UiO : University of Oslo

Aspects of Resurgence

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Theory Seminar

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University of Oslo

Key points

Perturbation theory is an important tool in physics

- Unfortunately, it generally diverges
- Not well-defined even after resummation

Non-perturbative effects should be included explicitly

- In particular: quantum tunnelling

Resurgence sheds new light on these challenges in QM and QFT

- Perturbation theory encodes non-perturbative effects

Context

Quantum electrodynamics

- Excellent perturbative prediction of e^- magnetic moment

$$\left[\frac{1}{2}(g-2)\right]_{\text{th}} = 0.001\,159\,652\,181\,78(77)$$
$$\left[\frac{1}{2}(g-2)\right]_{\text{ex}} = 0.001\,159\,652\,180\,73(28)$$

Quantum chromodynamics

- Perturbativity breaks down as α_s grows towards low energy scales
- Recent progress in toy models using resurgence idea
  - Connected to QM with periodic potential

Outline

1. Can perturbation theory make sense?
2. Non-perturbative instanton effects
3. The cosine potential, perturbatively
4. Resurgence to the rescue

Can perturbation theory make sense?

Perturbation theory: traditional recipe gone wrong

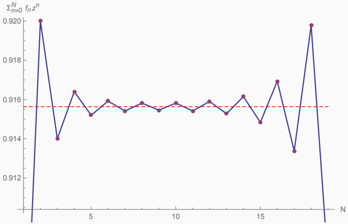
Find perturbative series through iterative procedures

$$E(g^2) = \sum_{n=0}^{\infty} a_n (g^2)^n$$

Due to factorial growth of number of Feynman diagrams: $a_n \sim n!$

$\rightarrow$ Generally divergent, but asymptotic series

- Converge at first, then diverge $\longrightarrow$ truncate optimally



Partial sum $\sum_{n=0}^{N} n! \left(-\frac{1}{10}\right)^n$ vs. N

2nd term: 0.02

10th term: 0.0004

20th term: 0.0243

Wait, really divergent?!

Yep. Look at quantum mechanical ground states

$$E(g^2) = \sum_{n=0}^{\infty} a_n (g^2)^n$$

Some physical examples:

Context	$a_n \sim \ldots$
Zeeman effect	$(-1)^n(2n)!$
Stark effect	$(2n)!$
Cubic oscillator	$\Gamma(n+\frac{1}{2})$
Quartic oscillator	$(-1)^n\Gamma(n+\frac{1}{2})$
Double well	$n!$
Periodic cosine well	$n!$

Really, really divergent

Also, Dyson (1952) asserted QED perturbation theory *must* diverge.

Physical argument: the perturbative expansion

$$F(e^2) = a_0 + a_1 e^2 + a_2 e^4 + \dots$$

is a power series converging inside some open disk.
However, for $e^2 < 0$:

"[...] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization."

Asymptotic Series 101

Perturbative expansions are generally divergent asymptotic series:

$$\sum_{n=0}^{\infty} f_n (z - z_0)^n = \sum_{n=0}^{N-1} f_n (z - z_0)^n + R_N(z)$$

- For any <u>fixed N</u>: when $z \to z_0$, remainder $|R_N(z)| \ll |z - z_0|^N$

$\leftrightarrow$ Convergent series: at <u>fixed z</u>, remainder $|R_N(z)| \to 0$ when $N \to \infty$

- Exponential accuracy possible when truncated optimally

Superasymptotics!

Consider the series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim (-1)^n n!$$

- Optimal truncation just before least term: $N \approx 1/z$, because

$$\frac{\mathrm{d}}{\mathrm{d}n} \ln |f_n z^n| \sim \ln nz$$

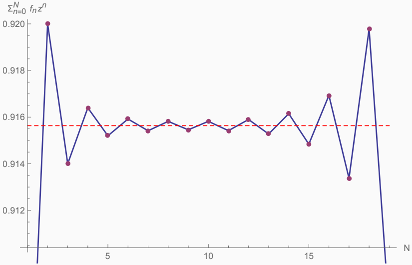
- The error is exponentially small:

$$|R_N(z)| \approx |f_N z^N| \sim N!\, N^{-N} \sim \sqrt{N} \mathrm{e}^{-N} \approx \frac{\mathrm{e}^{-1/z}}{\sqrt{z}}$$

