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# Universal Limits on Quantum Noise

UiO, 04.02.2026

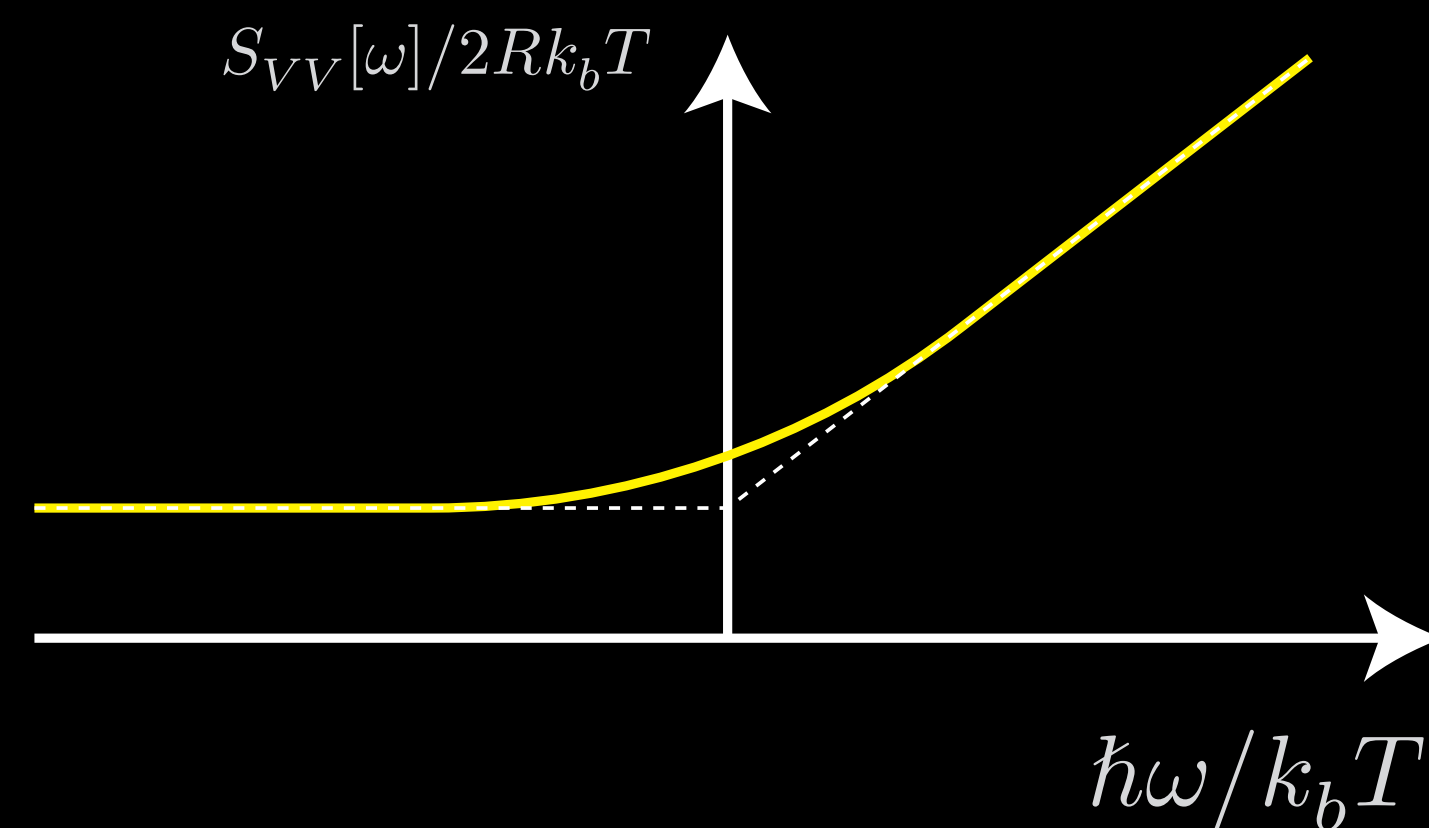
F. Massel



# QUANTUM vs CLASSICAL REGIMES

Quantum behavior:

- interference
- entanglement
- uncertainty product of incompatible observables  $\Delta X \Delta P \geq \hbar/2$



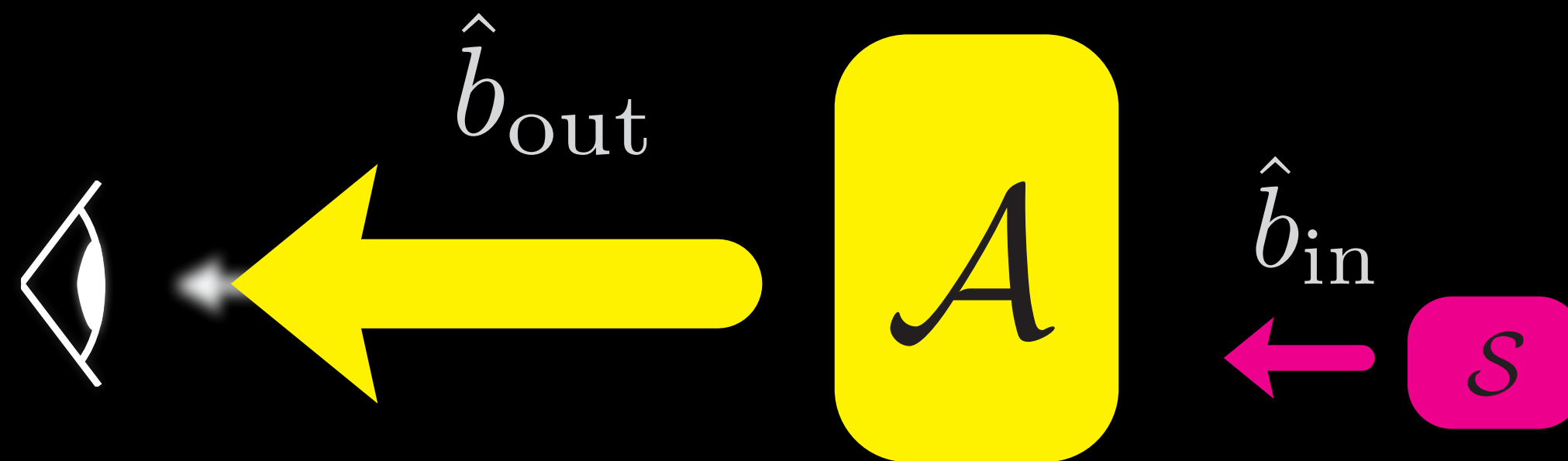
A diagram illustrating the Feynman path integral. It shows two white circular points representing initial and final states, labeled  $\psi(q_i, t_i)$  and  $\psi(q_f, t_f)$ . A solid red curve with an arrow represents the classical path. Two dashed black curves with arrows represent alternative paths. The equation below the diagram is:

$$K(q_f, t_f; q_i, t_i) = \int_{q_i}^{q_f} Dq \exp\left(\frac{i}{\hbar} S[q]\right) \xrightarrow{\hbar \rightarrow 0} A(q_f, q_i) \exp\left[\frac{i}{\hbar} S_{cl}(q_f, t_f; q_i, t_i)\right]$$

# QUANTUM AMPLIFICATION

Quantum / classical boundary conceptually relevant, but also important for quantum technology.

