Calculation of Momentum Distribution Function of a Non-Thermal Fermionic Dark Matter

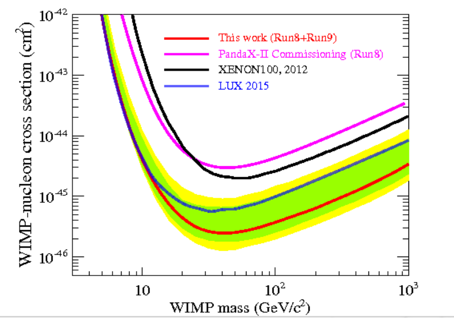
, March 8, 2017.

arXiv:1612.02793, with Anirban Biswas.

Aritra Gupta

Why Non-Thermal ?

- The most widely studied dark matters are the thermal WIMPS ⇒ relic satisfied by thermal freeze-out.
- Direct detection experiments however yielded null results ⇒ upper bound on dark matter nucleon cross section.



- Non observation of dark matter may indicate that these particles are more feebly interacting than we think.

PandaX-II *collaboration*, arXiv:1607.07400

Thermal Freeze-Out : A Brief Recap

- Particles which were in thermal equilibrium in early universe decouple or *freezes out* when the their rate of interactions can not keep up with the expansion of the universe.
- Boltzmann Equation :

$$\hat{L} f_a = \left(\frac{\partial}{\partial t} - H\, p_a \frac{\partial}{\partial p_a} \right) f = \mathscr{C}_{a+b \leftrightarrow i+j}\, [f_a] \tag{1}$$

where,

$$\mathscr{C}\,[f_a] \simeq \int \frac{d^3 p_b}{2E_b} \frac{d^3 p_i}{2E_i} \frac{d^3 p_j}{2E_j} \times (2\pi)^4\, \delta^4(p_a + p_b - p_i - p_j) \times \left(\overline{|\mathscr{M}|^2}_{a+b \to i+j}\, f_a\, f_b\, (1 \pm f_i)\, (1 \pm f_j) - \overline{|\mathscr{M}|^2}_{i+j \to a+b}\, f_i\, f_j\, (1 \pm f_a)\, (1 \pm f_b) \right) \tag{2}$$

- Eq.(2) can be simplified further using some approximations ...

- CP invariance is assumed i.e.

$\overline{|\mathscr{M}|^2}_{i+j\,\to\,a+b} = \overline{|\mathscr{M}|^2}_{a+b\,\to\,i+j} \equiv \overline{|\mathscr{M}|^2}$ (say)

$\Rightarrow$

$$\mathscr{C}[f] \simeq \int \frac{d^3p_a}{2E_a}\frac{d^3p_b}{2E_b}\frac{d^3p_i}{2E_i}\frac{d^3p_j}{2E_j} \times (2\pi)^4\,\delta^4(p_a+p_b-p_i-p_j) \times$$

$$\overline{|\mathscr{M}|^2}\left(f_a\,f_b\,(1\pm f_i)\,(1\pm f_j)\;-\;f_i\,f_j\,(1\pm f_a)\,(1\pm f_b)\right)$$

- **Pauli blocking and stimulated emission terms are neglected.**

$\Rightarrow$

$$\mathscr{C}[f] \simeq \int \frac{d^3p_a}{2E_a}\frac{d^3p_b}{2E_b}\frac{d^3p_i}{2E_i}\frac{d^3p_j}{2E_j} \times (2\pi)^4\, \delta^4(p_a+p_b-p_i-p_j) \times \overline{|\mathscr{M}|^2}\left(f_a f_b - f_i f_j\right) \tag{3}$$

- If particles i and j belong to the thermal soup, they are assumed to follow the classical Maxwell-Boltzmann distribution function i.e. $f \sim e^{-\frac{E}{T}}$

$$\Rightarrow \quad f_i f_j = e^{-\frac{E_i+E_j}{T}} = e^{-\frac{E_a+E_b}{T}} = f_a^{\rm eq} f_b^{\rm eq}$$

