Fermi liquids and fractional statistics in one dimension

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JML Phys. Rev. B (April, 2017)

Related publications:

M Horsdal, M Rypestøl, T H Hansson, and J M Leinaas, Phys Rev B (2011)

J M Leinaas, M Horsdal, and T H Hansson, Phys Rev B (2009)

Brief outline of the talk

Fractional statistics in two and one dimensions

Luttinger liquid description of fermions in 1D

Charge fractionalization

Dressed fermions and Fermi liquids in 1D

Momentum transformation and generalized Pauli exclusion

The 1D Fermi fluid as particles with fractional charge and statistics

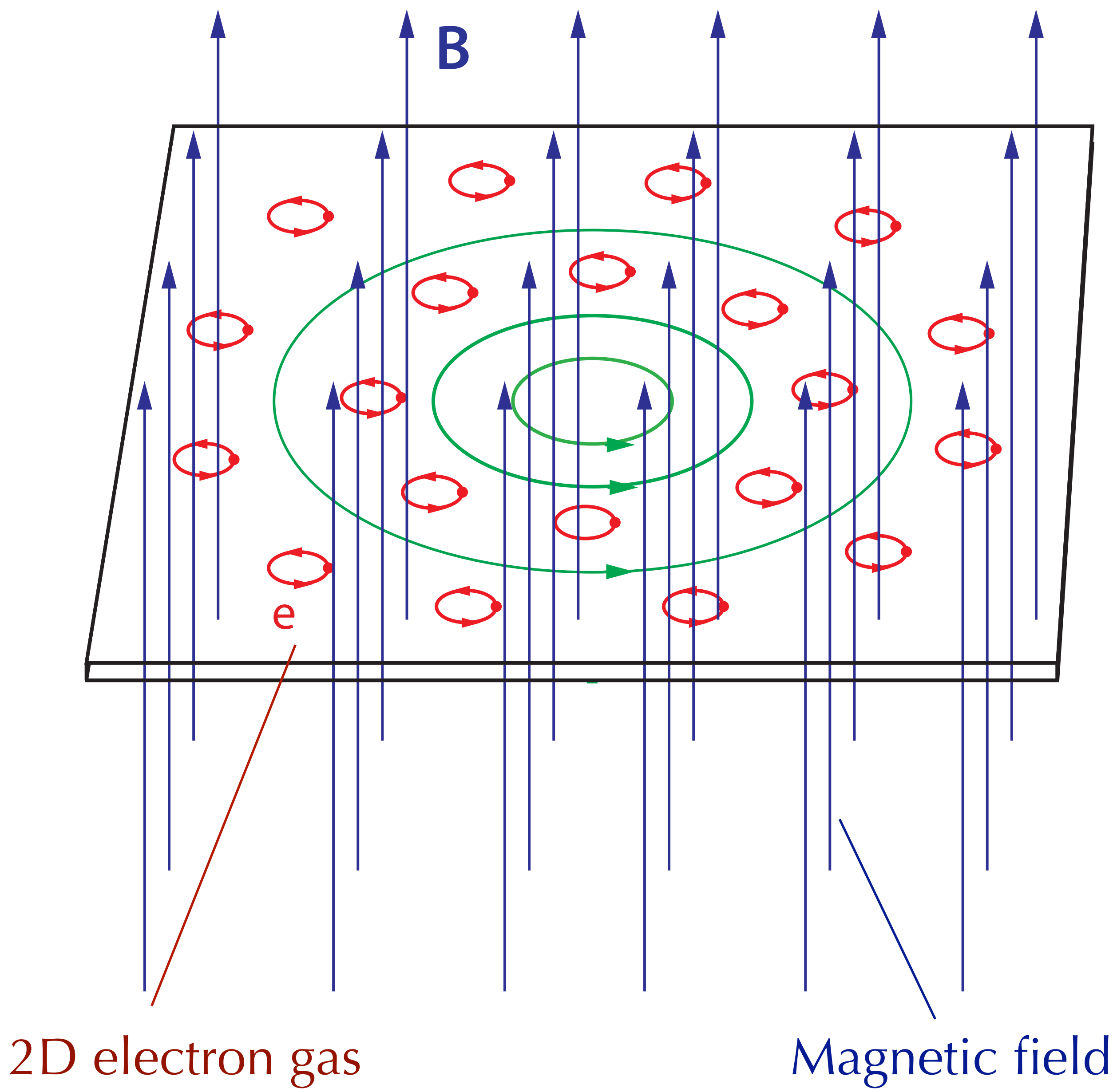
Anyons in the fractional QHE

quasihole =
vortex in the 2D electron gas

R.B. Laughlin, Phys. Rev. Lett. (1983)

Vortex carries
fractional charge
and fractional statistics

Topologically ordered systems:
Generally assumed to have
excitations with
fractional quantum numbers.