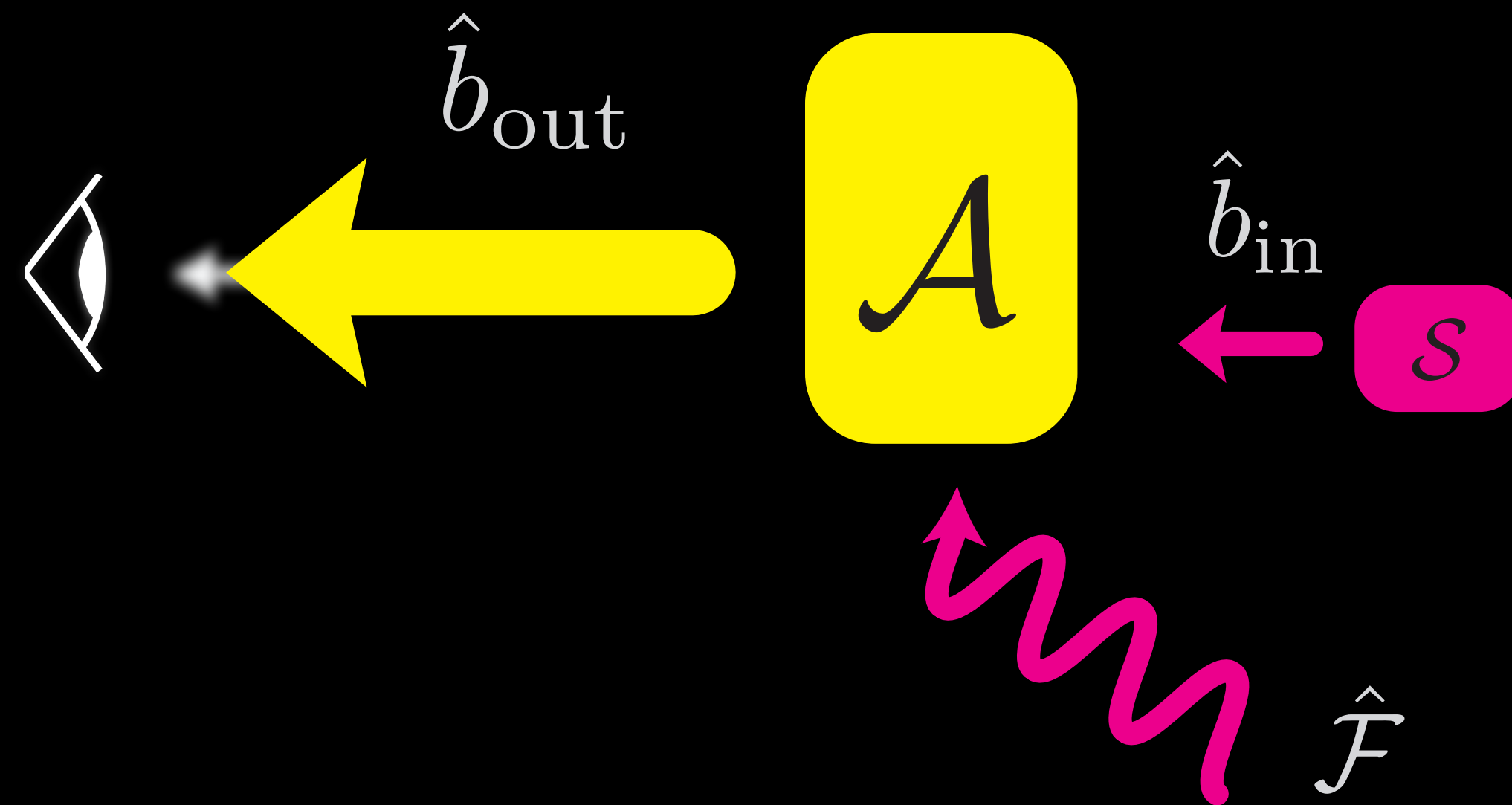
Measurement and characterization of quantum systems:  
access to quantum state by (classical) observer



$\hat{b}_{\text{in}}, \hat{b}_{\text{out}}$  are bosonic operators, characterizing the input and the output of an amplifier.

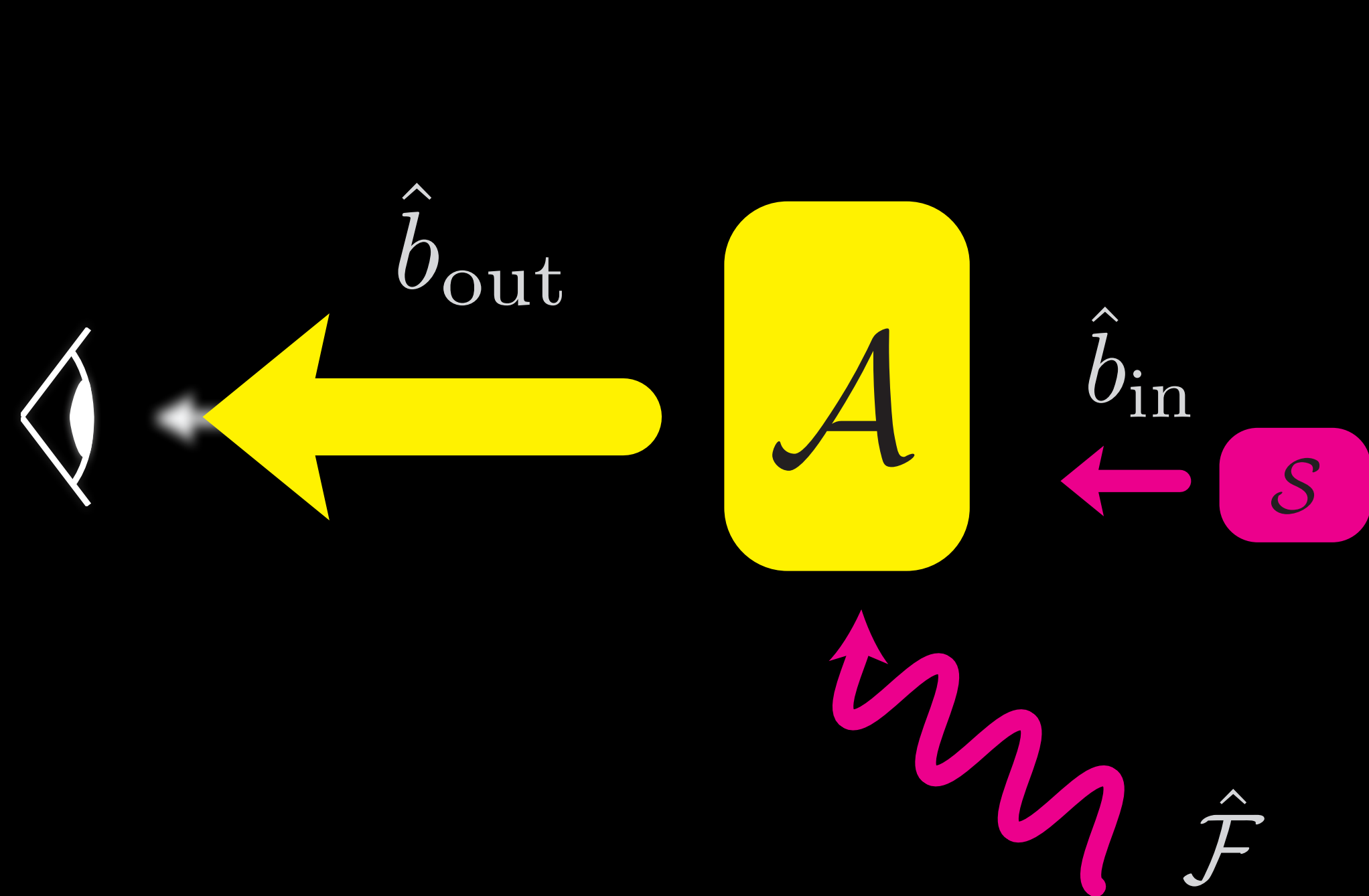
# QUANTUM AMPLIFICATION

Quantum limits on amplification [1].