- Eq.(3) simplifies to :

$$\mathscr{C}[f] \simeq \int \frac{d^3p_a}{2E_a} \frac{d^3p_b}{2E_b} \frac{d^3p_i}{2E_i} \frac{d^3p_j}{2E_j} \times (2\pi)^4 \, \delta^4(p_a+p_b-p_i-p_j) \times \overline{|\mathscr{M}|^2} \left(f_a f_b - f_a^{\rm eq} f_b^{\rm eq} \right) \tag{4}$$

- If a and b are both identical particles (say χ), then integrating Eq.(4) on both sides by $\int_0^\infty d^3p$, and remembering that $n_\chi \sim \int_0^\infty f(p)\, d^3p$, we finally arrive at the conventional form :

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle \left(n_\chi^2 - n_\chi^{2\,\mathrm{eq}} \right)$$

where, $\sigma = \sigma_{\chi\,\chi \to \mathrm{all}}$ and,

$$\langle \sigma v \rangle = \frac{1}{8\,T\,M_\chi^4\,K_2^2\left(\frac{M_\chi}{T}\right)} \int_{4M_\chi^2}^{\infty} \sigma \left(s - 4M_\chi^2 \right) \sqrt{s}\, K_1 \left(\frac{\sqrt{s}}{T} \right) ds$$

Cosmic abundances of stable particles: Improved analysis, Gelmini & Gondolo, 1990.

- If χ can decay then an extra term depletion term will be added to the "Rate Equation"..

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma v\rangle\left(n_\chi^2 - n_\chi^{2\,\mathrm{eq}}\right) - \langle\Gamma_\chi\rangle\left(n_\chi - n_\chi^{\mathrm{eq}}\right) \qquad (5)$$

where, Γ_χ is the relevant width of χ ,

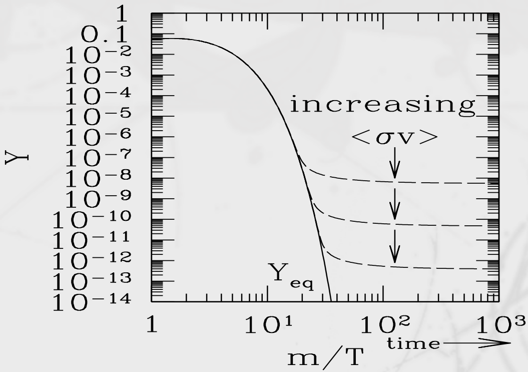
$$\langle\Gamma_\chi\rangle = \frac{K_1(T)}{K_2(T)}\Gamma_\chi$$

K_1 and K_2 are modified Bessel function of order 1 and 2.

- In terms of Y,

$$\frac{dY_\chi}{dx} = -\frac{s\langle\sigma v\rangle}{Hx}\left(Y_\chi^2 - Y_\chi^{2\,\mathrm{eq}}\right)$$

- A formal solution of this equation is given by :



The Early Universe, Kolb & Turner

Non-Thermal Dark Matter : Basic Concepts

- Condition for thermalisation of a system of particles in the background of an expanding universe :

$$\frac{\Gamma}{H} \gg 1$$

where,

Γ = Interaction Rate = $n_{\rm eq} \langle \sigma v \rangle$, H = Expansion Rate.

- Freeze-out : $\frac{\Gamma}{H} \ll 1$ (T < $\text{T}_{\rm fo}$)
- Freeze-in : $\frac{\Gamma}{H} \ll 1$ (at all epochs)

- Number density of non-thermal dark matter is very small $\Rightarrow$ initial abundance is almost negligible.
- Non-thermal dark matters hence needed to be produced $\Rightarrow$ comoving number density gradually rises and finally saturates to satisfy relic density.
- Usually the most dominant production channels are from decay of heavier particles.
- The decaying particle may be in thermal equilibrium or can itself be out-of-equilibrium.

- If the decaying particle is in thermal equilibrium, then the usual form of rate equation in terms of n (or Y) can be used.

- For example, we can use : $\langle\Gamma\rangle = \frac{K_1(T)}{K_2(T)}\Gamma$ etc ...

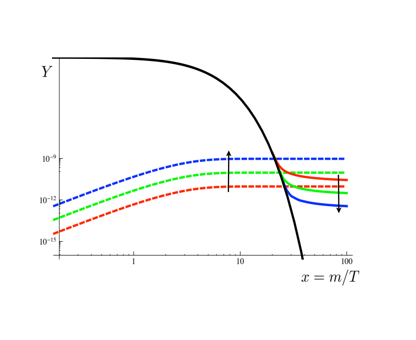
- But if the mother particle is not in equilibrium, then such an equation cannot be used.