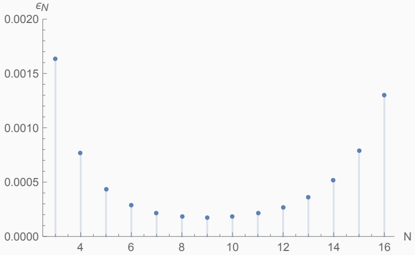
Superasymptotics: exponential accuracy

Consider the series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim (-1)^n n!$$



Partial sum for $z = \frac{1}{10}$

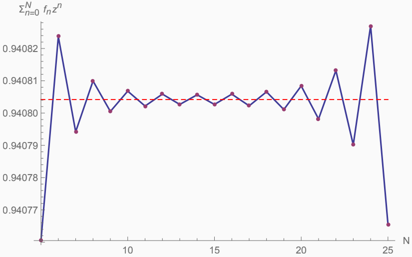


Remainder $R_N(z = \frac{1}{10})$

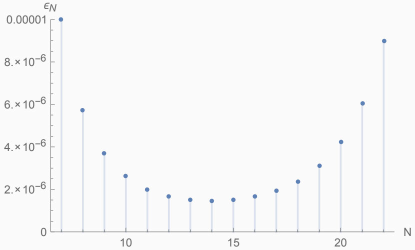
Superasymptotics: exponential accuracy

Consider the series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim (-1)^n n!$$



Partial sum for $z = \frac{1}{15}$



Remainder $R_N(z = \frac{1}{15})$

Borel resummation

Consider a factorially divergent series:

$$\sum_{n=0}^{\infty} f_n z^n$$

Step 1: Borel transform of the series:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n$$

Step 2: Resummation of the series:

$$\mathcal{S}[f](z) = \frac{1}{z} \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t/z} \, \mathcal{B}[f](t)$$

Borel resummation: example

Example of an alternating asymptotic series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim (-1)^n n!$$

→ Borel transform:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$$

→ Resummation:

$$\mathcal{S}[f](z) = \frac{1}{z}\int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t/z} \frac{1}{1+t} = \frac{1}{z}\mathrm{e}^{1/z} E_1\left(\frac{1}{z}\right)$$

Borel resummation: example

Example of an alternating asymptotic series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim (-1)^n n!$$

→ Borel transform:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$$

→ Resummation:

$$\mathcal{S}[f](z) = \frac{1}{z} \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t/z} \frac{1}{1+t} = \frac{1}{z} \mathrm{e}^{1/z} E_1\left(\frac{1}{z}\right)$$

→ Surprise:

$$\sum_{n=0}^{\infty} (-1)^n n! = 1 - 1 + 2 - 6 + 24 - 120 + \ldots = \mathrm{e}\, E_1(1) \approx 0.596$$

Borel resummation: another example

Example of a non-alternating asymptotic series

$$\sum_{n=0}^{\infty} f_n z^n \text{ with } f_n \sim n!$$

→ Borel transform:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

→ Resummation:

$$\mathcal{S}[f](z) = \frac{1}{z}\int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t/z} \frac{1}{1-t} = ?!$$

Directional resummation: a tale of two possibilities

Borel transform and resummed perturbation series:

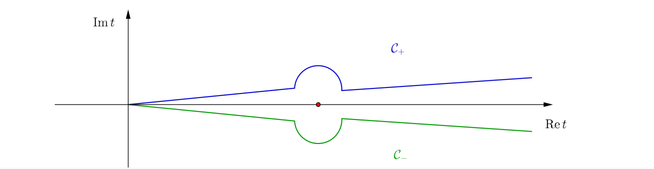
$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n \quad \longrightarrow \quad \mathcal{S}[f](z) = \frac{1}{z} \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t/z} \, \mathcal{B}[f](t)$$

Beware of singularities on the integration path!

$\rightarrow$ Non-Borel summable series

Avoiding singularities yields an ambiguous imaginary contribution

$\rightarrow$ Proportional to residue: $\mathrm{Im}\left[\sum_{n=0}^{\infty} n!\, z^n\right] = \pm\pi\, \frac{1}{z} \mathrm{e}^{-1/z}$



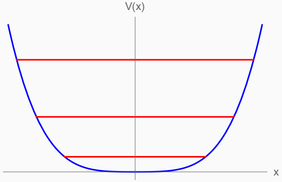
Imaginary energies, sure . . .