For (linear) quantum amplifiers, the laws of qm impose that the amplification process adds a minimal amount of noise.

# QUANTUM AMPLIFICATION



$$b_{\text{out}} = M b_{\text{in}} + L b_{\text{in}}^{\dagger} + \mathcal{F}$$

Narrowband, linear amplifier

$\in \mathbb{C}$

The requirement that I/O the operators are *bona fide* bosonic operators ( $[b_{\text{out}}, b_{\text{out}}^{\dagger}] = [b_{\text{out}}, b_{\text{out}}^{\dagger}] = 1$ ) imposes that

$$1 = |M|^2 - |L|^2 + [\mathcal{F}, \mathcal{F}^{\dagger}]$$

Physical realizability (PR)  
(unitarity for Caves)

# QUANTUM AMPLIFICATION

We can write any operator  $\mathcal{F}$  as

$$\mathcal{F} = \frac{1}{\sqrt{2}} (F_R + iF_I)$$

If we define

$$\text{Var} [\mathcal{F}] \doteq \frac{1}{2} \langle \mathcal{F} \mathcal{F}^\dagger + \mathcal{F}^\dagger \mathcal{F} \rangle - \langle \mathcal{F}^\dagger \rangle \langle \mathcal{F} \rangle$$

the generalized Heisenberg uncertainty principle

$$\text{Var} [F_R] \text{Var} [F_I] \geq |[F_R, F_I]|$$

$$\text{gives } \text{Var} \mathcal{F} \geq \frac{1}{2} |\langle [\mathcal{F}, \mathcal{F}^\dagger] \rangle|$$

PR condition  $1 = |M|^2 - |L|^2 + [\mathcal{F}, \mathcal{F}^\dagger]$  therefore forces

$$\text{Var} \mathcal{F} \geq \frac{1}{2} \left| 1 - |M|^2 - |L|^2 \right|$$

# QUANTUM AMPLIFICATION

For a phase-preserving amplifier (equations independent of I/O phase choice),  $L = 0$

$$b_{\text{out}} = M b_{\text{in}} + \mathcal{F}$$

(and  $\mathcal{F}$  does not depend on phase either  $G = |M|^2$ ), we have

$$\text{Var } b_{\text{out}} = G \text{Var } b_{\text{in}} + \text{Var } \mathcal{F}$$

the relation  $\text{Var } \mathcal{F} \geq \frac{1}{2} \left| 1 - |M|^2 - |L|^2 \right|$  allows us to write

$$\text{Var } b_{\text{out}} \geq \frac{G}{2} + \frac{1}{2} (G - 1) \xrightarrow{G \gg 1} \frac{G}{2} + \frac{G}{2}$$

