# Emergence of anyons and ground-state properties of the anyon gas

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*based on work in collaborations with*
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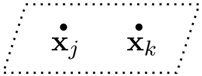
# Outline of Talk

1. Introduce 2D anyons — ideal or extended
2. Emergence of anyons in physics
3. The ideal anyon gas
4. Discussion

# Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi\colon (\mathbb{R}^2)^N \to \mathbb{C}$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) \in \mathbb{C}$$



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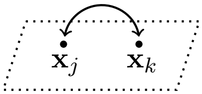
$$|\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_k,\ldots,\mathbf{x}_N)|^2 = |\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_k,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_N)|^2$$



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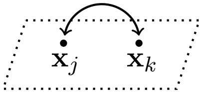
$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\theta}\, \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$



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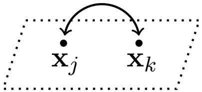

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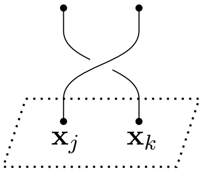
$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$




$e^{i\alpha\pi} \in U(1)$ **any** phase

$\alpha = 0$: bosons

$\alpha = 1$: fermions

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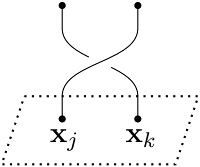


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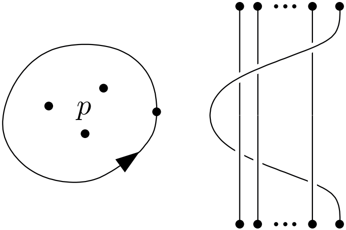
**anyons**: 'fractional'-statistics quasiparticles in confined systems — expected to arise e.g. in fractional quantum Hall systems

$\sim$1970 Souriau; Streater & Wilde ... Leinaas & Myrheim '77; Goldin, Menikoff & Sharp '81; Wilczek '82 ...

Reviews by Fröhlich '90, Wilczek '90, Lerda '92, Myrheim '99, Khare '05, Ouvry '07, Stern '08, Hansson et al '17...

Past rigorous QM studies by Baker, Canright & Mulay '93, Dell'Antonio, Figari & Teta '97

# Modelling anyons concretely — anyon gauge

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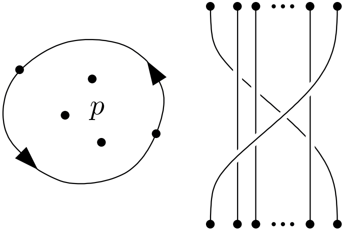


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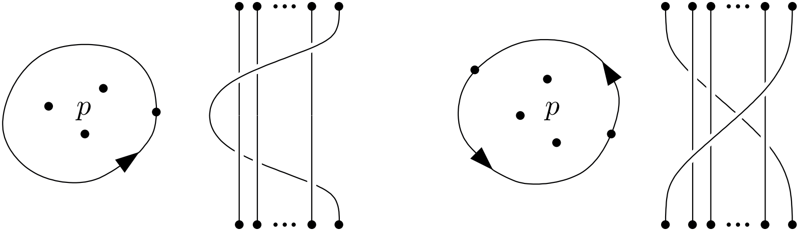


$e^{i2p\alpha\pi}$ $\qquad$ $e^{i(2p+1)\alpha\pi}$

*Think:* free kinetic energy $\hat{T}_0 = \frac{\hbar^2}{2m}\sum_{j=1}^N (-i\nabla_j)^2$ acting on multi-valued

$$\Psi_\alpha := U^\alpha\, \Psi_0, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|}.$$

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Bosons ($\Psi \in L^2_{\mathrm{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{\hbar^2}{2m} \sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j(\mathbf{x}_j), \quad \mathbf{A}_j(\mathbf{x}) = \sum_{k \neq j} \frac{(\mathbf{x} - \mathbf{x}_k)^\perp}{|\mathbf{x} - \mathbf{x}_k|^2}$$

