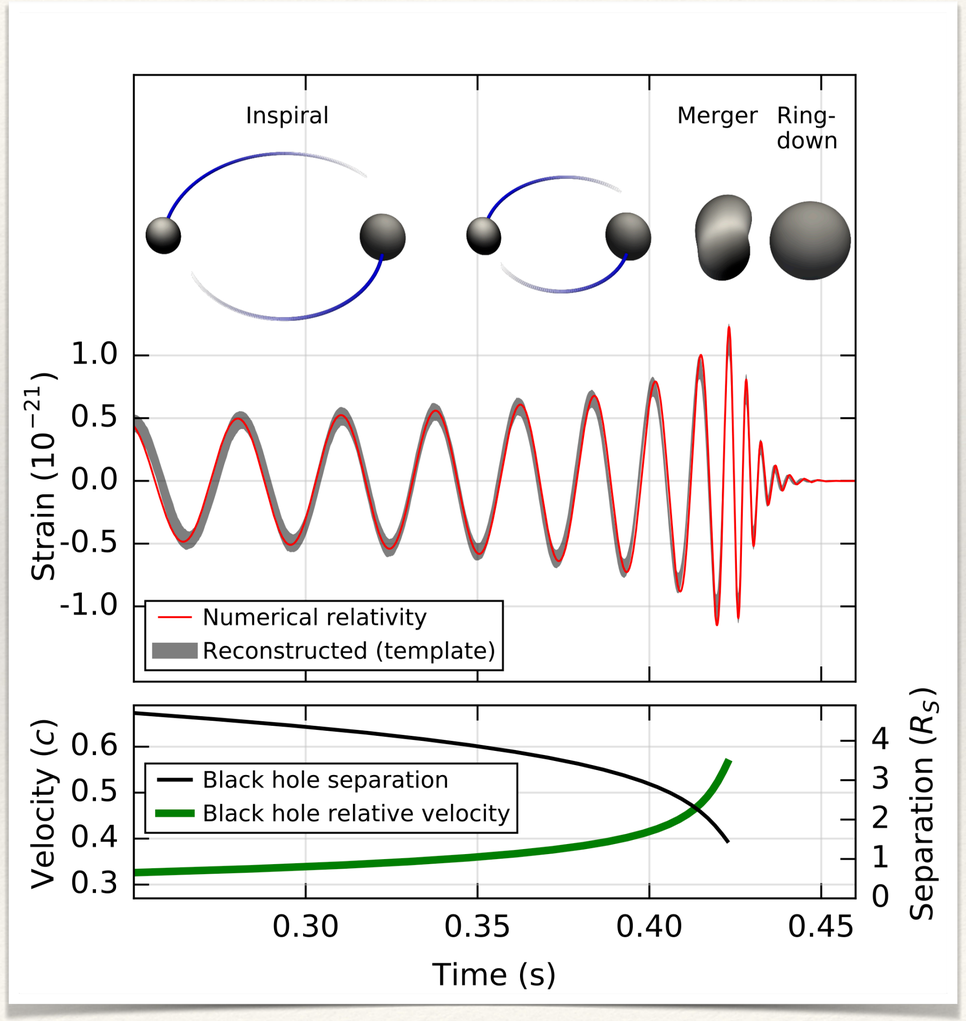
Predicting the gravitational-wave signal

Precision calculations



post-Kerr?

State of the art $\mathcal{O}(G^2 s^\infty)$ [Aoude, Haddad, & AH, PRL '22]



Scattering amplitudes

"The theory"

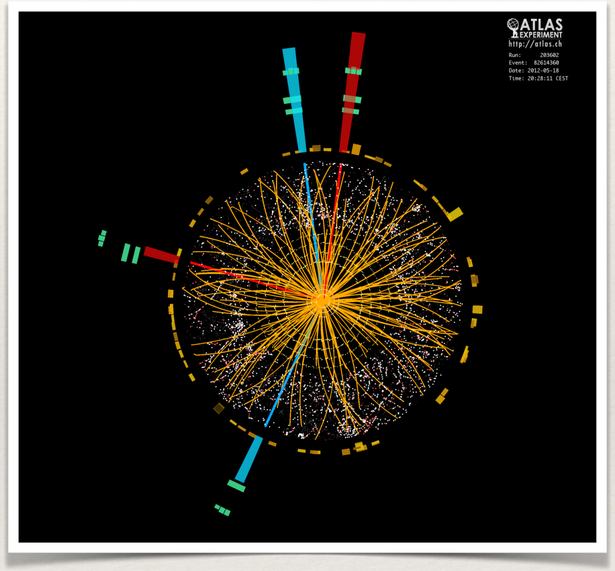
Lagrangian



"Observables"

Amplitudes

$$\frac{d\sigma}{d\Omega} \sim \sum |\mathcal{A}|^2$$



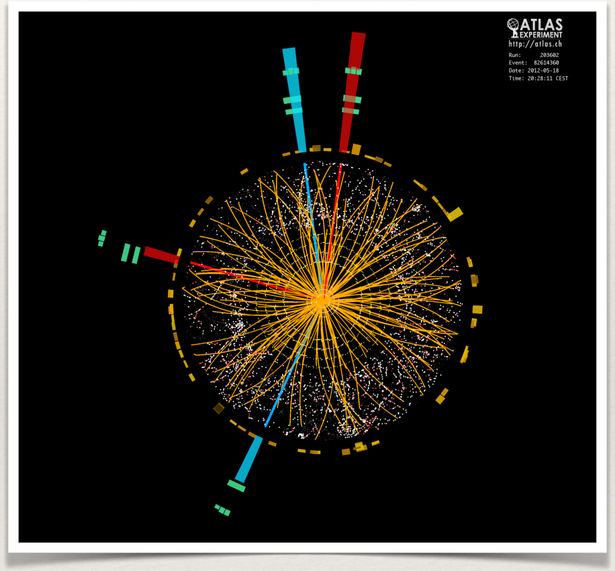
Scattering amplitudes

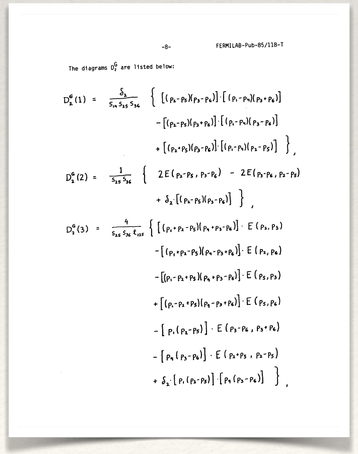
"The theory"