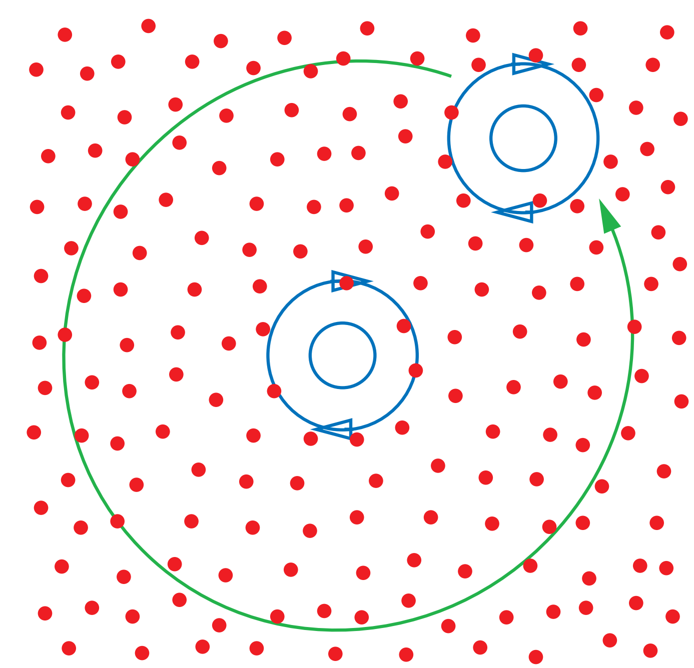
Laughlin theory and fractional statistics

Ground state at filling ν=1/m of the lowest Landau level

$$\psi_m(z_1, z_2, ..., z_N) = \prod_{i \neq j}(z_i - z_j)^m \exp(-\frac{1}{2}\sum_k |z_k|^2)$$

complex electron coordinates

m = 1, 3, 5, ...