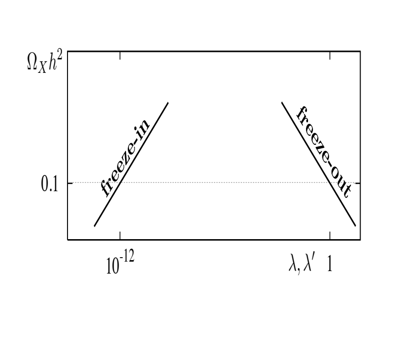
- Now, $\langle\Gamma\rangle = \frac{\int \Gamma\, f_{\rm DM}(p)\, d^3p}{\int f_{\rm DM}(p)\, d^3p}$ (from definition only...)

$\Rightarrow$ Knowledge of the non-equilibrium distribution function, $f_{\rm mother}$ is necessary to solve for $f_{\rm DM}$.

$\Rightarrow$ Need for coupled set of Boltzmann equations...

- The difference between the freeze-in and freeze-out scenario is best illustrated by the following plots :





Hall *et.al.*, arXiv:0911.1120

A *new* $U(1)_{\rm B-L}$ model

- **Gauge Group :** $SU(2)_L \times U(1)_Y \times U(1)_{B-L}$
- **The Lagrangian :**

$$\mathscr{L}_{\rm BL} = i\,\overline{\eta_{\rm L}}\,\gamma_\mu\,D^\mu_\eta\,\eta_{\rm L} + i\,\overline{\xi_{\rm L}}\,\gamma_\mu\,D^\mu_\xi\,\xi_{\rm L} + i\sum_{i=1}^{2}\overline{\chi_{i\rm R}}\,\gamma_\mu\,D^\mu_{\chi_i}\,\chi_{i\rm R} - \frac{1}{4}F^{\mu\nu}_{Z_{\rm BL}}F_{Z_{\rm BL},\,\mu\nu} + \sum_{i=1}^{2}(D^\mu_{\phi_i}\phi_i)^\dagger(D_{\phi_i\,\mu}\phi_i) - \sum_{i=1}^{2}\Big(y_{\xi\,i}\,\overline{\xi_{\rm L}}\,\chi_{i\rm R}\,\phi_2 + y_{\eta\,i}\,\overline{\eta_{\rm L}}\,\chi_{i\rm R}\,\phi_1 + h.c.\Big) - V(H,\,\phi_1,\,\phi_2) + \mathscr{L}_\Delta$$

Sudhanwa Patra *et.al.*, arXiv:1607.04029 where,

$$V(H,\phi_1,\phi_2) = \mu_H^2 H^\dagger H + \lambda_H (H^\dagger H)^2 + \mu_1^2 \phi_1^\dagger \phi_1 + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \rho_1 (H^\dagger H)(\phi_1^\dagger \phi_1) + \rho_2 (H^\dagger H)(\phi_2^\dagger \phi_2) + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \mu \left(\phi_2 \phi_1^{\dagger 2} + \phi_2^\dagger \phi_1^2 \right)$$

- The model is well motivated since :
  - It is anomaly free.
  - It can give rise to neutrino mass.
  - It can accommodate a thermal dark matter.

- The charge assignment is given in the following table :

	Field	$SU(2)_L$ charge	$U(1)_Y$ charge	$U(1)_{B-L}$ charge	VEV
SM Fermions	$l_L \equiv (\nu_L \;\; e_L)^T$	2	$-\frac{1}{2}$	-1	0
	$Q_L \equiv (u_L \;\; d_L)^T$	2	$\frac{1}{6}$	$\frac{1}{3}$	
	e_R	1	-1	-1	
	u_R	1	$\frac{2}{3}$	$\frac{1}{3}$	
	d_R	1	$-\frac{1}{3}$	$\frac{1}{3}$	
BSM Fermions	ξ_L	1	0	$\frac{4}{3}$	0
	η_L	1	0	$\frac{1}{3}$	
	χ_{1R}	1	0	$-\frac{2}{3}$	
	χ_{2R}	1	0	$-\frac{2}{3}$	
Scalars	H	2	$\frac{1}{2}$	0	v
	ϕ_1	1	0	1	v_1
	ϕ_2	1	0	2	v_2
	Δ	3	1	-2	v_Δ