Lagrangian

"Observables"

Amplitudes





-8- FERMILAB-Pub-85/118-T

The diagrams D_n^0 are listed below:

$$D_n^0(1) = \frac{5}{s_{n1}s_{45}s_{56}}\Big\{[(p_1\cdot p_n)(p_5\cdot p_6)]\cdot[(p_1\cdot p_6)(p_5\cdot p_n)] - [(p_1\cdot p_n)(p_5\cdot p_6)]\cdot[(p_1\cdot p_5)(p_6\cdot p_n)] + [(p_1\cdot p_6)(p_5\cdot p_n)]\cdot[(p_1\cdot p_5)(p_6\cdot p_n)]\Big\},$$

$$D_n^0(2) = \frac{1}{s_{23}s_{45}}\Big\{2E(p_1,p_2,p_3,p_4) - 2E(p_1,p_4,p_3,p_2) + \delta_n[(p_1\cdot p_3)(p_5\cdot p_n)]\Big\},$$

$$D_n^0(3) = \frac{4}{s_{23}s_{45}\epsilon_{1234}}\Big\{[(p_1\cdot p_3\cdot p_5)(p_1\cdot p_2\cdot p_n)]\cdot E(p_1,p_2) - [(p_1\cdot p_2\cdot p_5)(p_1\cdot p_3\cdot p_n)]\cdot E(p_1,p_3) - [(p_1\cdot p_3\cdot p_5)(p_1\cdot p_2\cdot p_n)]\cdot E(p_2,p_3) + [(p_1\cdot p_2\cdot p_5)(p_1\cdot p_3\cdot p_n)]\cdot E(p_2,p_3) - [p_1(p_1\cdot p_5)]\cdot E(p_1,p_2,p_3,p_4) - [p_4(p_5\cdot p_n)]\cdot E(p_1,p_2,p_3,p_5) + \delta_n[p_1(p_1\cdot p_5)]\cdot[p_4(p_5\cdot p_n)]\Big\},$$

+ 8 more pages

$$\mathcal{A}_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Parke-Taylor amplitude

$$p^\mu = \frac{1}{2}\langle p|\sigma^\mu|p]$$

[Parke & Taylor, PRL '86]

Gravity from QCD

Double copy = amplitude 'magic'

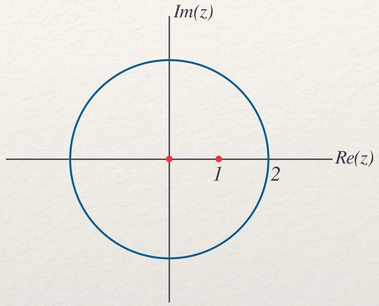
$$\mathcal{A}(1_a^-2_b^+3_c^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} f_{abc} \qquad \mathcal{M}(1^-2^+3^+) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2}$$

$$\mathcal{A}(1_a^+2_b^-3_c^-) = \frac{[12]^3}{[13][32]} f_{abc} \qquad \mathcal{M}(1^+2^-3^-) = \frac{[12]^6}{[13]^2[32]^2}$$

Simply replace color factor with kinematic factor

Building amplitudes recursively

We can compute higher-multiplicity amplitudes using on-shell recursion relations

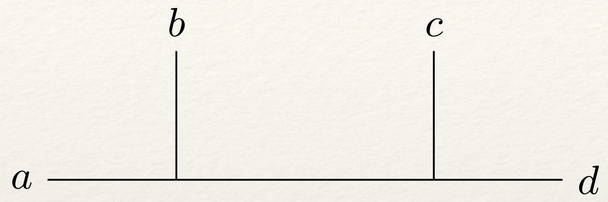


$$A(0) = \oint \frac{dz}{2\pi i} \frac{A(z)}{z} = -\sum_{\text{res}} A(z) = -\sum \frac{A_L A_R}{P^2} + B_\infty$$

Complex momenta + Cauchy's theorem

[Britto et.al., NPB '05]

Color-Kinematics Duality



QCD Amplitude

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

Color

$$c_s = f_{a_1 a_2 b} f_{b a_3 a_4}$$

Jacobi identity

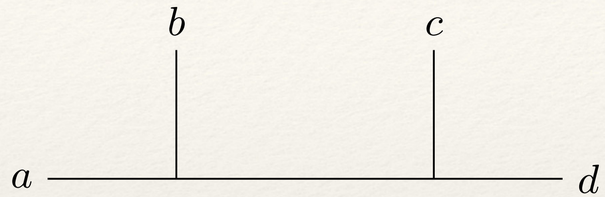
$$c_s + c_t + c_u = 0$$

Kinematic Jacobi identity

$$n_s + n_t + n_u = 0$$

[Bern et.al., PRL '10]