Quasihole-state

quasihole coordinate

$$\psi_m(Z; z_1, z_2, ...) = \prod_i (Z - z_i)\psi_m(z_1, z_2, ...)$$

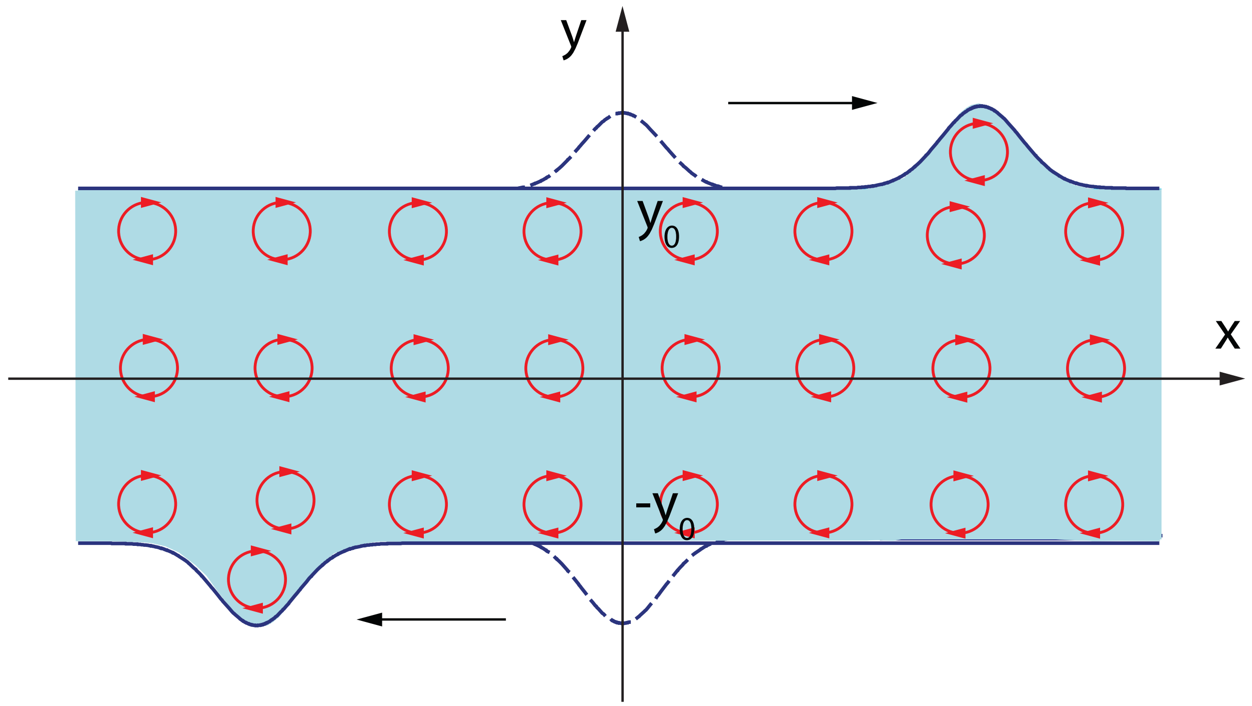
Quasihole has fractional charge q=e/m

and fractional statistics θ=π/m

Arovas, Schrieffer and Wilczek 1984 :

Statistics determined by Berry-phase calculation

The 2D electron fluid as a 1D system



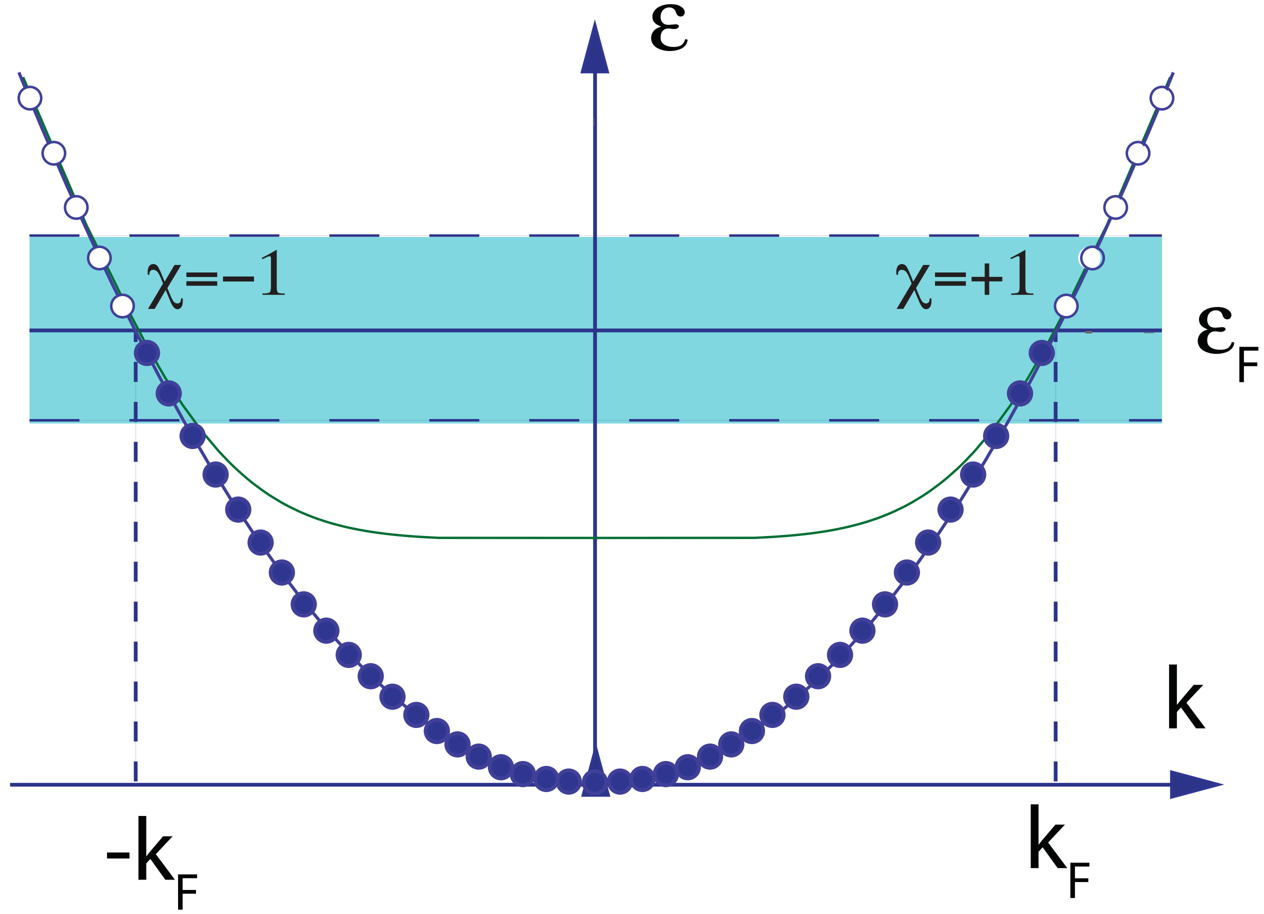
In the lowest Landau level:

$$[x, y] = -i\hbar \frac{1}{eB} \quad \Rightarrow \quad y = -\frac{1}{eB}\hbar k$$

k conjugate momentum to x

Fermi momentum: $k_F = -\frac{eB}{\hbar} y_0$

Interacting fermions in 1D



Low energy fermion Hamiltonian (on a circle of length L)

$$H = v_F \sum_{\chi,k} (\chi k - k_F) :c^\dagger_{\chi,k} c_{\chi,k}: + \frac{1}{4L} \sum_{\chi,q} [V_1(q)\rho_{\chi,q}\rho_{\chi,-q} + V_2(q)\rho_{\chi,q}\rho_{-\chi,-q}]$$

Luttinger liquids

Fermi liquids in one dimension:
problematic infrared divergences in perturbative expansions

Luttinger liquids:
reformulation in terms of bosonic variables (Haldane 1981)

$$a_q = \sqrt{\frac{2\pi}{|q|L}} \sum_{\chi} \theta(\chi q) \rho_{\chi,q} \quad \Rightarrow \quad \left[a_q, a^\dagger_{q'}\right] = \delta_{qq'}$$

Hamiltonian in bosonic variables $\quad N = \sum_{\chi} N_\chi \,, \qquad J = \sum_{\chi} \chi N_\chi$

$$H = \frac{1}{2} \sum_{q \neq 0} |q| \left[(v_F + \frac{V_1(q)}{4\pi})(a^\dagger_q a_q + a_q a^\dagger_q) + \frac{V_2(q)}{4\pi}(a^\dagger_q a^\dagger_{-q} + a_q a_{-q}) \right] + \frac{\pi}{2L}(v_N N^2 + v_J J^2)$$

$$v_N = v_F + \frac{1}{4\pi}(V_1(0) + V_2(0)) \,, \qquad v_J = v_F + \frac{1}{4\pi}(V_1(0) - V_2(0))$$

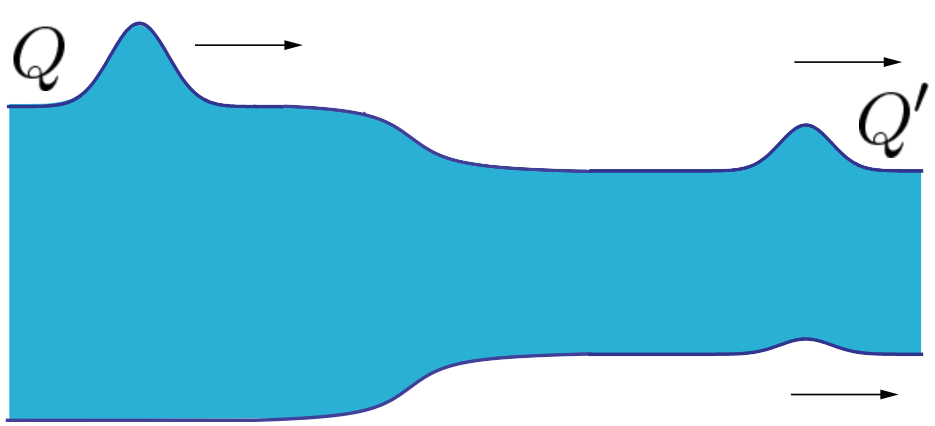
The transformation to free form

$$H = \sum_{q \neq 0} \omega_q \, b_q^\dagger b_q + \frac{\pi}{2L}(v_N N^2 + v_J J^2)$$