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These are **ideal** anyons. One can also model $R$-**extended** anyons:

$$\mathbf{A}_j^R(\mathbf{x}) := \sum_{k \neq j} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|_R^2}, \qquad |\mathbf{x}|_R := \max\{|\mathbf{x}|, R\}$$

$$\Rightarrow \quad \operatorname{curl} \alpha \mathbf{A}_j^R = 2\pi\alpha \sum_{k \neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} \xrightarrow{R \to 0} 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_k}$$

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We would like to understand the $N$-anyon ground state $\Psi_0$ and energy

$$E_0(N) := \inf \operatorname{spec} \hat{H}_N, \quad \hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left( \frac{\hbar^2}{2m} D_j^2 + V(\mathbf{x}_j) \right)$$

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- Need several particles!
- Need 2D!

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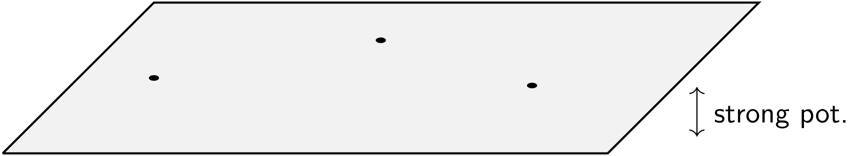
cf. e.g. Forte, 1991



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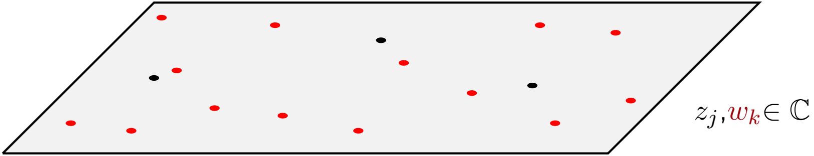


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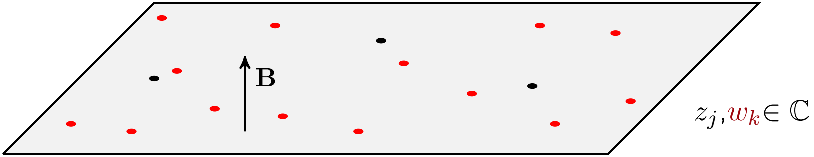


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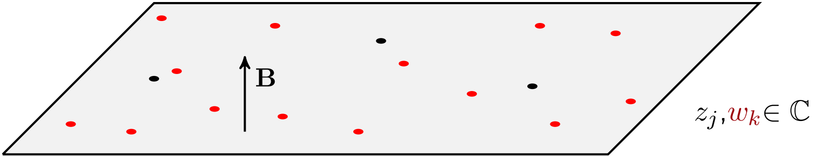


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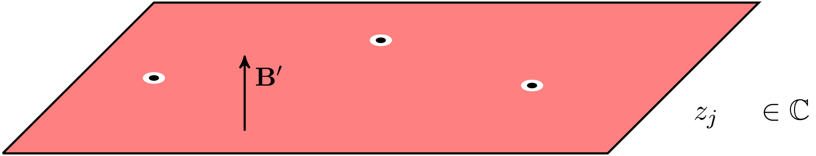


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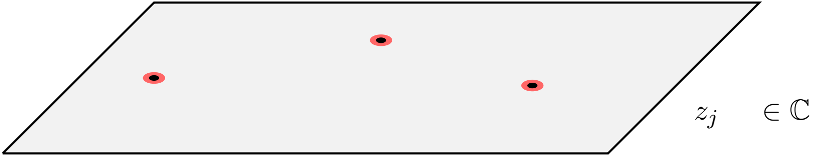


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Take two different species of quantum particles in a strong magnetic field $B > 0$: 'tracer' particles at $\mathbf{x}_{j=1\ldots M} \in \mathbb{R}^2$ in a large sea of 'bath' particles at $\mathbf{y}_{k=1\ldots N} \in \mathbb{R}^2$, $N \gg M$.