Color-Kinematics Duality



QCD Amplitude

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

Gravity

$$\mathcal{M}_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

Color

$$c_s = f_{a_1 a_2 b} f_{b a_3 a_4}$$

Jacobi identity

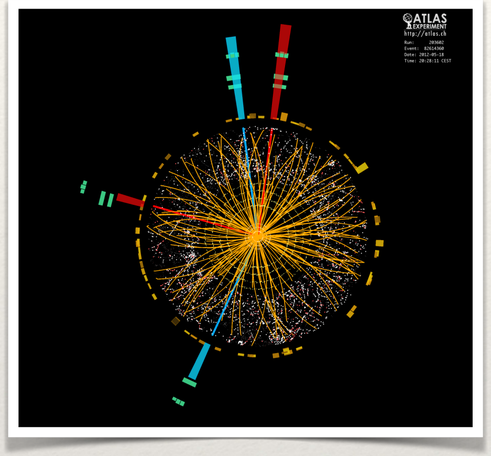
$$c_s + c_t + c_u = 0$$

Kinematic Jacobi identity

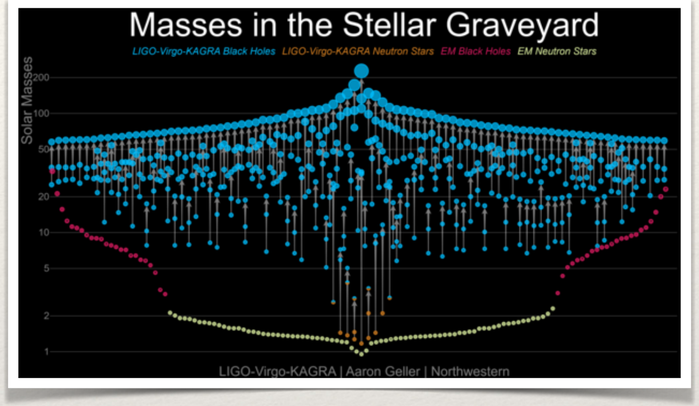
$$n_s + n_t + n_u = 0$$

[Bern et.al., PRL '10]

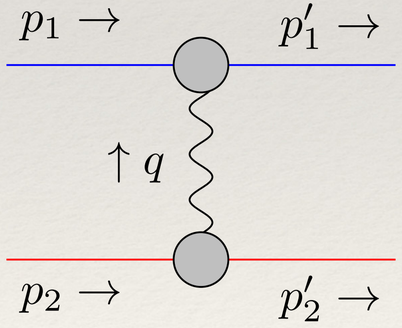
Particle physics for black holes



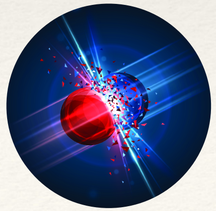
Gravitational dynamics of black holes are captured with tools from particle physics!



Particle collisions



Gravity!





Classical potential from scattering amplitudes

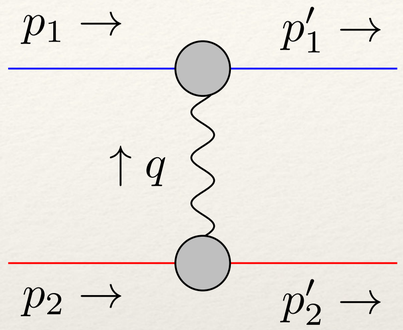
Tree-level scattering amplitude

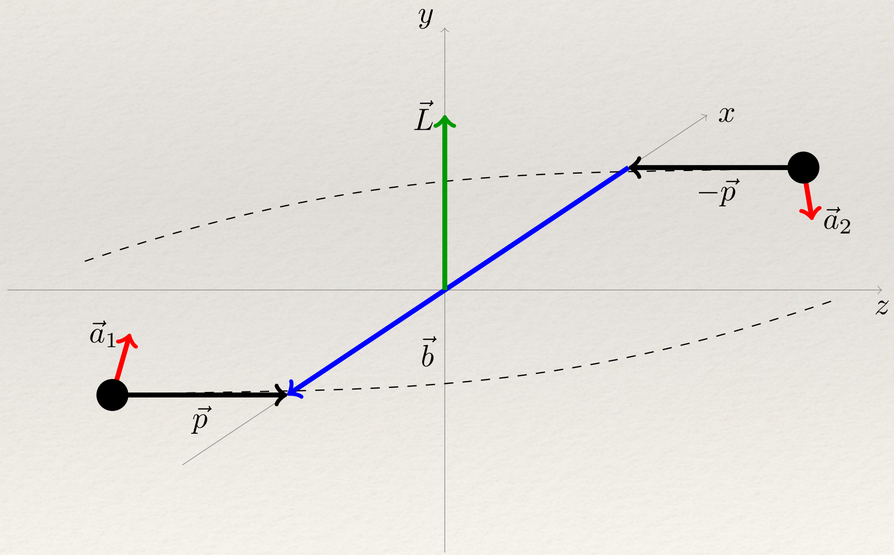
$$\mathcal{A} = \frac{4\pi G m_1 m_2}{\vec{q}^2}$$

Fourier transform

$$V(r) = -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \mathcal{A} = -\frac{G m_1 m_2}{r}$$

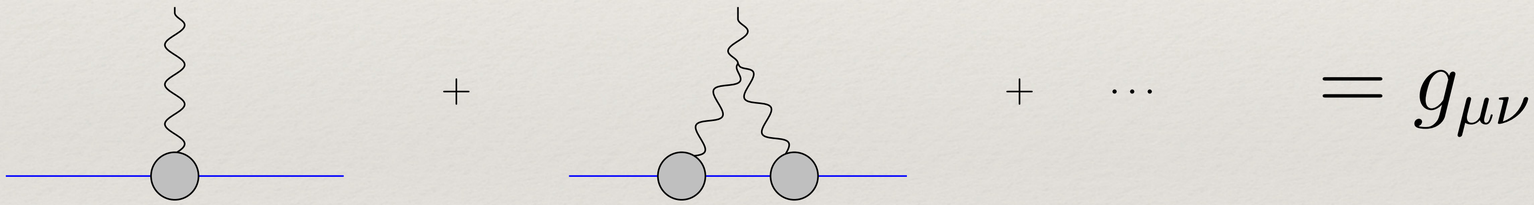
This is Newton's potential



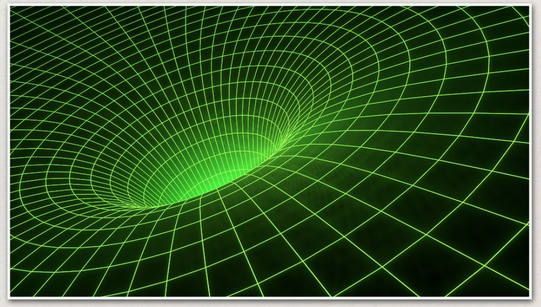


Black holes from scattering amplitudes

We can compute the space-time metric from amplitudes



$$+ \quad + \quad \cdots \quad = g_{\mu\nu}$$



[Duff, PRD '73]