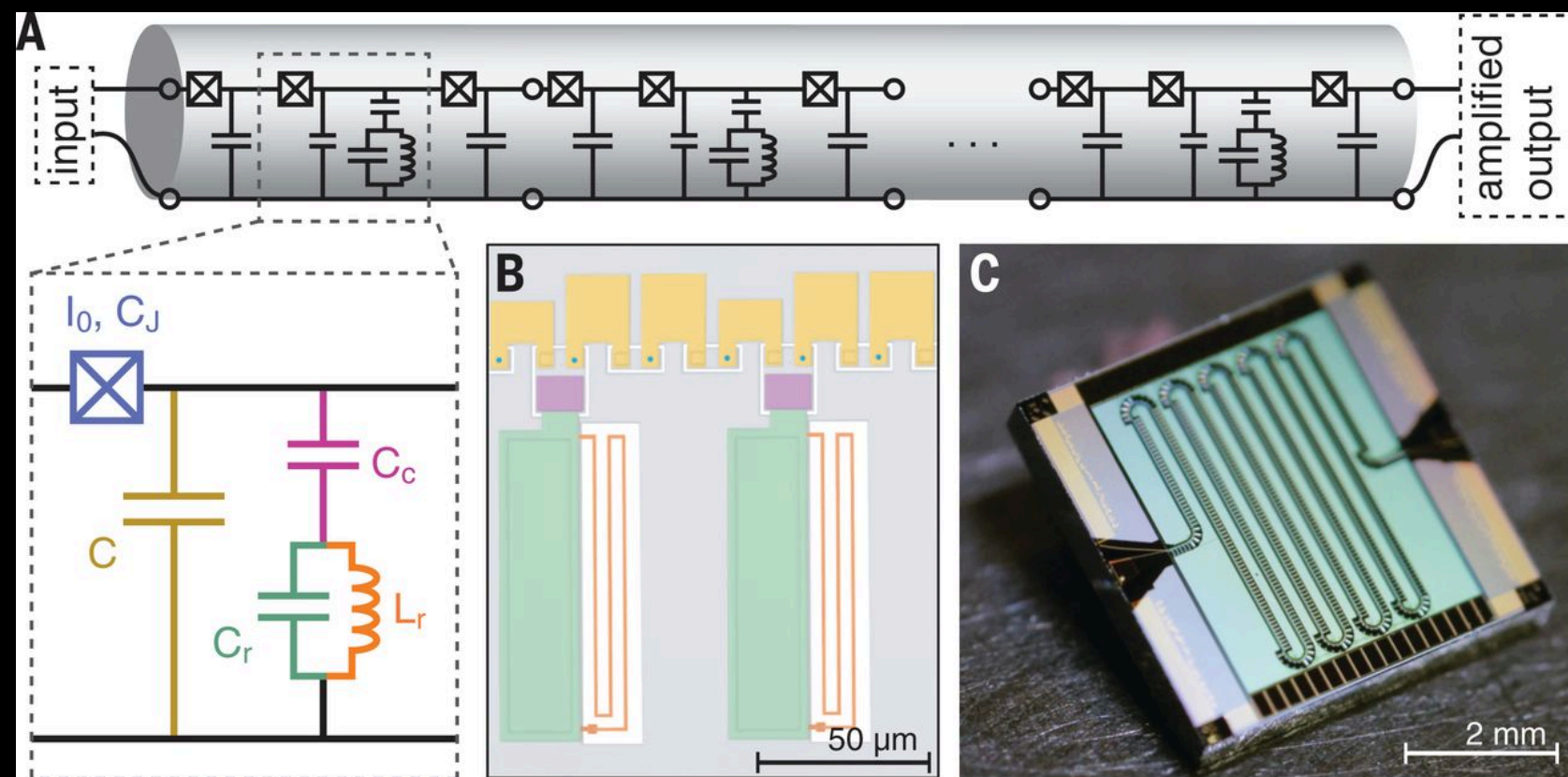
input noise

added noise

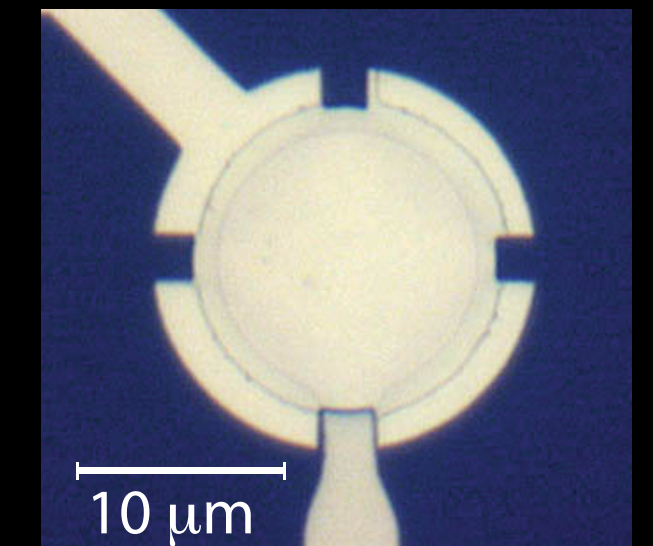
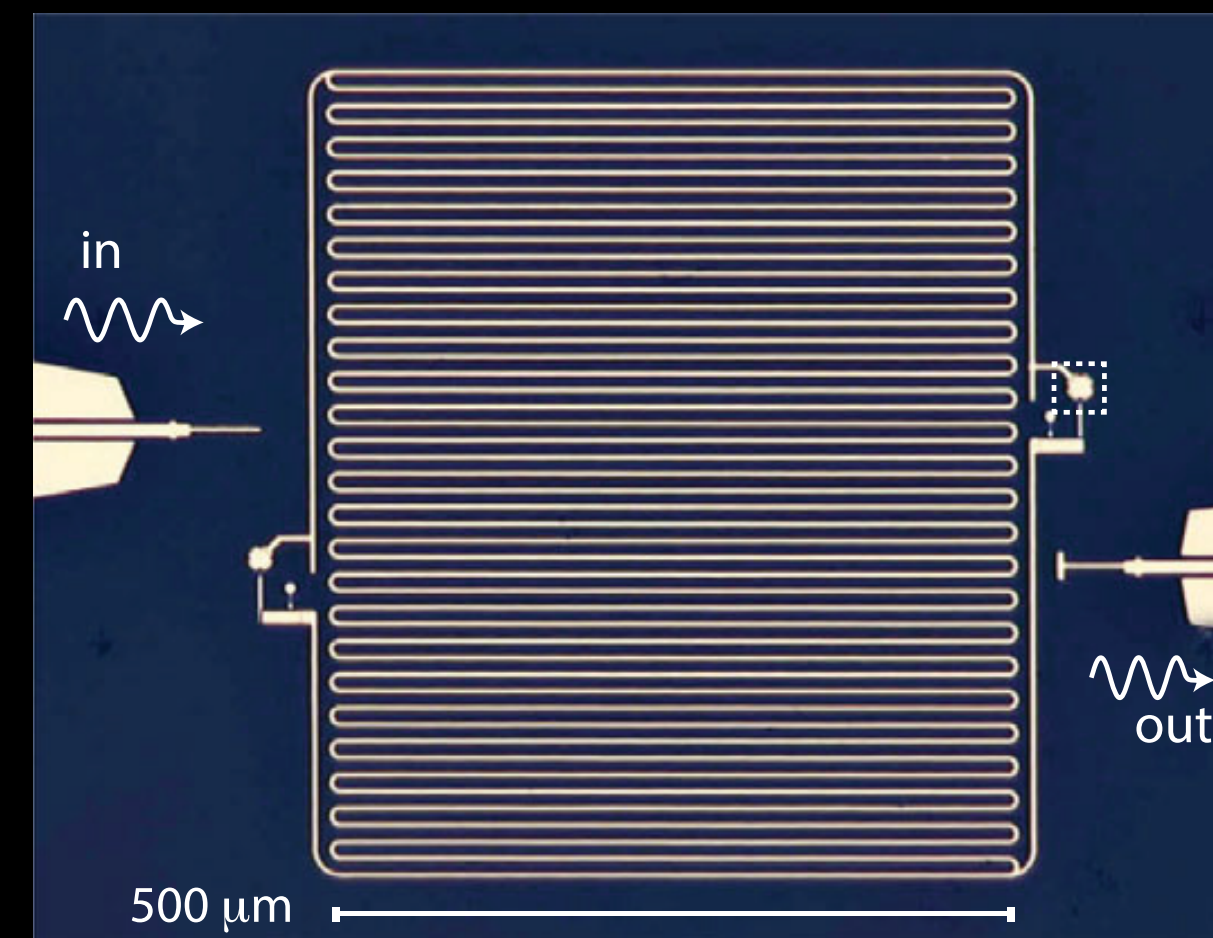
# QUANTUM AMPLIFICATION

Examples of (nearly) q-limited amplifiers

Josephson parametric amplifiers [1]



Optomechanical amplifiers [2]



[1] C. Macklin, et al, A near-quantum-limited Josephson traveling-wave parametric amplifier, *Science* **350**, 307 (2015).

[2] F. Massel, et al, Microwave amplification with nanomechanical resonators, *Nature* **480**, 351 (2011).

# PHYSICAL REALIZABILITY

Let's generalize the previous statement concerning the commutators of input and output fields.

Let's consider a linear quantum network

$$\dot{\xi} = \bar{A}\xi + D\xi_{\text{in}}$$

with

$$\xi = [a_1, \dots, a_N; a_1^\dagger, \dots, a_N^\dagger]$$

$$[\xi(t), \xi^\dagger(t)] = J, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$[\xi_{\text{in}}(t), \xi_{\text{in}}^\dagger(t')] = J_{\text{in}} \delta(t - t')$$

$\bar{A}$  drift matrix

$$D = \begin{bmatrix} D_a \\ D_c \end{bmatrix}$$

$$D_a, D_c = \begin{bmatrix} \sqrt{\gamma_1} & & & \sqrt{\gamma_1} & & \\ & \sqrt{\gamma_2} & & & \sqrt{\gamma_2} & \\ & & \ddots & & & \\ & & & \ddots & & \end{bmatrix}$$

# PHYSICAL REALIZABILITY

In order to be physical a linear quantum network

$$\dot{\xi} = \bar{A}\xi + D\xi_{\text{in}}$$

Must fulfill the following relation

$$\bar{A}J + J\bar{A}^\dagger + D J_{\text{in}} D^\dagger = 0$$

Physical realizability

generalization of the condition  $1 = |M|^2 - |L|^2 + [\mathcal{F}, \mathcal{F}^\dagger]$

# PHYSICAL REALIZABILITY

For a PR system, I can then use  $\bar{A}$  to evaluate the matrix  $X_i$  (“how much ‘commutator budget’ goes to mode  $i$ ”)

$$\bar{A}^\dagger X_i + X_i \bar{A} + \Pi_i^\dagger \Sigma_2 \Pi_i = 0.$$

$\Pi_i$  mode selector  
 $[a_i, a_i^\dagger]^T = \Pi_i \xi$

Commutator kernel (for mode  $i$ )

$$\Sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$K_i \doteq D_a^\dagger X_i D_a$$

Interpretation:  $(K_i)_{jj}$  is the “CCR share” supplied by annihilation port  $j$  to the internal mode  $i$ . Under PR + stability you get  $K_i \succeq 0$  and, with a complete port set, and  $[a_i, a_i^\dagger] = 1$ ,  $\text{Tr } K_i = 1$ .