$$\mathcal{H}^{M+N} = L^2_{\mathrm{sym}}(\mathbb{R}^{2M}) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$$

$$H_{M+N} = H_M \otimes \mathbb{1} + \mathbb{1} \otimes H_N + \sum_{j=1}^{M} \sum_{k=1}^{N} W_{12}(\mathbf{x}_j - \mathbf{y}_k),$$

$$H_M = \sum_{j=1}^{M} \frac{1}{2m} \left(\mathbf{p}_{\mathbf{x}_j} + e\mathbf{A}(\mathbf{x}_j)\right)^2 \quad + \sum_{1 \le i < j \le M} W_{11}(\mathbf{x}_i - \mathbf{x}_j),$$

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**Ansatz:** $\Psi(\mathbf{X},\mathbf{Y}) = \Phi(\mathbf{X}) c_{\mathrm{qh}}(\mathbf{X}) \overline{\Psi^{\mathrm{qh}}(\mathbf{X},\mathbf{Y})}$,

with a Laughlin wave function coupled to quasi-holes at $\mathbf{x}_j \equiv z_j$:

$$\Psi^{\mathrm{qh}}(\mathbf{X},\mathbf{Y}) = \prod_{j=1}^{M} \prod_{k=1}^{N} (z_j - w_k)^q \prod_{1 \le i < j \le N} (w_i - w_j)^{n} \, e^{-B \sum_{j=1}^{N} |w_j|^2/4}.$$

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**Claim:** for $N \gg M$:

$$\langle \Psi, H_{M+N}\Psi\rangle \approx \left\langle \Phi, H_M^{\mathrm{eff}}\Phi \right\rangle + BN/2,$$

where

$$H_M^{\mathrm{eff}} = \sum_{j=1}^{M}\frac{1}{2m}\left(\mathbf{p}_{\mathbf{x}_j} + \frac{B}{2}(e - \frac{1}{n})\mathbf{x}_j^{\perp} + \alpha \mathbf{A}_j^R(\mathbf{x}_j)\right)^2 + \sum_{1\le i<j\le M} W_{11}(\mathbf{x}_i - \mathbf{x}_j)$$

is an effective Hamiltonian describing $M$ anyons with $\alpha = 1/n$, $R = \sqrt{2/B}$.

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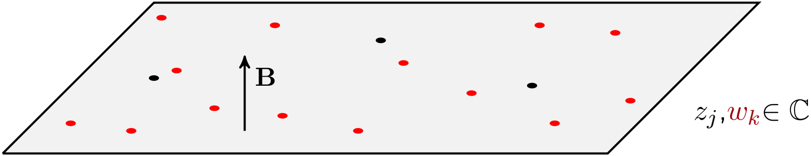


- Two species of particles in a plane (bosons or fermions)
- Strong perpendicular magnetic field $B \;\Rightarrow\;$ LLL
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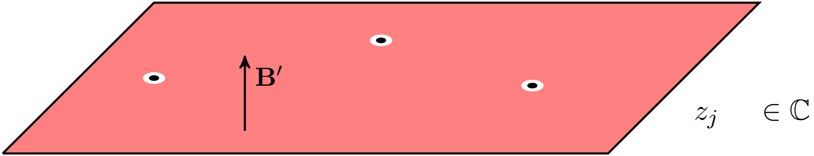


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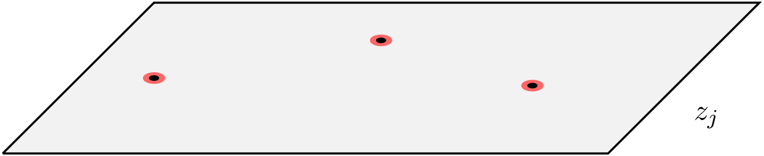