QFT Toolbox

Results using QFT tools

- Generalized unitarity
- Multiloop integration
- Double Copy
- Effective Field Theory
- Heavy limit
- On-shell variables
- Hilbert series for tidal effects

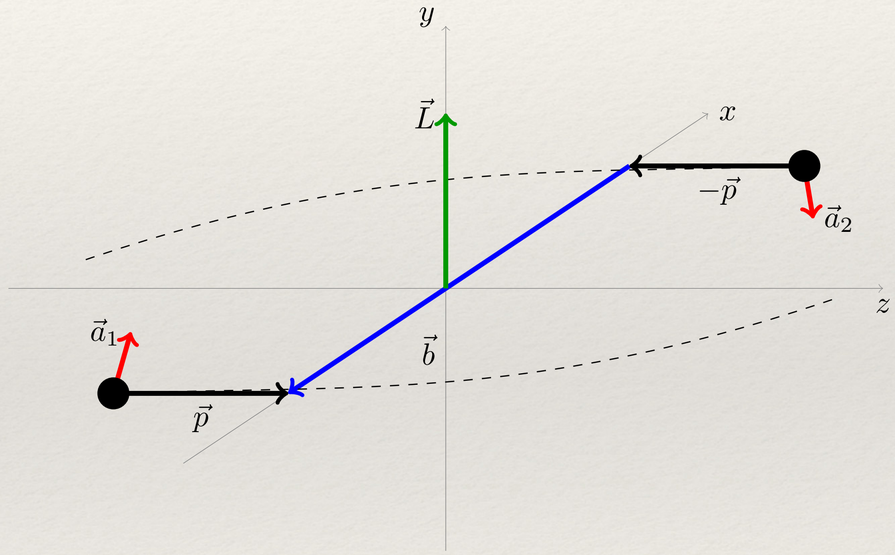


Heavy Particles and Classical Physics

But what does this have to do with classical physics?

$$\frac{Gm^2}{\hbar} \underset{\text{classical}}{\gg} 1 \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1$$

$$\frac{\hbar}{m} \ll Gm \ll b$$



Heavy limit tailored for classical physics $\quad \dfrac{\hbar}{m} \ll b \Longleftrightarrow \dfrac{\hbar \bar{q}}{m} \ll 1$

Heavy Quark Effective Theory

Heavy quark (mass M) interacting with light quarks and gluons

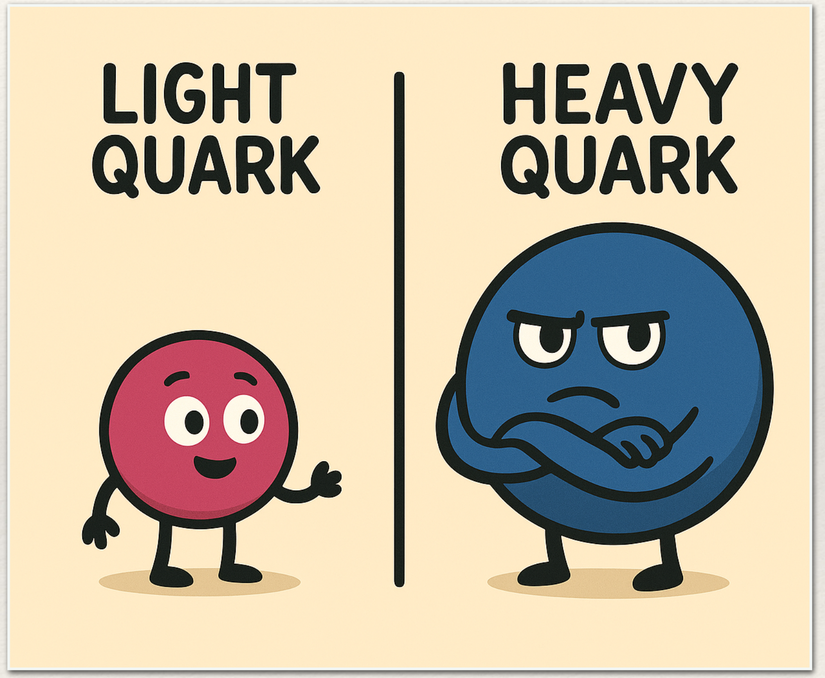
Hierarchy of scales

$$\Lambda_{\rm QCD} \sim |k| \ll M$$

Momentum of heavy quark

$$p^\mu = M v^\mu + k^\mu$$

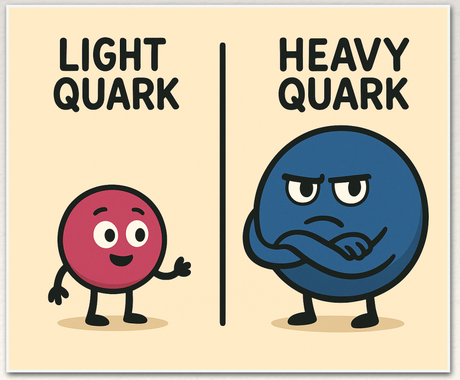
Expand in $\frac{k}{M}$



Heavy Quark Effective Theory

QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(i\gamma^\mu D_\mu - M)\psi$$