# PHYSICAL REALIZABILITY

The commutator channel  $K_i$  encodes properties about the commutators.  
We want to relate this to bounds on expectation values in various settings.

$$a_i = \int \frac{d\omega}{2\pi} \left[ M_{i,1}(\omega) a_{1,in} + L_{i,1}(\omega) a_{1,in}^\dagger + \dots \right]$$

In the absence of Bogoliubov terms  
( $L_{i,j}(\omega) = 0$ ), we have that

$$\langle a_i^\dagger a_i \rangle = \int \frac{d\omega}{2\pi} \left[ M_{i,1}(\omega) \langle a_{1,in}^\dagger a_{1,in} \rangle + \dots \right]$$

$$\int \frac{d\omega}{2\pi} |M_{i,j}(\omega)|^2 = (K_i)_{jj} \leq 1$$

i.e. the power “gain” for each  
path  $i \leftarrow j$  is bounded by  
the  $j,j$  element of the  
commutator channel  $K_i$ .

## From PR to observable bounds

For  $L_{i,j} = 0$ , this is true for the quadratures  $\text{Var } X_i \doteq \frac{1}{2} \left[ \langle (a^\dagger + a)(a^\dagger + a) \rangle - \langle a^\dagger + a \rangle^2 \right]$  as well

$$\text{Var } X_i = \int \frac{d\omega}{2\pi} \left[ |M_{i,1}(\omega)|^2 \text{Var } X_{1,in} + \dots \right]$$

$$\int \frac{d\omega}{2\pi} |M_{i,j}(\omega)|^2 = (K_i)_{jj} \leq 1$$

# QLE in optomechanics

QM Hamiltonian

$$H = \omega_c a^\dagger a + \omega_m b^\dagger b + g a^\dagger a (b^\dagger + b)$$

Let's write the EOMs for the fields  $a$  and  $b$

$$\begin{cases} \dot{a}_t &= -i\omega_c a_t - \frac{\kappa}{2} a_t - ig_0 a_t (b_t^\dagger + b_t) + \sqrt{\kappa} a_t^{\text{in}} \\ \dot{b}_t &= -i\omega_m b_t - \frac{\gamma}{2} b_t - ig_0 a_t^\dagger a_t + \sqrt{\gamma} b_t^{\text{in}} \end{cases}$$

Where do we go from here?

Impose a strong **coherent drive** to the “optical reservoir”: its form determines most of the interesting results in the field of optomechanics of the last few years.

$$a_t^{\text{in}} \rightarrow \alpha^{\text{in}} \exp[-i\omega_p t] + a_t^{\text{in}}$$

## Quantum Langevin equations

The idea is that one can linearize the EOMs for the quantum fluctuations.

$$a_t^{\text{in}} \rightarrow \alpha^{\text{in}} \exp[-i\omega_p t] + a_t^{\text{in}}$$

The easiest way of seeing this is to rewrite the radiation-pressure interaction Hamiltonian  $H_I = g_0 a^\dagger a (b^\dagger + b)$  ( $a \rightarrow \alpha \exp[-i\omega_p t] + a, b \rightarrow \beta + b$ )

$$H_I = g_0 \alpha^* \alpha (\beta^* + \beta + b^\dagger + b) + g_0 \alpha^* \exp[i\omega_p t] a (\beta^* + \beta + b^\dagger + b) + \text{h.c.} + \text{higher order}$$

relevant terms

Depending on  $\omega_p$ , we have that terms like  $a^\dagger b$  (beam splitter) or  $a^\dagger b^\dagger$  (parametric-down converter) survive (rotating-wave approximation).

# Quantum Langevin equations

Linearized EOMs for the quantum fluctuations.

$$\dot{a} = -i\omega_c a - \frac{\kappa}{2}a - iG(b^\dagger + b) + \sqrt{\kappa}a_{\text{in}}$$

$$\dot{b} = -i\omega_m a - \frac{\gamma}{2}b - i(Ga^\dagger + G^*a) + \sqrt{\gamma}b_{\text{in}}$$

with  $G = g_0\alpha \exp[i\omega_p t]$

We can control the interaction between the modes as an external “knob”.

## Quantum Langevin equations

If  $\omega_p \simeq \omega_c - \omega_m$  (red-detuned pump)

$$\dot{a} = -i\omega_c a - \frac{\kappa}{2}a - iG(b^\dagger + b) + \sqrt{\kappa}a_{\text{in}}$$

$$\dot{b} = -i\omega_m a - \frac{\gamma}{2}b - i(Ga^\dagger + G^*a) + \sqrt{\gamma}b_{\text{in}}$$

which corresponds to the QLEs we would obtain for  $H_I = G a^\dagger b + \text{h.c.}$  (beam-splitter Hamiltonian).

This allows the cooling of the mechanics mode  $b$  to the ground state (since  $\omega_c \gg \omega_m$ ).

# Quantum Langevin equations

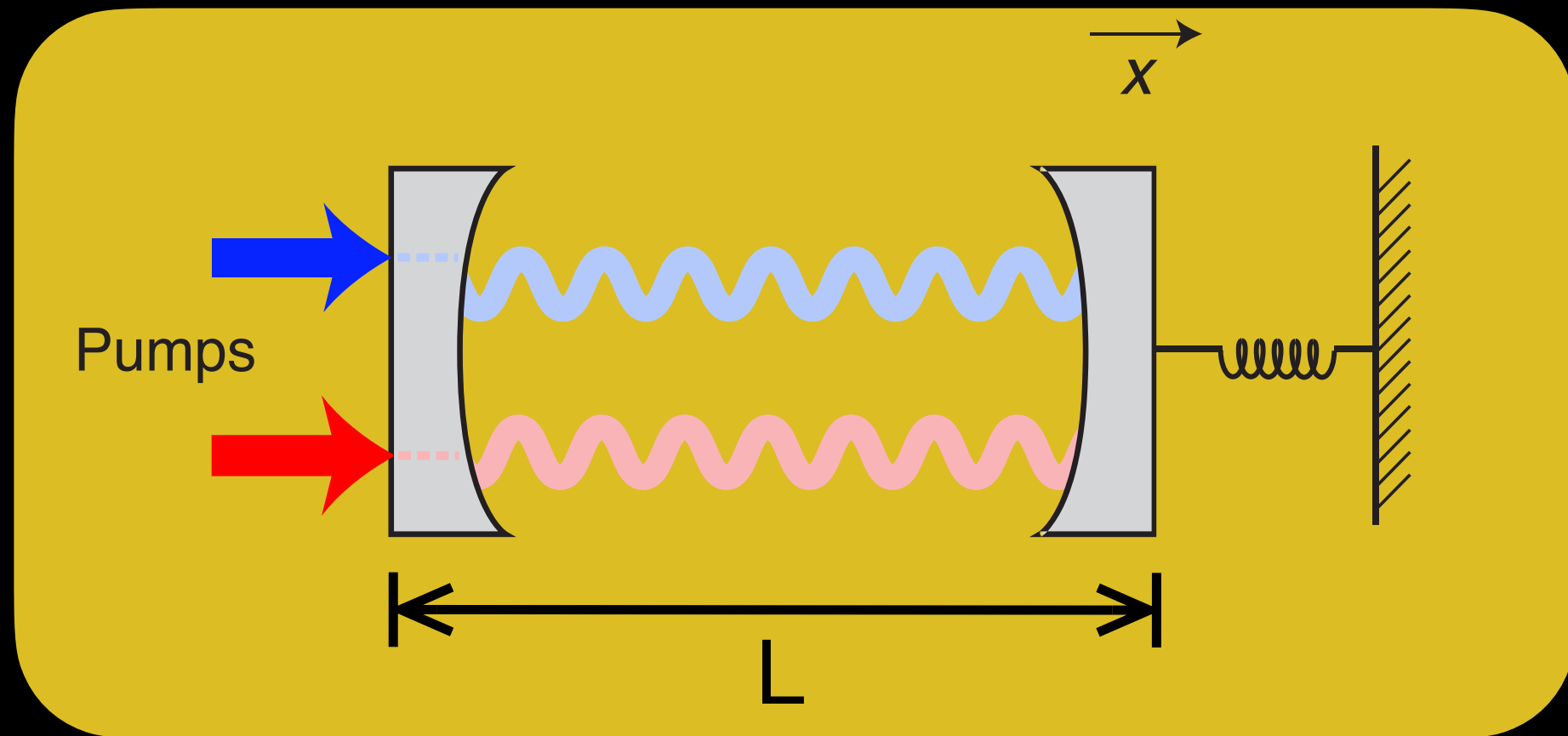
If  $\omega_p \simeq \omega_c + \omega_m$  (blue-detuned pump)