For unstable states, energy $E = \text{Re } E - i\frac{\Gamma}{2}$

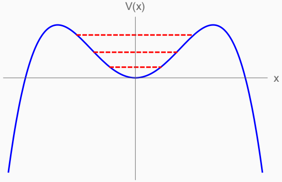
$\rightarrow$ Usual time evolution: $e^{-iEt} = e^{-it\text{Re } E} e^{-\Gamma t/2}$

$\rightarrow$ Lifetime $\tau = 1/\Gamma$

Example: quartic anharmonic oscillator $V(x) = \frac{1}{2}m\omega^2 x^2 + gx^4$



$g > 0$
stable



$g < 0$
unstable

Stable or unstable?

Context	$a_n \sim \dots$	Stability
Zeeman effect	$(-1)^n(2n)!$	stable
Stark effect	$(2n)!$	unstable
Quartic oscillator	$(-1)^n\Gamma(n+\frac{1}{2})$	stable
Cubic oscillator	$\Gamma(n+\frac{1}{2})$	unstable
Double well	$n!$	stable?!
Periodic cosine well	$n!$	stable?!

Not all imaginary ambiguities can be explained!

$\rightarrow$ Perturbation theory is ill-defined. What is missing?

Non-perturbative instanton effects

Quantum effects are important

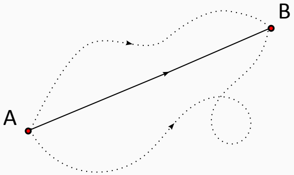
Path integrals determine quantum amplitudes as a sum over all possible paths

- Each path contribution is weighted by $e^{iS/\hbar}$

$$\text{Action } S[x(t)] = \int_{t_i}^{t_f} \mathrm{d}t \; [T - V]$$

- The classical path dominates the oscillatory integral
- But tunnelling trajectories (*instantons*) give quantum corrections!

Path integrals determine the energy spectrum via partition functions



Path integration

Action of trajectory $x(t)$:

$$S[x(t)] = \int_{-t_0/2}^{+t_0/2} \mathrm{d}t \left[\frac{1}{2} \left((x')^2 - V(x) \right) \right]$$

The Feynman path integral in Minkowski space:

$$\langle x_f | \mathrm{e}^{-iHt_0/\hbar} | x_i \rangle = N \int \mathcal{D}[x(t)] \; \mathrm{e}^{iS[x(t)]/\hbar}$$

- Oscillatory behaviour
- Semiclassical expansion in $\hbar$ starts with classical trajectory $x_{cl}(t)$

Path integration

After Wick rotation ($t = -i\tau$) to Euclidean space:

$$S_E[x(\tau)] = \int_{-T/2}^{T/2} \mathrm{d}\tau \left[\frac{1}{2}\left(\dot{x}^2 + V(x)\right)\right]$$

$$\langle x_f | \mathrm{e}^{-HT} | x_i \rangle = N \int \mathcal{D}[x(\tau)]\ \mathrm{e}^{-S_E[x(\tau)]/\hbar}$$

- Non-oscillatory, exponential weighting factor
- Inverted potential

Energy spectrum follows from partition function:

$$\sum_n \mathrm{e}^{-E_n\tau/\hbar} = N \int_{x_i = x_f} \mathcal{D}[x(\tau)]\ \mathrm{e}^{-S_E[x(\tau)]/\hbar}$$

Instantons in the double well

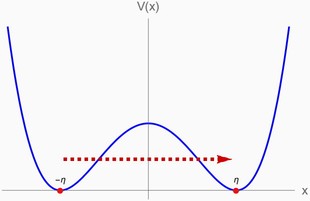
Classically, doubly-degenerate ground state

However, instantons change that picture qualitatively!