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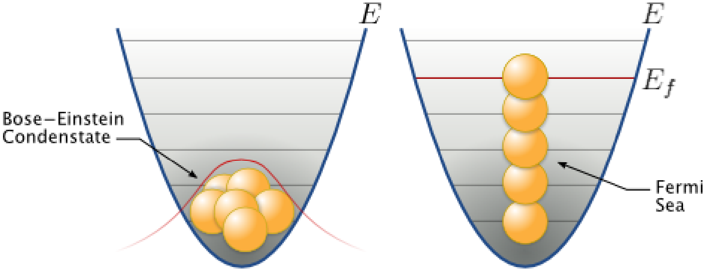
# Understanding the zero-temperature ideal anyon gas




cold **bosons** cold **fermions**

**anyons?**

2-body: Leinaas, Myrheim, 1977; Wilczek, 1982; Arovas, Schrieffer, Wilczek, Zee, 1985
3- and 4-body numerics: Sporre, Verbaarschot, Zahed, 1991-92; Murthy, Law, Brack, Bhaduri, 1991
Approximations: average-field theory, lowest Landau level, dilute
Hundreds of papers...

# Compare with the ideal Bose gas in 2D

Know: $\Psi_0 = \otimes^N \varphi_0$, $\quad \varphi_0$ lowest state of $\hat{H}_1 = -\Delta_{\mathbb{R}^2} + V(\mathbf{x})$

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The free Bose gas in a box $Q = [0, L]^2$:

$$\hat{H}_1 = (-\Delta_Q)^{\text{Dirichlet}}, \qquad \varphi_0(x, y) = \sin(\pi x/L)\sin(\pi y/L),$$

$$E_0(N, L) = \frac{\langle \Psi_0, \hat{H}_N \Psi_0 \rangle}{\|\Psi_0\|^2} = N\lambda_0 = N\frac{2\pi^2}{L^2} = 2\pi^2 \bar{\varrho}$$

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$\Rightarrow$ Energy per area:

$$\frac{E_0(N, L)}{L^2} = \frac{2\pi^2 \bar{\varrho}}{L^2} \to 0,$$

as $N \to \infty$ and $L \to \infty$ with fixed density $\bar{\varrho} = N/L^2$.

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$\Rightarrow$ Thomas–Fermi approximation: (Thomas, Fermi, 1927 — precursor to modern DFT)

$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big( 2\pi \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x}) \varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

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$$\langle \Psi_0, (\hat{T}_{\alpha=1} + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big( 2\pi \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x}) \varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x}$$

The Lieb–Thirring inequality: (Lieb, Thirring, 1975)

$$\langle \Psi, (\hat{T}_{\alpha=1} + \hat{V})\Psi \rangle \geq \int_{\mathbb{R}^2} \Big( C_{\mathrm{LT}}\, \varrho_{\Psi}(\mathbf{x})^2 + V(\mathbf{x}) \varrho_{\Psi}(\mathbf{x}) \Big) d\mathbf{x}$$

# Compare with a relaxed Pauli principle

If $\nu$ particles allowed in each state: $\Psi_0 = \bigotimes^{\nu} \bigwedge^{N/\nu} \varphi_k$,

The free Fermi gas in a box $Q \subset \mathbb{R}^2$: (Weyl asymptotics)

$$E_0(N,L) = \sum_{k=0}^{N-1} \lambda_k \sim 2\pi\nu \frac{(N/\nu)^2}{|Q|} = 2\pi\nu^{-1} \bar{\varrho}^2 \, |Q|$$

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# Average-field approximation

Huge past literature: see e.g. Wilczek 1990 review

For anyons one may consider an **average-field** approximation

$$\langle \Psi_0, (\hat{T}_\alpha + \hat{V})\Psi_0 \rangle \approx \int_{\mathbb{R}^2} \Big( 2\pi|\alpha| \varrho_{\Psi_0}(\mathbf{x})^2 + V(\mathbf{x}) \varrho_{\Psi_0}(\mathbf{x}) \Big) d\mathbf{x},$$

where $B = \mathrm{curl}\, \alpha \mathbf{A}_j \approx 2\pi\alpha\varrho$ with LLL energy/particle $\sim |B|$.