$$\begin{aligned}\dot{a} &= -i\omega_c a - \frac{\kappa}{2}a - iG(b^\dagger + \cancel{b}) + \sqrt{\kappa}a_{\text{in}} \\ \dot{b} &= -i\omega_m a - \frac{\gamma}{2}b - i(Ga^\dagger + \cancel{G^*a}) + \sqrt{\gamma}b_{\text{in}}\end{aligned}$$

which corresponds to the QLEs we would obtain for  $H_I = G a^\dagger b^\dagger + \text{h.c.}$

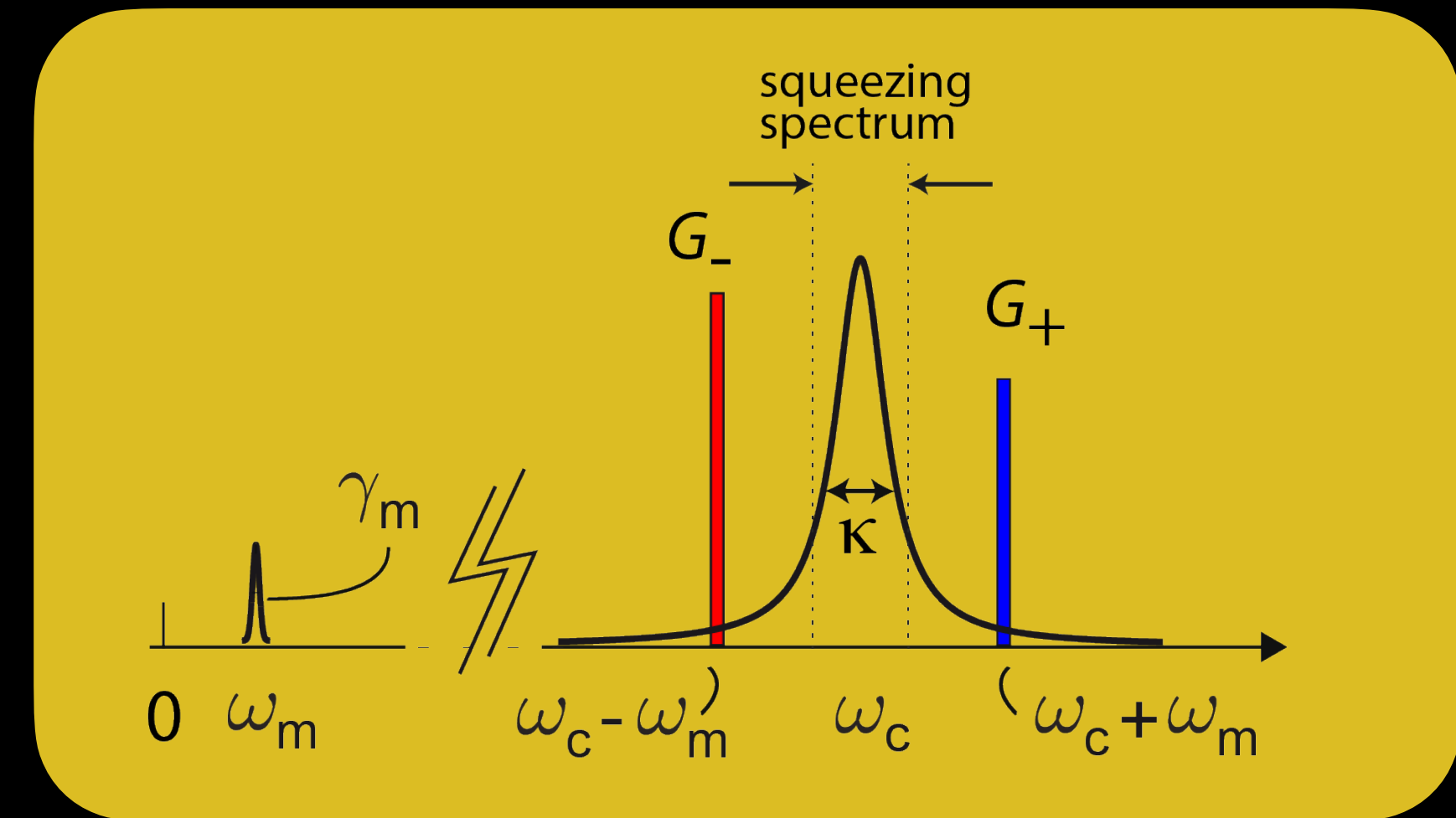
This leads to “mechanical lasing”.

# Squeezing of the mechanical motion



Imposing two tones at the appropriate frequencies, the Hamiltonian (for the field fluctuations) becomes

$$H = \omega_c a^\dagger a + \omega_m b^\dagger b + (G_+ a^\dagger b^\dagger + G_- a^\dagger b) + \text{h.c.}$$



Bogolyubov transformation

$$\beta = \frac{1}{\mathcal{G}} (G_+ b^\dagger + G_- b)$$

Cooling of a Bogolyubov mode

$$H = \omega_c a^\dagger a + \omega_m \beta^\dagger \beta + \mathcal{G} a^\dagger \beta + \text{h.c.}$$

Wollman, E. E., et al. Science, 349(6251), 952–955 (2015).