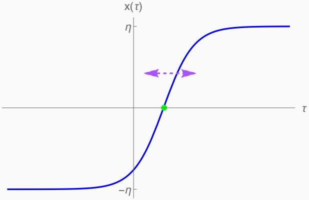
→ Non-perturbative splitting of ground-state energy level:

$$\Delta E \propto \hbar\, e^{-S_I/\hbar}$$

- S_I is the action of an instanton trajectory



double-well potential



instanton

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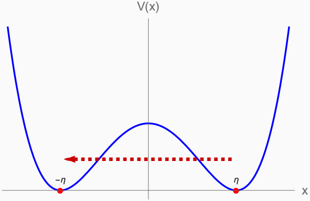
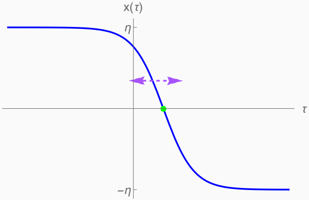


double-well potential



anti-instanton

Instantons in the double well

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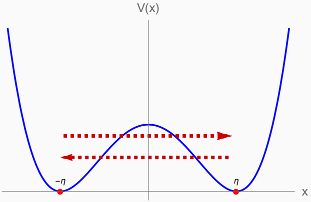
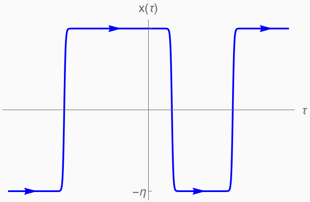


double-well potential



dilute instanton gas

Instantons in the cosine well

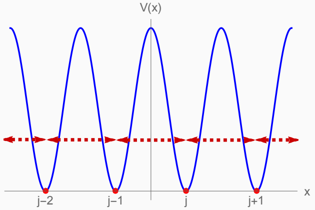
Classically, *infinitely*-degenerate ground state

However, instantons change that picture qualitatively!

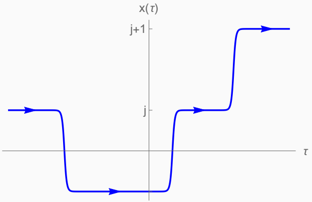
$\rightarrow$ Energy band structure for ground state:

$$\Delta E_{band} \propto \hbar\, e^{-S_I/\hbar}$$

- S_I is the action of an instanton trajectory



cosine potential



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Instantons in the cosine well

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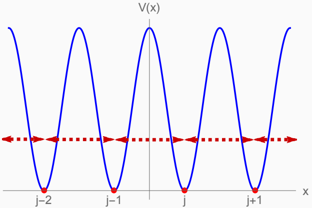
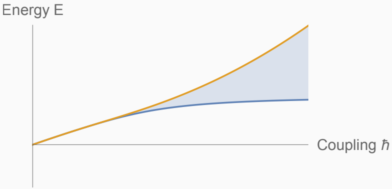


cosine potential



energy band

The cosine well, perturbatively

Perturbation theory for the cosine potential

Schrödinger equation

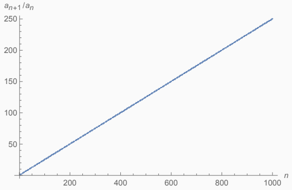
$$\left[-g^4\partial_x^2 + \cos(x)\right]\psi(y) = g^2 E\psi(x)$$

$\rightarrow$ Determine energy perturbatively:

$$E(g^2) = \sum_{n=0}^{\infty} a_n (g^2)^n$$

- For ground state, a_n computed to 1000 orders via *recurrence relations*
- Study of large-order behaviour:

$$a_n \sim \frac{n!}{4^n}$$



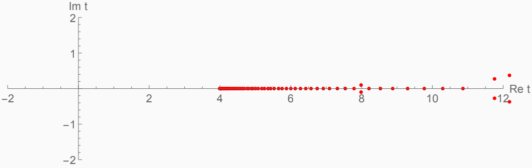
$\frac{a_{n+1}}{a_n}$ vs. n

Large-order behaviour of perturbation theory

Note the connection to a 2-instanton effect:

$$a_n \sim \frac{n!}{4^n} = \frac{n!}{(2S_I)^n}$$

- Action of 1 instanton in the cosine well: $S_I = 2$
- Non-alternating series $\longrightarrow$ singularities of Borel transform $\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$
  - Singularities converge to branch cut, starting at $t = 2S_I$



Poles of approximated Borel transform

Beyond leading-order behaviour

There are corrections to the leading behaviour:

$$a_n \sim \frac{n!}{(2S_I)^n}\left(1 - \frac{5}{2n} - \frac{13}{8n^2} - \dots\right)$$