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A particular **almost-bosonic** limit $\alpha = \beta/N \to 0$ leads to

$$\mathcal{E}^{\mathrm{af}}[\psi] := \int_{\mathbb{R}^2} \left( \left| \left( -i\nabla + \beta \mathbf{A}[|\psi|^2] \right) \psi(\mathbf{x}) \right|^2 + V(\mathbf{x})|\psi(\mathbf{x})|^2 \right) d\mathbf{x},$$

where $\operatorname{curl} \mathbf{A}[|\psi|^2] = 2\pi|\psi|^2$ and $\beta$ the only parameter.

DL, Rougerie, 2015; Correggi, DL, Rougerie, 2017; Chern-Simons coupled to non-rel. matter ∼1980

# Average-field approximation for almost-bosonic anyons

"Less bosonic" anyons would then amount to $\beta = \alpha N \to \infty$.

**Theorem:** As $\beta \to \infty$

$$E_0^{\mathrm{af}}(\beta) = E_0^{\mathrm{TF}}(\beta) + \text{lower order},$$

where $E_0^{\mathrm{TF}}(\beta)$ is the minimum of the Thomas–Fermi functional

$$\mathcal{E}^{\mathrm{TF}}[\varrho] := \int_{\mathbb{R}^2} \left( e(1,1)\beta\varrho(\mathbf{x})^2 + V\varrho(\mathbf{x}) \right) d\mathbf{x}, \qquad \int_{\mathbb{R}^2} \varrho(\mathbf{x}) d\mathbf{x} = 1.$$

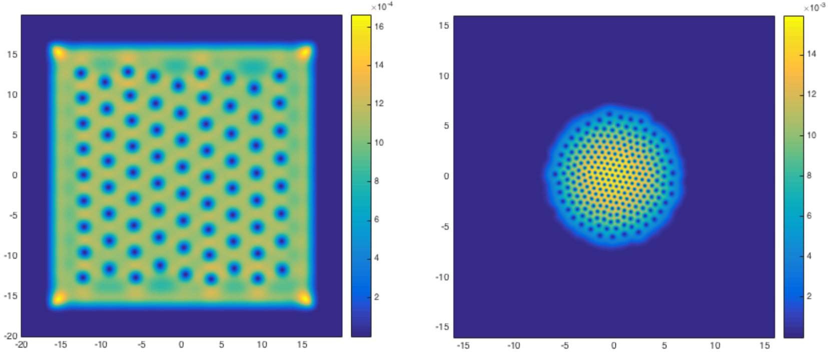
Furthermore, $e(1,1) \geq 2\pi$, with $e(\beta,\rho) = e(1,1)\beta\rho^2$ the energy per area of the homogeneous problem at density $\rho$.

**Conjecture:** $e(1,1) > 2\pi$

Correggi, DL, Rougerie, 2017

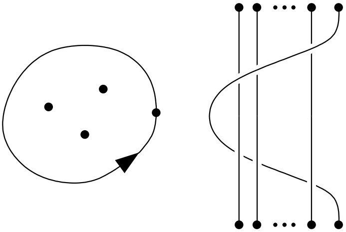
# Average-field approximation for almost-bosonic anyons

Continued study of the average-field functional $\mathcal{E}^{\mathrm{af}}[\psi]$ is work in progress with M. Correggi, R. Duboscq and N. Rougerie.

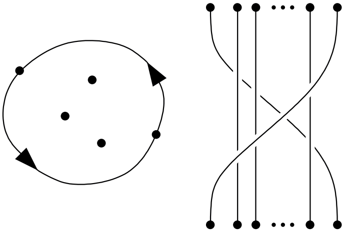



Numerical simulations of $|\psi_0|^2$ at $\beta = 318$ by Romain Duboscq.