Integrate out anti-particle

$$\psi = e^{-iMv\cdot x}\left(Q + \tilde{Q}\right)$$

HQET Lagrangian

$$\mathcal{L}_{\mathrm{HQET}} = \bar{Q}\left(iv\cdot D - \frac{D_\perp^2}{2M} - g\frac{\sigma_{\mu\nu}F^{\mu\nu}}{4M} + \dots\right)Q$$

Heavy Black Hole Effective Theory

Spin 0

$$\mathcal{L}_{\text{HBET}} = \sqrt{-g}\phi^* \left(g^{\mu\nu} i v_\mu \partial_\nu + \frac{1}{2} m (g^{\mu\nu} v_\mu v_\nu - 1) + \dots \right) \phi$$

Spin 1/2

$$\mathcal{L}_{\text{HBET}} = \sqrt{-g}\bar{\psi} i \gamma^a \left(\delta_a^\mu \nabla_\mu + (e_a^\mu - \delta_a^\mu)(\nabla_\mu - i m v_\mu) + \dots \right) \psi$$

Spin 1

$$\mathcal{L}_{\text{HBET}} = \sqrt{-g} \left(\frac{i}{2} \left[(\nabla_\mu B_\nu^*)(v_\rho B_\sigma) - (v_\mu B_\nu^*)(\nabla_\rho B_\sigma) \right] (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + \dots \right)$$

[Damgaard, Haddad, & AH, JHEP '19]

What is a black hole?

Image of a black hole



Schwarzschild metric

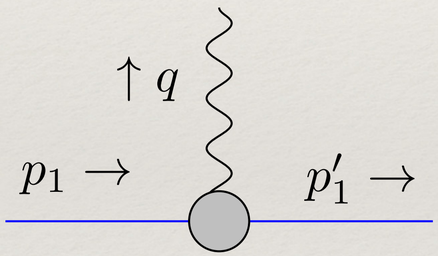
$$g_{\mu\nu}dx^\mu dx^\nu = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

Scattering amplitude

$$\mathcal{A}_3 = (v\cdot\epsilon)^2 e^{\pm q\cdot a}$$

Minimal three-particle amplitude

The minimal three-particle amplitude with one massless and two massive particles is



$$\mathcal{A}_3(1^s, 2^s, 3^+) = (v \cdot \epsilon)^2 \langle \mathbf{21} \rangle^{\mathbf{2s}}$$

On-shell variables for massive particles

$$p^\mu = \frac{1}{2}\langle p^I | \sigma^\mu | p_I] \qquad\qquad p^2 = m^2$$

[Arkani-Hamed et.al, JHEP '17]

On-shell variables for heavy particles

Massive and heavy Dirac spinors are related as

$$u_v(p) = \left(\frac{1+\gamma^\mu v_\mu}{2}\right) u(p) = \left(1 - \frac{\gamma^\mu k_\mu}{2m}\right) u(p)$$

We introduce heavy on-shell variables

$$\begin{pmatrix} |\mathbf{v}\rangle \\ |\mathbf{v}] \end{pmatrix} = \frac{1}{\sqrt{m_k}} \left(1 - \frac{\gamma^\mu k_\mu}{2m}\right) \begin{pmatrix} |\mathbf{p}\rangle \\ |\mathbf{p}] \end{pmatrix}$$

There are several advantages in using heavy spinors:

★ Exploits the hierarchy of scales (EFT)

★ Define spin wrt conserved velocity

★ Manifest spin exponentiation

★ Natural to combine with coherent states

★ Enables all-spin computations

[Aoude, Haddad, & AH, JHEP '20]

Spin in on-shell variables

The Pauli-Lubanski pseudo-vector

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}$$

In on-shell variables

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}[(\sigma^\mu)_{\alpha\dot\alpha}v^{\dot\alpha\beta} - v_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}]$$

Ring radius

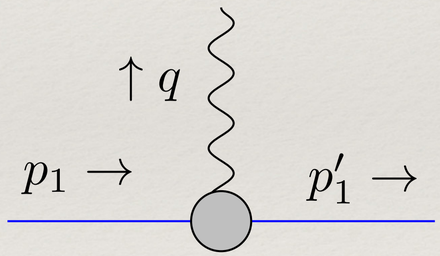
$$a^\mu = \frac{\langle\bar{\mathbf{v}}|S^\mu|\mathbf{v}\rangle}{m\langle\bar{\mathbf{v}}\mathbf{v}\rangle}$$



Heavy three-particle amplitude

The three general three-point amplitude is

$$\mathcal{A}_3(1^s,2^s,3^+) = \mathcal{A}_3(1,2,3^+)\sum_{k=0}^{2s} g_{s,k}\frac{(2s-k)!}{(2s)!}(q\cdot a)^k$$



For minimal amplitude $g_{s,k} = \binom{2s}{k}$

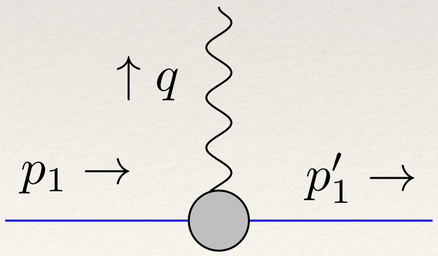
$$\mathcal{A}_3(1^s,2^s,3^+) = \mathcal{A}_3(1,2,3^+)\sum_{k=0}^{2s}\frac{1}{k!}(q\cdot a)^k = \mathcal{A}_3(1,2,3^+)e^{q\cdot a}$$

[Aoude, Haddad, & AH, JHEP '20]