$$b_q = U a_q U^\dagger \qquad U = \exp[-\sum_{q\neq 0} \frac{\xi_q}{2}(a_q^2 - a_q^{\dagger 2})]$$

U connects adiabatically the non-interacting and the interacting system

Adiabatic turning on the interaction in a quantum Hall bar



$$Q' \neq Q$$

Charge fractionalization

The elementary charged excitations in one dimension are typically split in chiral components with non-integer charge units.
Has been studied both theoretical and experimentally.

M Horsdal et. al. (2009, 2011):
Adiabatic switching on will separate the charge in a local non-integer charge, and compensating background charge.

Total charge: $\rho_0 = N$ Local charge: $\lim_{q \to 0} \rho_q$

$$\lim_{q \to 0} U^\dagger \rho_q U = \sqrt{g} \lim_{q \to 0} \rho_q \,, \quad g = \sqrt{v_J / v_N}$$

$$Q = 1 \quad \Rightarrow \quad Q' = \sqrt{g}$$

Fractional statistics

JML (2017)

«Dressed» fermion field $\phi(x) = U\psi(x)U^\dagger$

Low-energy approximation $\omega_q = v_s|q|$ $\quad v_s = \sqrt{v_J v_N}$

Hamiltonian

$$
\begin{aligned}
H &= U v_s(\sum_{q\neq 0}|q|a_q^\dagger a_q + \frac{\pi}{2L}(N^2+J^2))U^\dagger + \frac{\pi}{2L}((\frac{1}{g}-1)N^2+(g-1)J^2) \\
&= v_s\{\int_0^L dx : \sum_\chi \phi_\chi^\dagger(x)(-i\chi\partial_x - k_F)\phi_\chi(x) : + \frac{\pi}{2L}\sum_\chi[\lambda_1 N_\chi^2 + \lambda_2 N_\chi N_{-\chi}]\}
\end{aligned}
$$

$$
\lambda_1 = \frac{1}{2}(\frac{1}{g}+g-2)\,, \quad \lambda_2 = \frac{1}{2}(\frac{1}{g}-g)
$$

Fermi liquid description

Interacting fermion system adiabatically connected to non-interacting system (Landau 1956)

Energy as a functional of distribution of occupation numbers *n(k)*

$$\delta E = \sum_k \epsilon(k)\delta n(k) + \frac{1}{2}\sum_{k,k'} f(k,k')n(k)\delta n(k')$$

In the present case

$$\delta E = \sum_k v_s(|k| - k_F)\delta n(k) + v_s\frac{\pi}{L}\sum_{k,k'}(\lambda_1\theta(kk') + \lambda_2\theta(-kk'))\delta n(k)\delta n(k')$$

energy and interaction terms

$$\epsilon(k) = v_s(|k| - k_F) \qquad f(k,k') = v_s\frac{2\pi}{L}(\lambda_1\Theta(kk') + \lambda_2\Theta(-kk'))$$

Landau parameters

interactions between particles at the Fermi points

$$F_0 = \frac{L}{2\pi v_s}(f(k_F, k_F) + f(k_F, -k_F)) = \lambda_1 + \lambda_2 = \frac{1}{g} - 1$$

$$F_1 = \frac{L}{2\pi v_s}(f(k_F, k_F) - f(k_F, -k_F)) = \lambda_1 - \lambda_2 = g - 1$$

Relation $$1 + F_1 = \frac{1}{1 + F_0} = g$$

Change of momentum variables

N-particle state: $k_i = 2\pi n_i/L\,, \quad i = 1, 2, ..., N$

New variable: $\kappa_i \equiv k_i + \lambda \frac{\pi}{L} \sum_{j \neq i} \mathrm{sgn}(k_i - k_j)\,, \quad i = 1, 2, ...$