# Universal bounds: A local exclusion principle for anyons



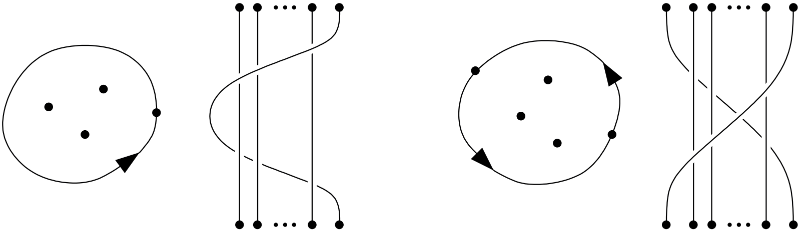
$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

*Recall:* 2-particle exchange phase $(2p+1)\alpha$ times $\pi$.
But anyons can also have pairwise relative angular momenta $\pm 2q$.

# Universal bounds: A local exclusion principle for anyons

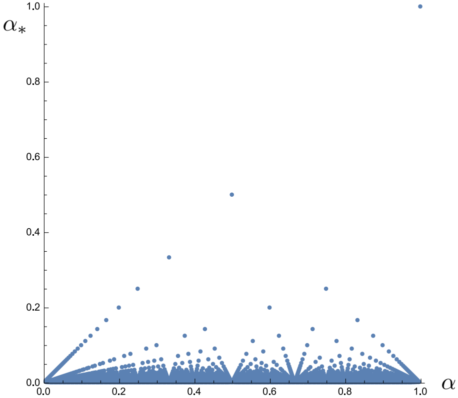


$e^{i2p\alpha\pi}$ $\qquad$ $e^{i(2p+1)\alpha\pi}$

*Recall:* 2-particle exchange phase $(2p+1)\alpha$ times $\pi$.
But anyons can also have pairwise relative angular momenta $\pm 2q$.
$\Rightarrow$ effective **statistical repulsion** DL, Solovej, 2013

$$V_{\text{stat}}(r) = |(2p+1)\alpha - 2q|^2 \frac{1}{r^2} \;\geq\; \frac{\alpha_N^2}{r^2}, \qquad r = |\mathbf{x}_j - \mathbf{x}_k|$$

# Universal bounds: A local exclusion principle for anyons




$$\alpha_N := \min_{p \in \{0,1,\ldots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|$$

$$\underset{N\to\infty}{\longrightarrow} \alpha_* := \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \textbf{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

# Universal bounds for the homogeneous anyon gas

DL, Solovej, 2011-'13, Larson, DL, 2016-'18, DL, Seiringer, 2017
Work in progress with Qvarfordt extends to non-abelian anyons.

Define the ground-state energy per particle and unit density

$$e(\alpha) := \liminf_{\substack{N,L\to\infty \\ N/L^2=\bar{\varrho}}} \frac{E_0(N,L)}{N\bar{\varrho}} \qquad \begin{aligned} e(0) &= 0, \\ e(1) &= 2\pi \end{aligned}$$

**Theorem:** There exist constants $0 < C_1 \leq C_2 < \infty$ such that for any $0 \leq \alpha \leq 1$,

$$C_1\alpha \leq e(\alpha) \leq C_2\alpha,$$

and as $\alpha \to 0$,

$$e(\alpha) \geq \frac{\pi}{4}\alpha\big(1 - O(\alpha^{1/3})\big)$$

$$e(\alpha) \geq \pi\alpha_*\big(1 - O(\alpha_*^{1/3})\big)$$

**Conjecture:** optimal $C_1$ and $C_2$ cannot both be $2\pi$.

# Lieb–Thirring inequalities for anyons

Dyson, Lenard, 1967
DL, Solovej, 2011-'13; LT with general local exclusion developed by Nam, Portmann, Solovej, 2013-'15;
Larson, DL, 2016-'18; DL, Seiringer, 2017

**Theorem (LT inequality for ideal anyons)**

*There exists a constant $0 < C \leq 2\pi$ such that for any $0 \leq \alpha \leq 1$ and any $N$-anyon wave function $\Psi$ on $\mathbb{R}^2$,*

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \ \geq \ C\alpha \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x}.$$

Hence

$$\langle \Psi, \hat{H}_N \Psi \rangle \geq \int_{\mathbb{R}^2} \Big( C\alpha \varrho_\Psi(\mathbf{x})^2 + V(\mathbf{x}) \varrho_\Psi(\mathbf{x}) \Big) d\mathbf{x}$$

i.e. a universal lower bound of the form of average-field theory.