J. M. Pirkkalainen et al., Phys. Rev. Lett. 115, 243601 (2015).

Lecocq, F., et al. (2015). Phys Rev X, 5(4), 041037 (2015)

## From PR to observable bounds

Let's focus on the cooling case (beam splitter Hamiltonian)

$$\begin{aligned}\dot{a}_1 &= -\frac{\gamma_1}{2}a_1 - iGa_2 + \sqrt{\gamma_1}a_{1,in} & a_1 &= \int \frac{d\omega}{2\pi} [M_{1,1}(\omega)a_{1,in} + M_{1,2}(\omega)a_{2,in}] \\ \dot{a}_2 &= -\frac{\gamma_2}{2}a_2 - iGa_1 + \sqrt{\gamma_2}a_{2,in} & a_2 &= \int \frac{d\omega}{2\pi} [M_{2,1}(\omega)a_{1,in} + M_{2,2}(\omega)a_{2,in}]\end{aligned}$$

In this case, it is easy to show that  $||M_{i \leftarrow j}||^2 \leq \gamma_j/(\gamma_i + \gamma_j)$ . From PR we have that

$$||M_{i \leftarrow i}||^2 + ||M_{i \leftarrow j}||^2 = \text{Tr}(K_i) = 1$$

Implying  $||M_{i \leftarrow i}||^2 \geq \frac{\gamma_i}{\gamma_i + \gamma_j}$

# From PR to observable bounds

For the variances

$$\text{Var } X_1 = \|M_{1 \leftarrow 1}\|^2 \text{Var } X_{in,1} + \|M_{1 \leftarrow 2}\|^2 \text{Var } X_{in,2}$$

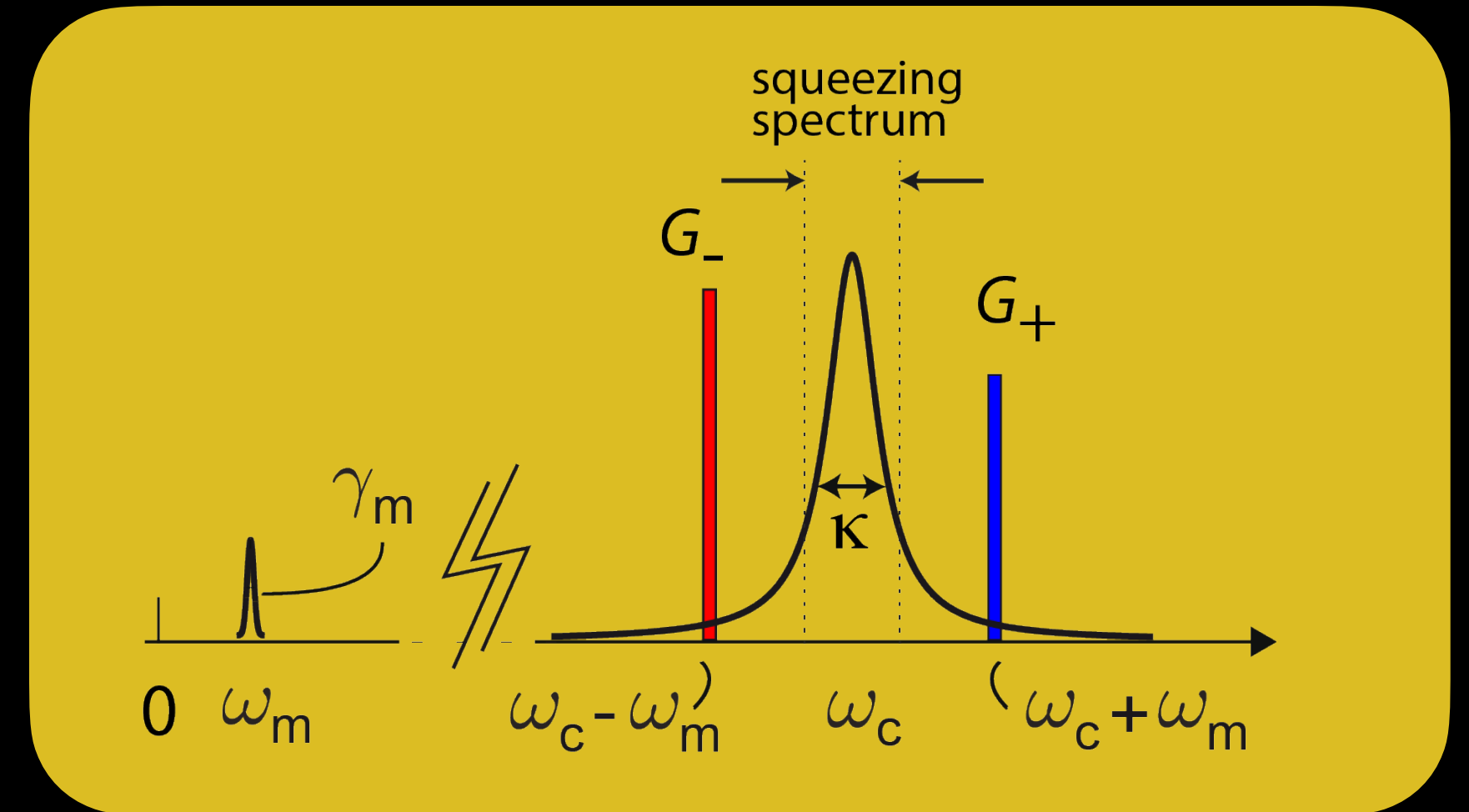
$$\|M_{i \leftarrow i}\|^2 \geq \frac{\gamma_i}{\gamma_i + \gamma_j} \text{ implies}$$

$$\frac{\text{Var } X_1}{\text{Var } X_{in,1}} \geq \frac{\gamma_1}{\gamma_1 + \gamma_2} \quad \frac{\text{Var } X_2}{\text{Var } X_{in,2}} \geq \frac{\gamma_2}{\gamma_1 + \gamma_2}$$

# From PR to observable bounds: dissipative squeezing

The EOMs in the original frame

$$\begin{aligned}\dot{a} &= -\frac{\gamma_1}{2}a - iG_+b - iG_-b^\dagger + \sqrt{\gamma_1}a_{in} \\ \dot{b} &= -\frac{\gamma_2}{2}b - iG_+a - iG_-a^\dagger + \sqrt{\gamma_2}b_{in}\end{aligned}$$



become

$$\begin{aligned}\dot{a} &= -\frac{\gamma_1}{2}a - i\mathcal{G}\beta + \sqrt{\gamma_1}a_{in} \\ \dot{\beta} &= -\frac{\gamma_2}{2}\beta - i\mathcal{G}a + \sqrt{\gamma_2}\beta_{in}\end{aligned}$$

$$\begin{aligned}\beta &= S(\xi)bS^\dagger(\xi) = b\cosh[\xi] + b^\dagger\sinh[\xi] \\ G_- &= \mathcal{G}\cosh[\xi], \quad G_+ = \mathcal{G}\sinh[\xi]\end{aligned}$$

$$\frac{\text{Var } X_\beta}{\text{Var } X_\beta|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1}$$

$$\frac{\text{Var } Y_\beta}{\text{Var } Y_\beta|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1}$$

## From PR to observable bounds: dissipative squeezing

Transforming back to the original frame, the bound is unchanged.

$$\begin{array}{ccc} \frac{\text{Var } X_\beta}{\text{Var } X_\beta|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1} & \xrightarrow{X_\beta = e^\xi X_b} & \frac{\text{Var}[X_b]}{\text{Var}[X_b]|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1} \\ \frac{\text{Var } Y_\beta}{\text{Var } Y_\beta|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1} & \xrightarrow{Y_\beta = e^{-\xi} Y_b} & \frac{\text{Var } Y_b}{\text{Var } Y_b|_0} \geq \frac{\gamma_2}{\gamma_2 + \gamma_1} \end{array}$$

valid for all quadratures

$$\left. \frac{\text{Var}[X_a]}{\text{Var}[X_a]|_0} \right|_{min} \geq \frac{\gamma_1}{\gamma_2 + \gamma_1}$$