Deduce imaginary energy contribution from large-order behaviour:

$$\operatorname{Im}\mathcal{S}[E](g^2) \sim \pm\frac{1}{g^2}\mathrm{e}^{-2S_I/g^2}\left(a_0^{I\bar{I}} + a_1^{I\bar{I}}g^2 + a_2^{I\bar{I}}g^4 + \dots\right)$$

→ Perturbative fluctuations around a 2-instanton sector

- The fluctuation series diverges too (4-instanton effect):

$$a_n^{I\bar{I}} \sim \frac{n!}{(2S_I)^n}$$

Resurgence to the rescue

Resurgence idea

Perturbation theory itself is incomplete

- Singularities of the Borel transform

Instanton gas picture incomplete too

- Missing correlated multi-instanton events

→ Solving both gives ambiguous, imaginary contributions

→ Sum over non-perturbative effects in a trans-series

$$E = E_{pert} + \sum_{k=1}^{\infty} e^{-S_k/g^2} \left[\sum_{n=0}^{\infty} a_n^{(k)} g^{2n} \right]$$

→ Resurgence: exact cancellations yield unambiguous, real observables

- Perturbation theory and $[\mathcal{I}\bar{\mathcal{I}}]$, $[\mathcal{I}\bar{\mathcal{I}}\mathcal{I}\bar{\mathcal{I}}]$, … sectors are connected

Resurgent trans-series via uniform WKB

Wave function for potentials with harmonic vacua:

$$\psi(y) = \frac{D_\nu(\frac{1}{g}u(y))}{\sqrt{u'(y)}}$$

Local analysis: $E(\nu, g^2) = \sum_{n=0}^{\infty} g^{2n} E_n(\nu)$

- Ansatz parameter ν not fixed, but close to integer N
- For $\nu = N$: $E(N, g^2) = E_{pert}(N, g^2)$

Global analysis: add Bloch condition $\psi(y+\pi) = e^{i\theta}\psi(y)$

- Resulting trans-series: perturbations, instantons, quasi-zero modes, Bloch

$$E^{(N)}(g^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\sum_{\substack{m=-k \\ \Delta=2}}^{k} c_{n,k,l}^{(m)}\, g^{2n} \left[\frac{1}{g^{2N+1}} e^{-S_I/g^2}\right]^k \ln\left(-\frac{1}{g^2}\right)^l e^{im\theta}$$

- Classification with topological charge $m = n_{\mathcal{I}} - n_{\overline{\mathcal{I}}}$

Resurgence triangle

Decode trans-series via resurgence triangle

- Number of $\mathcal{I}, \overline{\mathcal{I}}$ events $n = n_{\mathcal{I}} + n_{\overline{\mathcal{I}}}$
- Topological charge $m = n_{\mathcal{I}} - n_{\overline{\mathcal{I}}}$
- $\xi^n \sim e^{-nS_I/g^2}$ and $f(n,m)$ is perturbative fluctuation series
- Intricately related sectors with same topological charge can communicate to cancel imaginary ambiguities

$$f_{(0,0)}$$

$$\xi e^{i\theta} f_{(1,1)} \qquad \xi e^{-i\theta} f_{(1,-1)}$$

$$\xi^2 e^{2i\theta} f_{(2,2)} \qquad \xi^2 f_{(2,0)} \qquad \xi^2 e^{-2i\theta} f_{(2,-2)}$$

$$\xi^3 e^{3i\theta} f_{(3,3)} \qquad \xi^3 e^{i\theta} f_{(3,1)} \qquad \xi^3 e^{-i\theta} f_{(3,-1)} \qquad \xi^3 e^{-3i\theta} f_{(3,-3)}$$

$$\xi^4 e^{4i\theta} f_{(4,4)} \qquad \xi^4 e^{2i\theta} f_{(4,2)} \qquad \xi^4 f_{(4,0)} \qquad \xi^4 e^{-2i\theta} f_{(4,-2)} \qquad \xi^4 e^{-4i\theta} f_{(4,-4)}$$

Conclusions

Key points

Perturbation theory is an important tool in physics

- Unfortunately, it generally diverges
- Not well-defined even after resummation

Non-perturbative effects should be included explicitly

- In particular: quantum tunnelling

Resurgence sheds new light on these challenges in QM and QFT

- Perturbation theory encodes non-perturbative effects

Questions?