First Post-Minkowskian amplitude

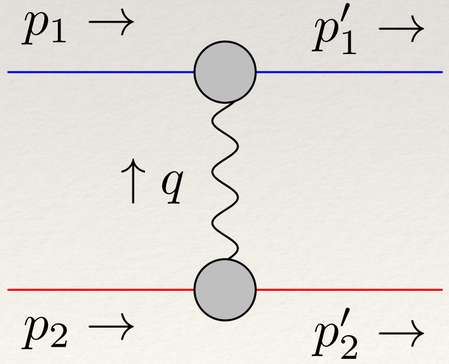
From three-point amplitude

$$\mathcal{A}_3(1^s,2^s,3^+) = \mathcal{A}_3(1,2,3^+)\sum_{k=0}^{2s}\frac{1}{k!}(q\cdot a)^k = \mathcal{A}_3(1,2,3^+)e^{q\cdot a}$$



to 1PM amplitude

$$\mathcal{A}_{1PM} = -\frac{\kappa^2 m_1 m_2}{q^2}\sum_{\pm}(\gamma \pm \sqrt{\gamma^2-1})^2 e^{\pm q\cdot a}$$

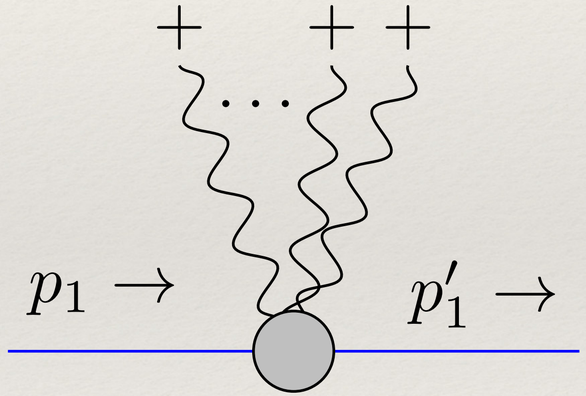


[Aoude, Haddad, & AH, JHEP '20]

Compton Scattering – same helicity

When helicity is equal: spin exponentiates!

$$\mathcal{A}_4^{+,s} = \mathcal{A}_4^{+,0} e^{q\cdot a}$$



This is true to all multiplicity!

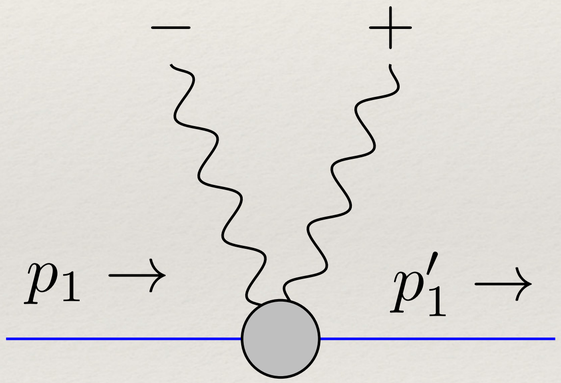
$$\mathcal{A}_n^{+,s} = \mathcal{A}_n^{+,0} e^{q\cdot a}$$

[Aoude, Haddad, & AH, JHEP '20]

Compton Scattering – opposite helicity

The opposite-helicity amplitude has a problem

$$\mathcal{A}_4^{-+,s} = \mathcal{A}_4^{-+,0} e^{(q_4 - q_3)\cdot a} e^{(t_{14} - t_{13})\frac{[4|a|3\rangle}{[4|v|3\rangle}}$$



Spurious pole

$$\frac{1}{[4|v|3\rangle}$$

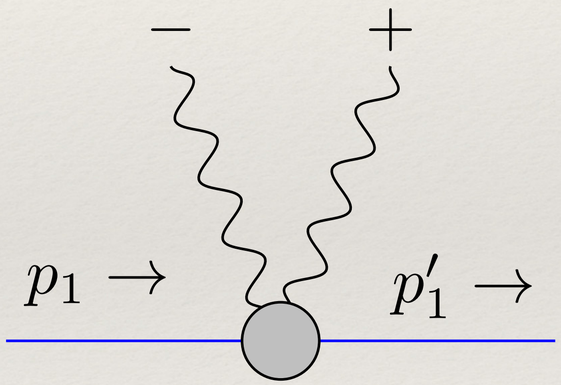
Compton Scattering – opposite helicity fixed

Solution: Add missing boundary term!

$$\mathcal{A}_4^{-+,s} = e^{(q_4-q_3)\cdot a}\sum_{n=0}^{2s}\frac{1}{n!}\bar{K}_n + C$$

$$K_n = \frac{[4|p_1|3\rangle^4}{s_{34}t_{13}t_{14}}\left(\frac{(t_{13}-t_{14})}{2}\frac{[4|a|3\rangle}{[4|p_1|3\rangle}\right)^n$$

$$\bar{K}_n = \begin{cases} K_n, & n \leq 4 \\ K_4 L_{n-4} - K_3 s_2 L_{n-5}, & n > 4 \end{cases}$$



Remarkably simple!

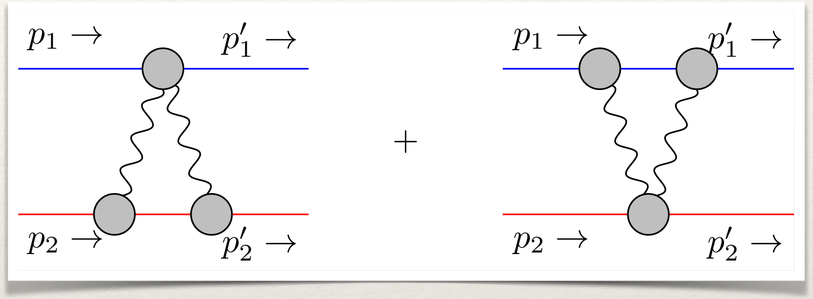
(Surprising!) Shift symmetry $\quad a^\mu \to a^\mu + q^\mu$

[Aoude, Haddad, & AH, JHEP '22]

Second Post-Minkowskian Amplitude

The amplitude is remarkably simple!