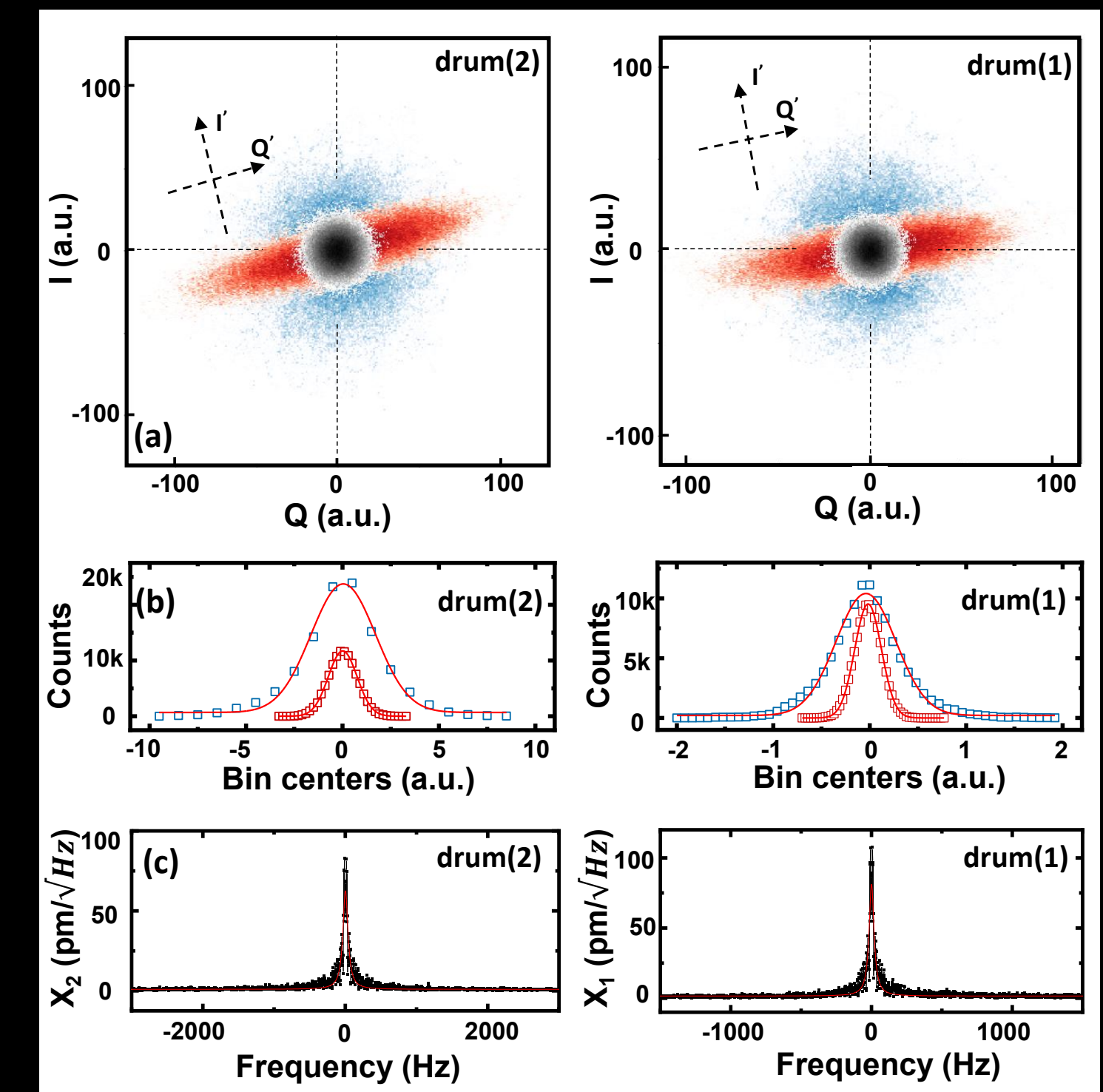
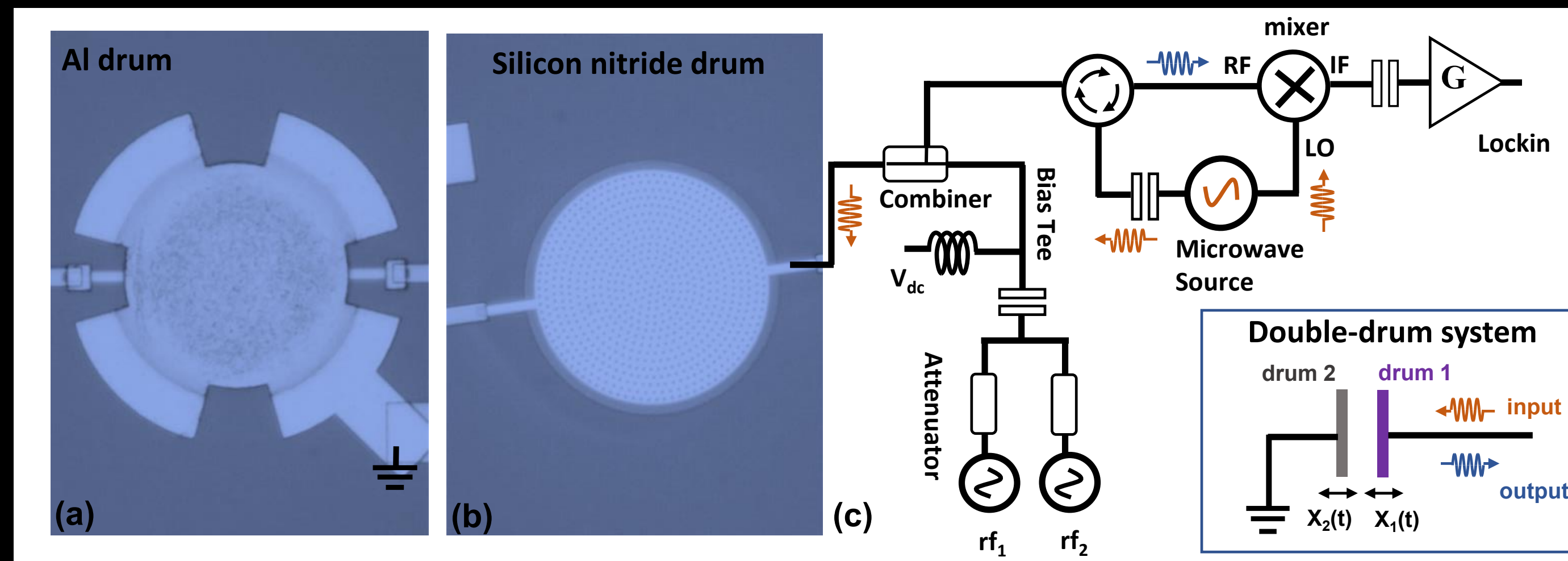
minimally squeezed quadratures

$$\left. \frac{\text{Var}[X_a]}{\text{Var}[X_a]|_0} \right|_{min} + \left. \frac{\text{Var}[X_b]}{\text{Var}[X_b]|_0} \right|_{min} \geq 1$$

# From PR to observable bounds: dissipative squeezing

$$\left. \frac{\text{Var}[X_a]}{\text{Var}[X_a]|_0} \right|_{min} + \left. \frac{\text{Var}[X_b]}{\text{Var}[X_b]|_0} \right|_{min} \geq 1 \quad \text{from PR (so from CCR)}$$

can be observed in an experiment @ room temperature.



# Take-home message

- Canonical commutation relations impose bounds on bosonic modes.
- This has consequences at the macroscopic level
  - Quantum limits on amplification
  - Squeezing power in dissipative squeezing setups
  - Parametric squeezing setups (or combined) can be discussed in the same framework
  - CV entanglement as well\*

\*to be done.