IF THERE ARE NO QUESTIONS, WE'LL MOVE ON TO THE NEXT CHAPTER.
I HAVE A QUESTION.
CERTAINLY, CALVIN. WHAT IS IT?
WHAT'S THE POINT OF HUMAN EXISTENCE?
I MEANT ANY QUESTIONS ABOUT THE SUBJECT AT HAND.
OH.
FRANKLY, I'D LIKE TO HAVE THE ISSUE RESOLVED BEFORE I EXPEND ANY MORE ENERGY ON THIS.
3-6

References

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- M. Mariño, **Lectures on non-perturbative effects in large N gauge theories, matrix models and strings**, arXiv:1206.6272, 2012.
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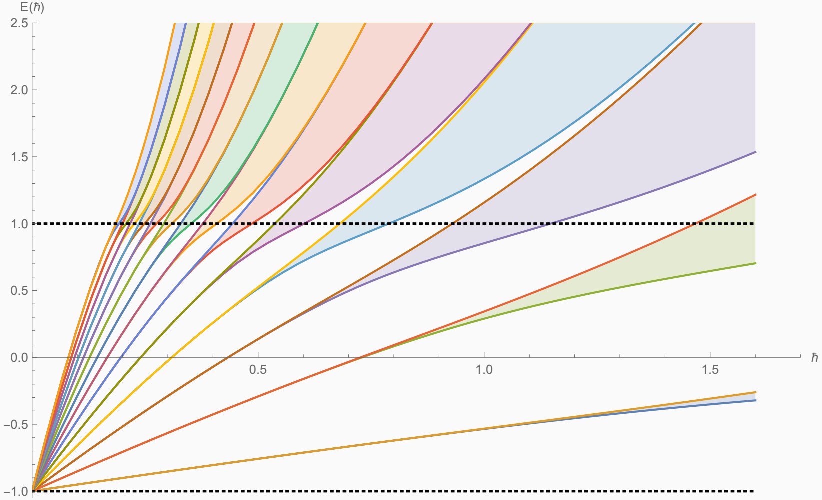
Backup slides

Energy bands for the cosine well



Asymptotic series

Perturbative expansions are generally divergent asymptotic series

- $\sum_{n=0}^{\infty} f_n z^n$ is asymptotic to $f(z)$ when for $z \to 0$ and any N,

$$\left| f(z) - \sum_{n=0}^{N-1} f_n z^n \right| \leq C_N |z|^N$$

$\leftrightarrow$ Convergence: at fixed z, remainder vanishes for $N \to \infty$
Exponential accuracy possible when truncated optimally

- Typically factorial divergence: $f_n \sim n!$

$\to$ Borel resummation idea: fix analytical transgression

$$\int_0^\infty \mathrm{d}t \; \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{f_n}{n!} (tz)^n = \sum_{n=0}^{\infty} \frac{n!\, f_n}{n!} z^n = \sum_{n=0}^{\infty} f_n z^n$$

Borel resummation

Borel transform and resummed perturbation series:

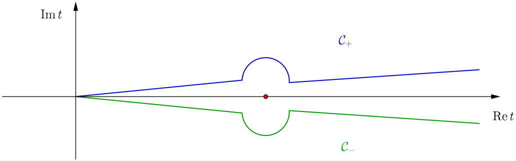
$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n \quad \longrightarrow \quad \mathcal{S}[f](z) = \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t}\, \mathcal{B}[f](tz)$$

Problem: non-Borel summable series, e. g. non-alternating $\sum_{n=0}^{\infty} n!\, z^n$

- Borel transform $\mathcal{B}[f](t) = \frac{1}{1-t}$ singular on integration contour

→ Lateral resummation yields ambiguous imaginary contribution:

$$\operatorname{Im} \mathcal{S}_{\pm}[f](z) = \pm \frac{\pi}{z} \mathrm{e}^{-1/z}$$