Points to highlight regarding the model :

- The scalars acquire vev after symmetry breaking :

$$
\begin{aligned}
H^0 &= \frac{1}{\sqrt{2}}(v+\tilde{h}) + \frac{i}{\sqrt{2}}\tilde{G}\,, \\
\phi_1 &= \frac{1}{\sqrt{2}}(v_1+\tilde{h}_1) + \frac{i}{\sqrt{2}}\tilde{A}_1\,, \\
\phi_2 &= \frac{1}{\sqrt{2}}(v_2+\tilde{h}_2) + \frac{i}{\sqrt{2}}\tilde{A}_2\,,
\end{aligned}
$$

- The fields $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$ mix with each other $\Rightarrow$ 3×3 PMNS-like mixing matrix parametrised by $\theta_{12}, \theta_{13}, \theta_{23}$.
- $\tilde{A}_1$ and $\tilde{A}_2$ also mix together to give rise to a massive physical pseudoscalar with mass M_A.

- **For the Fermions :**

$$\mathscr{L}_{\rm fermion-mass} = \left(\overline{\xi_{\rm L}} \quad \overline{\eta_{\rm L}} \right) \mathscr{M}_{\rm fermion} \begin{pmatrix} \chi_{1\rm R} \\ \chi_{2\rm R} \end{pmatrix}$$

where,

$$\mathscr{M}_{\rm fermion} = \begin{pmatrix} y_{\xi_1} v_2 & y_{\xi_2} v_2 \\ y_{\eta_1} v_1 & y_{\eta_1} v_1 \end{pmatrix}$$

- **The mass and gauge basis states are related by :**

$$\begin{pmatrix} \xi_{\rm L} \\ \eta_{\rm L} \end{pmatrix} = \mathscr{U}_{\rm L} \begin{pmatrix} \psi_{2\rm L} \\ \psi_{1\rm L} \end{pmatrix}, \qquad \begin{pmatrix} \chi_{1\rm R} \\ \chi_{2\rm R} \end{pmatrix} = \mathscr{U}_{\rm R} \begin{pmatrix} \psi_{2\rm R} \\ \psi_{1\rm R} \end{pmatrix}.$$

where,

$$\mathscr{U}_{\rm L,R} = \begin{pmatrix} \cos\theta_{\rm L,R} & \sin\theta_{\rm L,R} \\ -\sin\theta_{\rm L,R} & \cos\theta_{\rm L,R} \end{pmatrix}$$

- **The physical states are** $\psi_1 = \psi_{1\rm L} + \psi_{1\rm R}$ **and** $\psi_2 = \psi_{2\rm L} + \psi_{2\rm R}$.

- The breaking of $\mathrm{U}(1)_{\mathrm{B-L}}$ gauge group also gives the extra gauge boson its mass.

$$M_{Z_{\mathrm{BL}}}^2 = \left(\frac{g_{\mathrm{BL}}\, v_2}{\beta}\right)^2 (1+4\beta^2), \text{ where } \beta = \frac{v_2}{v_1}$$

- The set of independent parameters relevant for our analysis are as follows : $\theta_{12}, \theta_{13}, \theta_{23}, \theta_{\mathrm{L}}, \theta_{\mathrm{R}}, M_{h_2}, M_{h_3}, M_{\mathrm{A}}, M_{\psi_1}, M_{\psi_2}, M_{Z_{\mathrm{BL}}}, g_{\mathrm{BL}}$ and β.

- ψ_1 is our dark matter candidate.

- The dominant production modes of this dark matter are :

  $h_1 \to \psi_1 \psi_1,\, h_2 \to \psi_1 \psi_1,\, Z_{\rm BL} \to \psi_1 \psi_1 \,.$

- The decay widths strongly depend on $g_{\rm BL}$ and for non-thermality $g_{\rm BL} \sim 10^{-10}$.

- Very small $g_{\rm BL}$ $\Rightarrow$ $Z_{\rm BL}$ is also not in equilibrium.