Implies: $\kappa_{i+1} = \kappa_i + \frac{2\pi}{L}(\Delta n_i + \lambda)$

Modified Pauli exclusion with (1+ λ) as exclusion parameter

Continuum formulation

$$\kappa = k + \frac{1}{2}\lambda \int dk' n(k') \mathrm{sgn}(k - k') \qquad \nu(\kappa) d\kappa = n(k) dk$$

occupation densities

Transformation of energy $\epsilon(k)$ and interaction $f(k,k')$

Continuum expression for the energy functional

$$\begin{aligned}\delta E &= \frac{L}{2\pi}\int dk\,\epsilon(k)\delta n(k) + \frac{L^2}{8\pi^2}\iint dk dk'\, f(k,k')\delta n(k)\delta n(k') \\ &= \frac{L}{2\pi}\int d\kappa\,\tilde{\epsilon}(\kappa)\delta\nu(\kappa) + \frac{L^2}{8\pi^2}\iint d\kappa d\kappa'\,\tilde{f}(\kappa,\kappa')\delta\nu(\kappa)\delta\nu(\kappa')\end{aligned}$$

Change of integration variable $k \to \kappa$

$$\tilde{\epsilon}(\kappa') = \int dk\,\frac{\delta n(k)}{\delta\nu(\kappa')}\epsilon(k)$$

$$\tilde{f}(\kappa'',\kappa') = \frac{2\pi}{L}\int dk\,\frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')}\epsilon(k) + \iint d\bar{k}\,dk\,\frac{\delta n(\bar{k})}{\delta\nu(\kappa'')}\,\frac{\delta n(k)}{\delta\nu(\kappa')}f(\bar{k},k)$$

Results of the transformation

The transformation matrices

$$\begin{aligned}\frac{\delta n(k)}{\delta\nu(\kappa')} &= \frac{1}{2}\frac{d}{dk}[(1+\lambda n(k))\mathrm{sgn}(k-k')]\\ \frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')} &= \frac{1}{4}\lambda\frac{d^2}{dk^2}[(1+\lambda n(k))\mathrm{sgn}(k-k')\mathrm{sgn}(k-k'')]\end{aligned}$$

The energy function

$$\tilde{\epsilon}(\kappa) = \begin{cases}\epsilon(\kappa - \frac{\lambda}{1+\lambda}\kappa_F) & \kappa > \kappa_F\\ (1+\lambda)\epsilon(\frac{\kappa}{1+\lambda}) - \lambda\epsilon_F & |\kappa| < \kappa_F\\ \epsilon(\kappa + \frac{\lambda}{1+\lambda}\kappa_F) & \kappa < -\kappa_F\end{cases}$$

Interactions at the Fermi points (Landau parameters)

$$\begin{aligned}\widetilde{F}_0 &= F_0 - \lambda = \frac{1}{g}(1-g(1+\lambda))\\ \widetilde{F}_1 &= (1+\lambda)^2(F_1 + \frac{\lambda}{1+\lambda}) = -(1+\lambda)(1-g(1+\lambda))\end{aligned}$$

Absorbing the interaction

Define the value of λ

$$\lambda = \lambda_1 + \lambda_2 = \frac{1}{g} - 1 \quad \Rightarrow \quad \tilde{F}_0 = \tilde{F}_1 = 0$$

the interaction vanishes at the Fermi points:

$$\lim_{|\kappa''| \to k_F} \lim_{|\kappa'| \to k_F} \tilde{f}(\kappa'', \kappa') = 0$$

Interpretation:
In the transformed variables the system appears as consisting of weakly interacting particles with generalized Pauli exclusion.