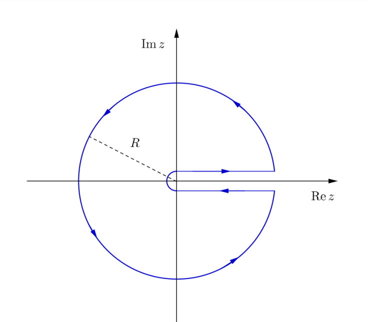


Dispersion relation

$$f(z_0) = \frac{1}{2\pi i} \oint \mathrm{d}z \frac{f(z)}{z - z_0} = \frac{1}{2\pi i} \int_0^\infty \mathrm{d}z \frac{\mathrm{Disc}_0\ f(z)}{z - z_0}$$

$$\mathrm{Disc}_0\ f(z) = \lim_{\varepsilon \to 0} \left[f(z + i\varepsilon) - f(z - i\varepsilon) \right]$$

$$f(z_0) = \sum_{n=0}^{\infty} f_n z_0^n \quad \longrightarrow \quad f_n = \frac{1}{\pi} \int_0^\infty \mathrm{d}z \frac{\mathrm{Im}\ f(z)}{z^{n+1}}$$

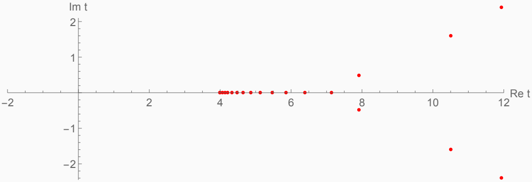


Large-order behaviour of perturbation theory

Note the connection to a 2-instanton effect:

$$a_n \sim \frac{n!}{4^n} = \frac{n!}{(2S_I)^n}$$

- Action of instanton trajectory in the cosine well: $S_I = 2$
- Non-alternating series $\longrightarrow$ singularities of Borel transform
  - Increasing number of terms, singularities converge to branch cut
  - Branch cut starts at $t = 2S_I$



Poles of Borel transform approximated with 70 terms

Large-order behaviour of perturbation theory

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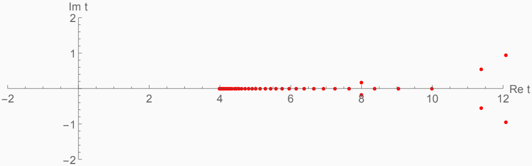

Poles of Borel transform approximated with 150 terms

Large-order behaviour of perturbation theory

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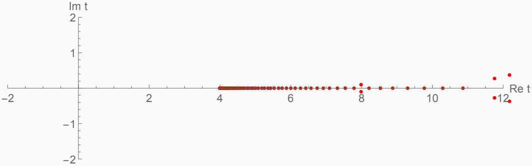


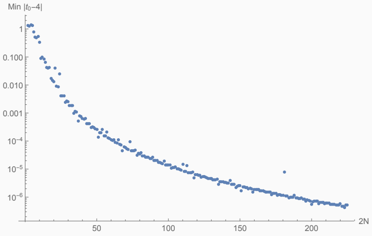
Poles of Borel transform approximated with 230 terms

Large-order behaviour of perturbation theory

Note the connection to a 2-instanton effect:

$$a_n \sim \frac{n!}{4^n} = \frac{n!}{(2S_I)^n}$$

- Action of instanton trajectory in the cosine well: $S_I = 2$
- Non-alternating series $\longrightarrow$ singularities of Borel transform
  - Increasing number of terms, singularities converge to branch cut
  - Branch cut starts at $t = 2S_I$



Nearest-pole distance to $t = 2S_I$

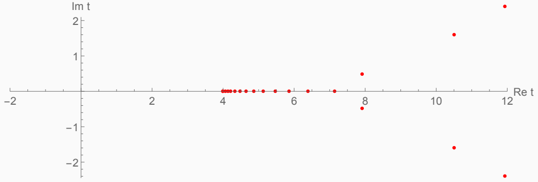
Large-order behaviour of perturbation theory

Pole structure of Borel transform confirms $I\bar{I}$ connection

- Approximated by Padé approximants of the Borel transform:

$$\sum_{n=0}^{N_*} \tilde{a}_n t^n \quad \longrightarrow \quad \mathcal{P}_{[N/M]}[\tilde{E}](t) = \frac{p_0 + p_1 t + \ldots + p_N t^N}{1 + q_1 t + \ldots + q_M t^M}$$

- Increasing N, M reveals poles converge to branch cut starting at $t = 2S_I$



Poles of $P_{[35/35]}[\tilde{E}](t)$

Large-order behaviour of perturbation theory

Pole structure of Borel transform confirms $I\bar{I}$ connection

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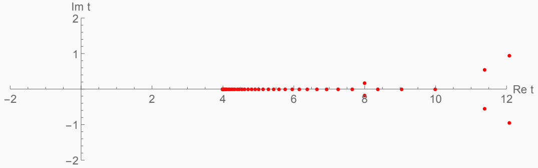