$Z_{\rm BL}$ production : $h_2 \to Z_{\rm BL} Z_{\rm BL} \,.$

$Z_{\rm BL}$ depletion : $Z_{\rm BL} \to f\bar{f}$ and $Z_{\rm BL} \to \psi_1 \bar{\psi}_1 \,.$

- h_2 can however equilibrate through their mixing with SM Higgs (h_1).

- We can not solve for Y_{ψ_1} directly, rather we need to find $f_{Z_{BL}}$ and consequently solve for $f_{\psi_1} \Rightarrow$ coupled Boltzmann equations.

$$\begin{aligned} \hat{L} f_{Z_{BL}} &= \mathscr{C}^{h_2 \to Z_{BL} Z_{BL}} + \mathscr{C}^{Z_{BL} \to all} \\ \hat{L} f_{\psi_1} &= \sum_{s=h_1,h_2} \mathscr{C}^{S \to \overline{\psi_1} \psi_1} + \mathscr{C}^{Z_{BL} \to \overline{\psi_1} \psi_1} \end{aligned}$$

- SM Higgs (h_1) gets its mass after EWPT. So, when $T > T_{\rm EWPT}$, h_1 decay is not allowed.

Simplifying the Liouville Operator

- For isotropic homogeneous universe, FRW metric gives :

$$\widehat{L} = \left(\frac{\partial}{\partial t} - H\, p \frac{\partial}{\partial p} \right)$$

- Using conservation of entropy :

$$\frac{dT}{dt} = -HT \left(1 + \frac{T g_s'(T)}{3 g_s(T)} \right)^{-1}$$

- Change of coordinates :

$$\begin{aligned} r &= \frac{m_0}{T}, \\ \xi_p &= \left(\frac{g_s(T_0)}{g_s(T)} \right)^{1/3} \frac{p}{T}, \end{aligned}$$

- The Liouville operator simplifies to :

$$\hat{L} \;=\; rH\left(1+\frac{T\,g_s'}{3\,g_s}\right)^{-1}\frac{\partial}{\partial r}\,,$$

- Chosen Benchmark ($\beta = 10^{-3}$) :

Input Parameters	Corresponding values
$M_{Z_{\rm BL}}$	1 TeV
M_{h_2}	5 TeV
M_{ψ_1}	10 GeV
$g_{\rm BL}$	1.75×10^{-11}
θ_{12}	0.1000 rad
θ_{13}	0.01 rad
θ_{23}	0.06 rad
$\theta_{\rm L}=\theta_{\rm R}$	$\pi/4$ rad

Calculation of Collision Terms

- $\mathbf{Z}_{\rm BL}(p) \to \mathbf{f}(p')\,\bar{\mathbf{f}}(q')$

$$\mathscr{C}^{Z_{\rm BL}\to f\bar{f}}[f_{Z_{\rm BL}}(p)] = \frac{1}{2E_p}\int \frac{g_f\, d^3p'}{(2\pi)^3\, 2E_{p'}}\frac{g_f\, d^3q'}{(2\pi)^3\, 2E_{q'}}\times$$

$$(2\pi)^4\,\delta^4(\tilde{p}-\tilde{p}'-\tilde{q}')\times\overline{|\mathscr{M}|^2}\times[-f_{Z_{\rm BL}}(p)]$$

$$= -f_{Z_{\rm BL}}(p)\times\frac{1}{2E_p}\int \frac{g_f\, d^3p'}{(2\pi)^3\, 2E_{p'}}\frac{g_f\, d^3q'}{(2\pi)^3\, 2E_{q'}}(2\pi)^4\,\delta^4(\tilde{p}-\tilde{p}'-\tilde{q}')\times\overline{|\mathscr{M}|^2}$$