$$\mathcal{A}_{\rm 2PM} = \frac{2G^2\pi^2 m_1^2 m_2^2}{\sqrt{-q^2}}\left(A^{\rm even}_{\rm 2PM} + i\epsilon(v_1 v_2 q a_1) A^{\rm odd}_{\rm 2PM}\right)$$



Odd-in-spin amplitude

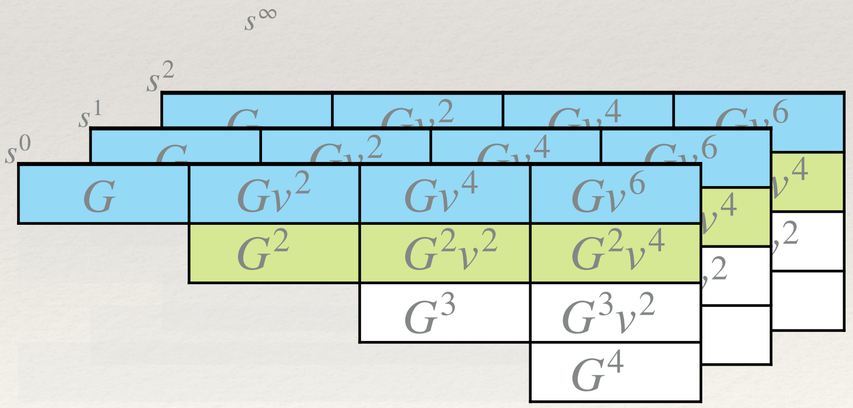
$$\mathcal{A}^{\rm odd}_{\rm 2PM} = -m_1\left(4\mathcal{F}_1 + \sum_{k=0}^{\infty}\frac{2\gamma^2-1}{(\gamma^2-1)^{k+1}}\frac{(-1)^k 8}{4^k(\tfrac{1}{2})_k}\mathcal{F}_k V^k\right) + m_2(\dots)$$

The spin enters in special functions

$$\mathcal{F}_j = {}_0F_1(j+1; Q/4)/\Gamma[j+1]$$

$$Q = (q\cdot a)^2 - q^2 a^2$$

$$V = q^2(v_2\cdot a)^2$$

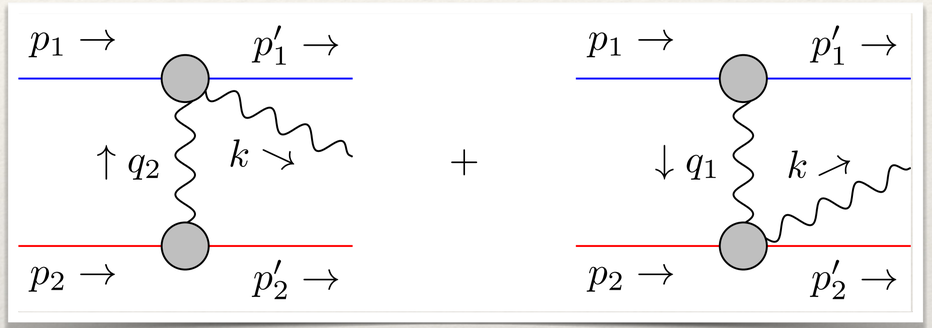


[Aoude, Haddad, & AH, PRL '22]

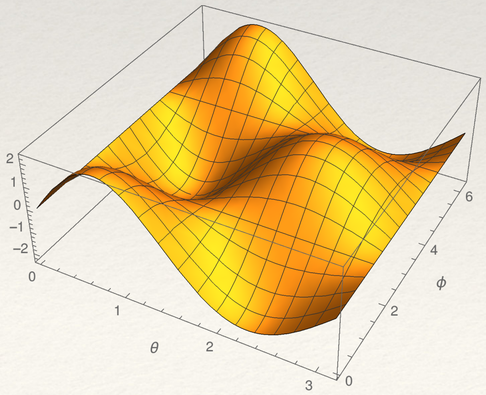
Leading-order gravitational radiation

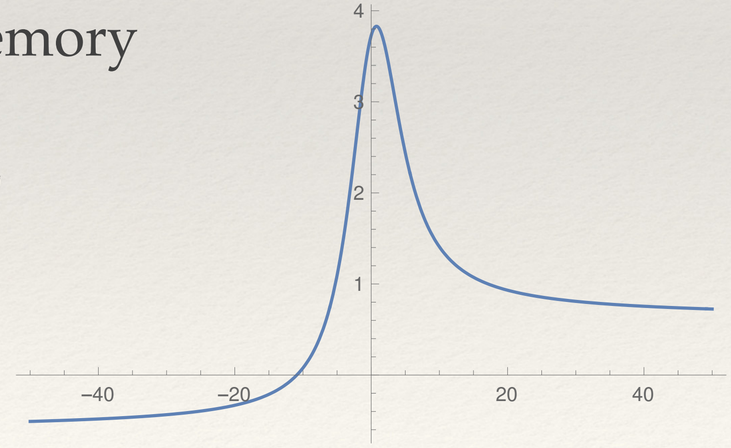
We use the same amplitude to compute the waveform