Poles of $P_{[75/75]}[\tilde{E}](t)$

Large-order behaviour of perturbation theory

Pole structure of Borel transform confirms $I\bar{I}$ connection

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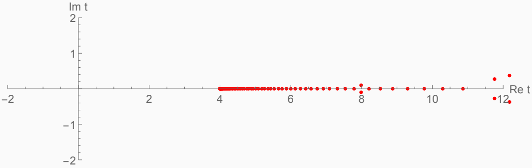
- Increasing N, M reveals poles converge to branch cut starting at $t = 2S_I$



Poles of $P_{[115/115]}[\tilde{E}](t)$

Large-order behaviour of perturbation theory

Pole structure of Borel transform confirms $I\bar{I}$ connection

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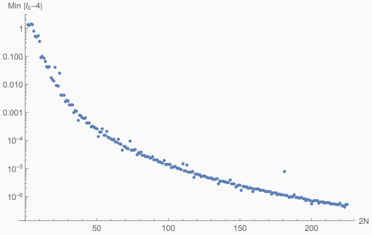

Log-plot of nearest-pole distance to $t = 4$

Large-order behaviour of perturbation theory

Pole structure of Borel transform confirms $I\bar{I}$ connection

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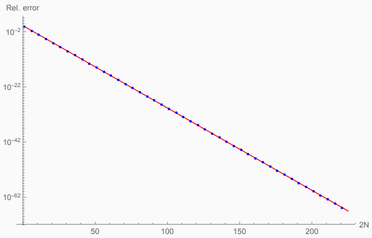
- Exponential convergence of (diagonal) Padé approximations:

$$\left| \frac{a^{\text{pred}}_{2N+1} - a^{\text{true}}_{2N+1}}{a^{\text{true}}_{2N+1}} \right| \sim 1.640 \; \mathrm{e}^{-1.376\, N}$$



Beyond leading-order behaviour

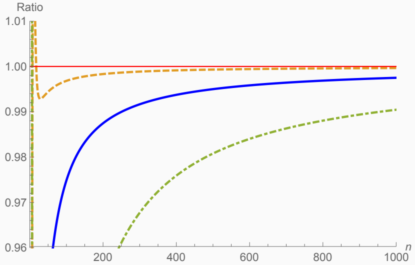
There are corrections to the leading behaviour:

$$a_n \sim -\frac{1}{\pi}\left(1 - \frac{5}{2n} - \frac{13}{8n^2} - \dots\right)\frac{n!}{(2S_I)^n}$$

Deduce imaginary energy contribution from large-order behaviour:

$$\operatorname{Im}\mathcal{S}[E](g^2) \sim \pm\frac{1}{g^2}\mathrm{e}^{-2S_I/g^2}\left(a_0^{I\bar{I}} + a_1^{I\bar{I}}g^2 + a_2^{I\bar{I}}g^4 + \dots\right)$$

→ Perturbative fluctuations around 2-instanton sector



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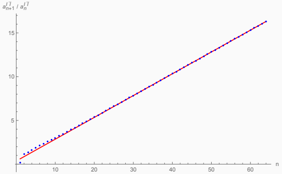
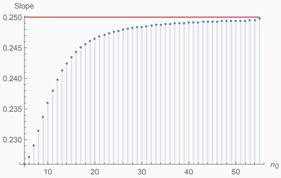
→ Perturbative fluctuations around 2-instanton sector

- The fluctuation series diverges as well, due to 4-instanton effect



$a_{n+1}^{I\bar{I}}/a_n^{I\bar{I}}$ vs. n



Slope converges to $\frac{1}{2S_I} = \frac{1}{4}$

Towards resurgence in the Mathieu spectrum

Perturbation theory itself is incomplete

- Ambiguous imaginary contribution to real ground-state energy
- Non-Borel summability when degenerate harmonic minima

Instanton gas picture incomplete too (Bogomolny/Zinn-Justin)

- $\mathcal{I}$ and $\overline{\mathcal{I}}$ attract when not widely separated

Regularise both problems by analytic continuation $g^2 \to -g^2$

- Both give ambiguous, imaginary non-perturbative contributions

$\to$ Resurgence: exact cancellation to all orders