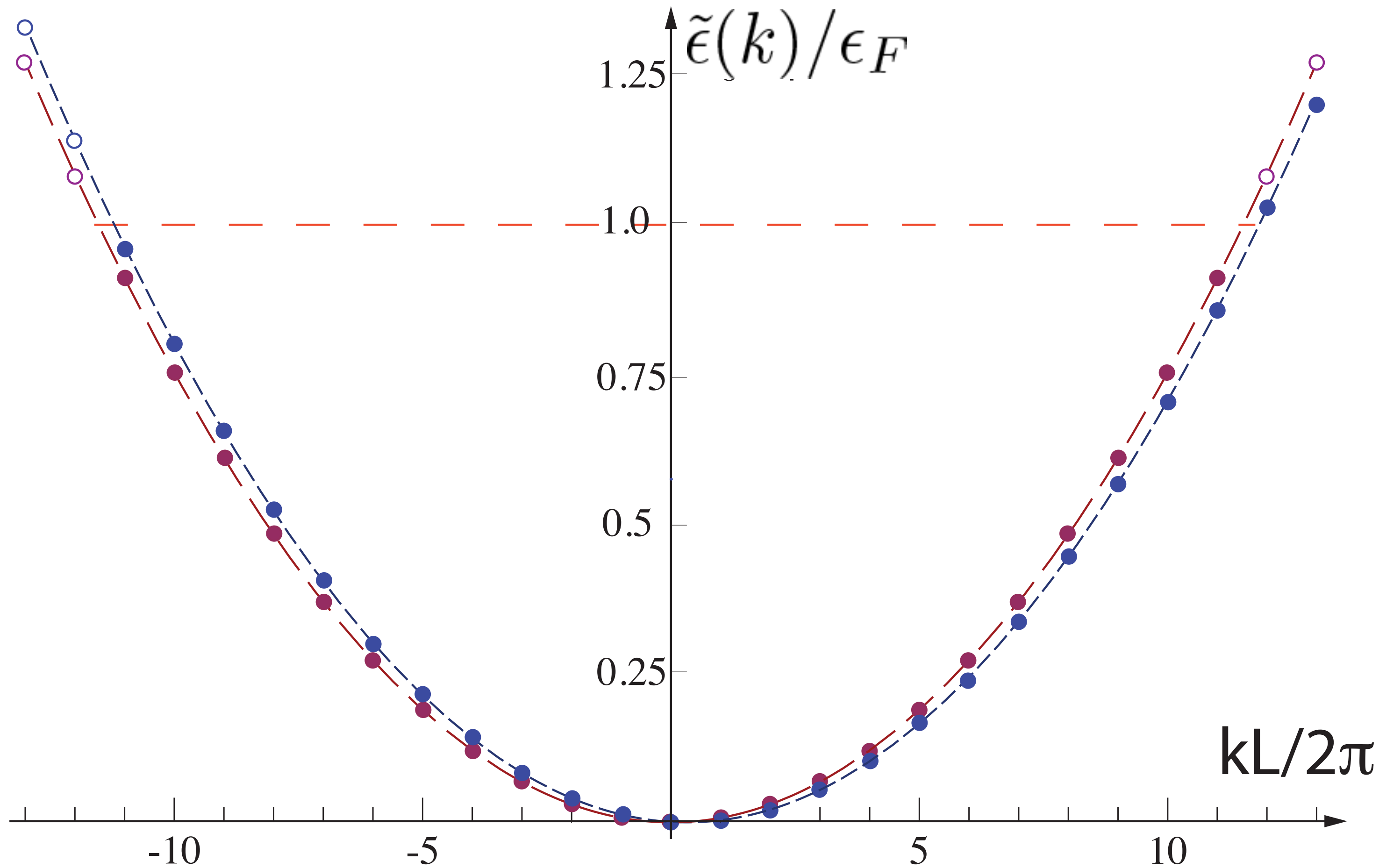
Low energy approximation:

$$E = \sum_i \tilde{\epsilon}(\kappa_i)\,, \qquad \kappa_{i+1} = \kappa_i + \frac{2\pi}{L}(\Delta n_i + \lambda)$$

Illustration: System with quadratic dispersion

Fermi sea (read) with N_0=23 particles,

Excited state (blue) with N=J=2 particles added



$$\kappa = (1+\lambda)k - \frac{\pi}{L}\lambda J$$

Summary

Fermion system in one dimension:
Luttinger liquid description with bosonic variables
Standard fermi liquid description problematic

Dressed fermions: adiabatically evolved from free fermions,
carry (locally) a fraction of the fermion charge

Modified fermi liquid description:
Absorb interaction in transformation of momentum variables

Transformation of the fermion system to a system of weakly interacting
particles wth fractional charge and fractional statistics