- `In a general frame moving with momentum 'p' :`

$$\Gamma'_{Z_{\rm BL}\to f\bar{f}} = \frac{1}{2E_p}\int \frac{g_f\, d^3p'}{(2\pi)^3\, 2E_{p'}}\frac{g_f\, d^3q'}{(2\pi)^3\, 2E_{q'}}(2\pi)^4\,\delta^4(\tilde{p}-\tilde{p}'-\tilde{q}')\times\overline{|\mathscr{M}|^2}\Bigg|_{Z_{\rm BL}\to f\bar{f}}$$

$$\mathscr{C}^{Z_{\rm BL}\to f\bar{f}}[f_{Z_{\rm BL}}(\xi_p)] = -f_{Z_{\rm BL}}(\xi_p) \times \left[\Gamma^{\rm rest}_{Z_{\rm BL}\to f\bar{f}} \times \frac{M_{Z_{\rm BL}}}{E_{Z_{\rm BL}}}\right]$$

$$= -f_{Z_{\rm BL}}(\xi_p) \times \Gamma^{\rm rest}_{Z_{\rm BL}\to f\bar{f}} \times \frac{r_{Z_{\rm BL}}}{\sqrt{\xi_p^2\,\mathscr{B}(r)^2 + r_{Z_{\rm BL}}^2}}$$

- **Hence,**

$$\mathscr{C}^{Z_{\rm BL}\to all} = -f_{Z_{\rm BL}}(\xi_p) \times \Gamma_{Z_{\rm BL}\to all} \times \frac{r_{Z_{\rm BL}}}{\sqrt{\xi_p^2\,\mathscr{B}(r)^2 + r_{Z_{\rm BL}}^2}}$$

where,

$$\left(\frac{g_s(T)}{g_s(T_0)}\right)^{1/3} = \left(\frac{g_s(M_{sc}/r)}{g_s(M_{sc}/r_0)}\right)^{1/3} \equiv \mathscr{B}(r)$$

- $\mathbf{h}_2(k) \rightarrow \mathbf{Z}_{\rm BL}(p)\,\mathbf{Z}_{\rm BL}(q')$

$$\mathscr{C}^{h_2\to Z_{\rm BL}Z_{\rm BL}}[f_{Z_{\rm BL}}(p)] = 2\times\frac{1}{2E_p}\int\frac{g_{h_2}\,d^3k}{(2\pi)^3\,2E_k}\,\frac{g_{Z_{\rm BL}}\,d^3q'}{(2\pi)^3\,2E_{q'}}\times$$

$$(2\pi)^4\,\delta^4(\tilde{k}-\tilde{p}-\tilde{q}')\times\overline{|\mathscr{M}|^2}\Big|_{h_2\to Z_{\rm BL}Z_{\rm BL}}\times$$

$$[f_{h_2}(1\pm f_{Z_{\rm BL}})(1\pm f_{Z_{\rm BL}})-f_{Z_{\rm BL}}\,f_{Z_{\rm BL}}\,(1\pm f_{h_2})]$$

$$=2\times\frac{1}{2E_p}\int\frac{g_{h_2}\,d^3k}{(2\pi)^3\,2E_k}\,\frac{g_{Z_{\rm BL}}\,d^3q'}{(2\pi)^3\,2E_{q'}}(2\pi)^4\,\delta^4(\tilde{k}-\tilde{p}-\tilde{q}')\times\overline{|\mathscr{M}|^2}\times[f_{h_2}(k)]\,.$$

- Here,

$$\overline{|\mathscr{M}|^2}\Big|_{h_2\to Z_{\rm BL}Z_{\rm BL}} = \frac{g^2_{h_2Z_{\rm BL}Z_{\rm BL}}}{2\times 9}\left(2+\frac{\left(E_pE_{q'}-\vec{p}\,.\,\vec{q}'\right)^2}{M^4_{Z_{\rm BL}}}\right)$$

$$\mathscr{C}^{h_2\to Z_{\rm BL}Z_{\rm BL}}[f_{Z_{\rm BL}}(p)] = \frac{g^2_{h_2 Z_{\rm BL} Z_{\rm BL}}}{6\,(4\pi)^2}\,\frac{1}{E_p}\int \frac{k^2 dk\ d(\cos\theta)}{E_k\,E_{q'}(p,k,\cos\theta)}\times$$

$$\delta\big(E_k - E_p - E_{q'}(p,k,\theta)\big)\times\left(2+\frac{\left(E_p E_{q'}(k,p,\theta)+p^2-p\,k\cos\theta\right)^2}{M^4_{Z_{\rm BL}}}\right)\times[f_{h_2}(k)]$$

where, $E_{q'} = \sqrt{k^2+p^2+M^2_{Z_{\rm BL}}-2p\,k\cos\theta}$

- In terms of the chosen coordinates,

$$E_{q'} = T\sqrt{\xi_k^2\,\mathscr{B}(r)^2+\xi_p^2\,\mathscr{B}(r)^2+r^2_{Z_{\rm BL}}-2\mathscr{B}(r)^2\,\xi_k\,\xi_p\,\cos\theta}$$