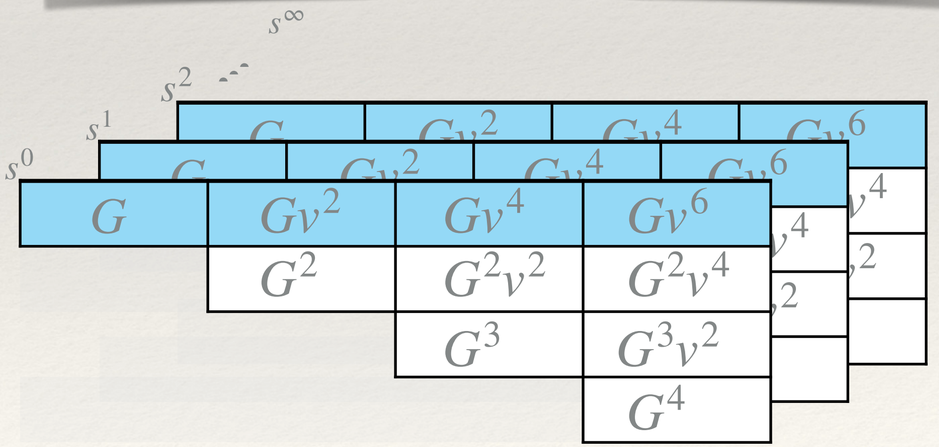
$$\mathcal{A}_5^{\text{cut}} = -\frac{\kappa^3 q_2^\mu q_2^\nu}{8q_2^2} \sum_{h=\pm,n=0}^{4} \frac{r_{(1),n}^{h,\mu\nu}}{2^n} e^{-hq_2\cdot a_2} \mathcal{A}_4^{h,(n)} + (1 \leftrightarrow 2)$$



Gravitational memory







[Aoude, Haddad, Heissenberg, & AH, PRD '23]

Neutron stars

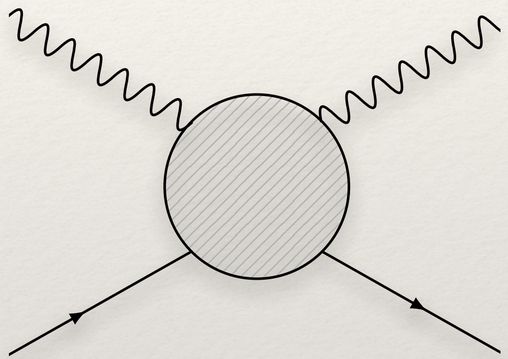
We model neutron stars as point particles.

Tidal deformability = effective operators.

$$\mathcal{O}^{(L)}_{\text{tidal}} = c_{E^{L+1}} \phi^2 \text{tr}[E^{L+1}]$$

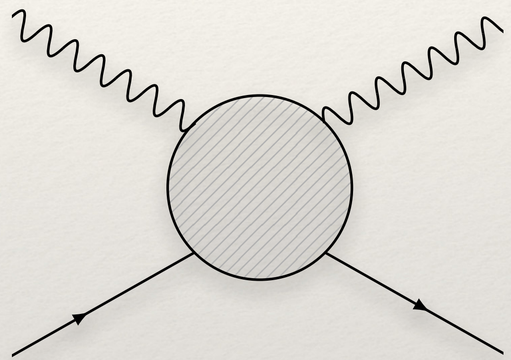
Tidal operators are of the form $\phi^2 R^2 D^{2n}$

How to construct them systematically?



Hilbert series for tidal effects

Hilbert series $$\mathcal{H} = \int d\mu \frac{1}{P} \mathrm{PE}[\chi_\phi]$$



- Generate tensor product of fields and derivatives $\mathrm{PE}[\chi_\phi]$
- Remove total derivative $\frac{1}{P}$
- Pick out invariant operators $\int d\mu$

Group theory problem: construct basis of effective operators

$$\mathcal{H}^{C^2}_{6+2n} = \lfloor \tfrac{n+2}{2} \rfloor (C_L^2 + C_R^2)\phi^2\nabla^{2n} + \lfloor \tfrac{n}{2} \rfloor C_L C_R \phi^2 \nabla^{2n}$$

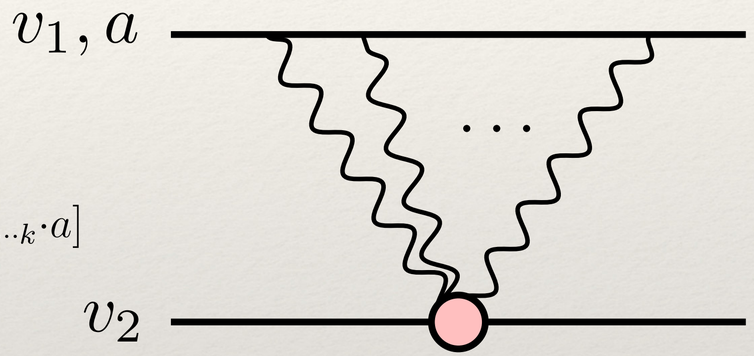
[Haddad & AH, JHEP '20]

Tidal probe in Kerr background

[Aoude & AH, '25]

Kerr generating functions

$$G^{(L,k)}(a) \equiv \int \hat{d}^D\ell_1 \cdots \hat{d}^D\ell_L \frac{1}{(2v_1\cdot\ell_1)\cdots(2v_1\cdot\ell_{1\cdots L})} \times \frac{1}{\ell_1^2\cdots\ell_L^2(q-\ell_{1\cdots L})^2} \exp[\ell_{1\cdots k}\cdot a]$$



Neutron star scattering

$$\chi_{\rm Kr} = \frac{1}{2}\sum_{\pm} \chi^{(k=0,s=0)}_{\varphi^2 R^{L+1}}(b \mp ia)$$

$$\chi^{(k=0,s=0)}_{\varphi^2 R^{L+1}} = \frac{c_{E^{L+1}} c_L (\kappa^2 m_1^2/16)^{L+1}}{4m_1 m_2\sqrt{\gamma^2-1}} \left(\frac{1}{\pi b^2}\right)^{\frac{3L}{2}+1} \frac{(-1)^{L+1}\pi^{L/2}\Gamma[\frac{3L}{2}+1]^2}{3\Gamma[L]\Gamma[L+2]} \left(\delta_{L,1} + \frac{4}{(9L^2-1)}\right)$$

Future of gravitational-wave science

The future of gravitational-wave science is bright. Einstein Telescope, LISA and more!

Urgent need for more high-precision calculations of black holes in extreme mass ratio binaries.

Combine QFT/amplitudes with gravity to optimize our theoretical toolkit.