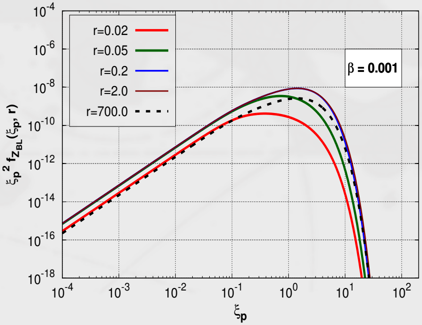
- Finally, doing the $\cos\theta$ integral and hence removing the delta function, we arrive at :

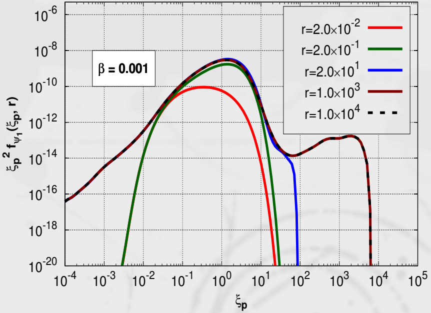
$$\mathscr{C}^{h_2\to Z_{\rm BL}Z_{\rm BL}} = \frac{r}{8\pi M_{sc}} \frac{\mathscr{B}^{-1}(r)}{\xi_p\sqrt{\xi_p^2\mathscr{B}(r)^2+\left(\frac{M_{Z_{\rm BL}}\,r}{M_{sc}}\right)^2}} \times$$

$$\frac{g^2_{h_2 Z_{\rm BL}Z_{\rm BL}}}{6}\left(2+\frac{(M^2_{h_2}-2M^2_{Z_{\rm BL}})^2}{4M^4_{Z_{\rm BL}}}\right)\times$$

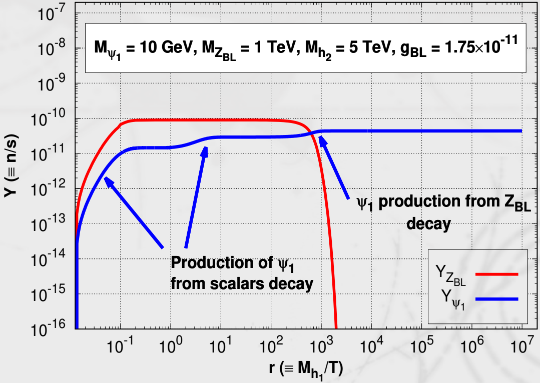
$$\left(e^{-\sqrt{\left(\xi_k^{\rm min}(\xi_p,r)\right)^2\mathscr{B}(r)^2+\left(\frac{M_{h_2}\,r}{M_{sc}}\right)^2}} - e^{-\sqrt{\left(\xi_k^{\rm max}(\xi_p,r)\right)^2\mathscr{B}(r)^2+\left(\frac{M_{h_2}\,r}{M_{sc}}\right)^2}}\right)$$

- **The distribution functions :**



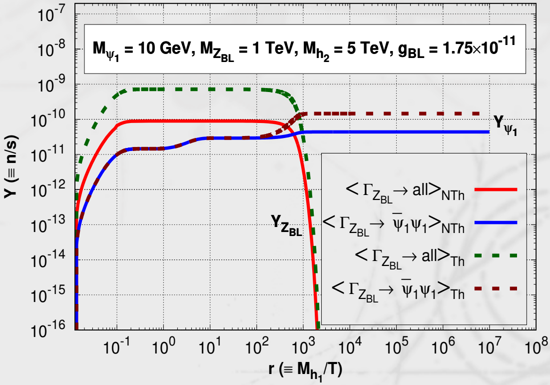


- Knowing the distribution function, we are finally able to calculate the comoving number density $Y \equiv \frac{n}{s}$.



Comparing with the approximate solution

- What if we had assumed that the system is close to equilibrium and used rate equations in terms of Y.



THANK YOU