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D. L., *Many-anyon trial states*, Phys. Rev. A 96 (2017) 012116.

M. Correggi, D. L., N. Rougerie, *Local density approximation for the almost-bosonic anyon gas*, Analysis & PDE 10 (2017) 1169.

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# Discussion

- Extended case
- Harmonic trap
- Clustering trial states

# Hardy inequality

Statistical repulsion gives rise to the following "Hardy inequality":

$$\hat{T}_\alpha \geq \frac{4\alpha_N^2}{N} \sum_{1 \leq j < k \leq N} |\mathbf{x}_j - \mathbf{x}_k|^{-2}$$

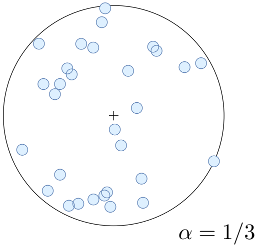
# Extended case

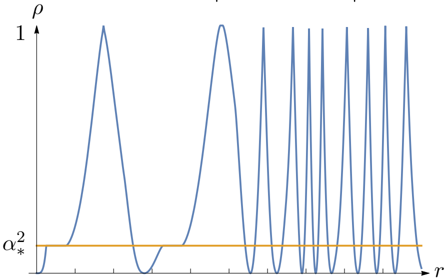
We use a magnetic Hardy inequality **with symmetry**

(cf. Laptev, Weidl, 1998; Hoffmann-Ostenhof$^2$, Laptev, Tidblom, 2008; Balinsky...)

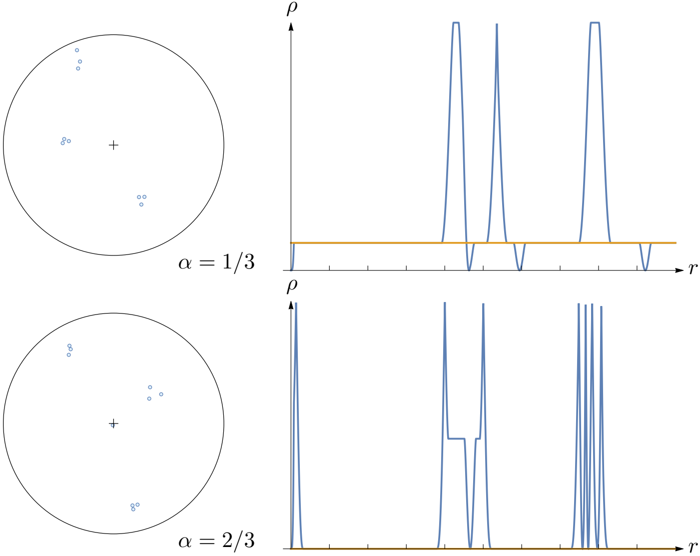
to consider the enclosed flux inside a two-particle exchange loop, subtracted with arbitrary pairwise angular momenta. Unwanted oscillation can be controlled by smearing (but analysis is tricky!)

$$V_{\mathrm{stat}}(r) = \rho(r)\frac{1}{r^2}, \qquad \rho(r) = \min_{q\in\mathbb{Z}} \left| \frac{\Phi(r)}{2\pi} - 2q \right|^2$$



# Extended case (clustering)



# Universal bounds for the extended anyon gas

Consider ground-state energy on a box $Q \subset \mathbb{R}^2$:

$$E_0(N, Q, \alpha, R) := \inf \Big\{ \langle \Psi, \hat{T}_\alpha^R \Psi \rangle \, : \, \Psi \in L_c^2(Q^N) \, , \, \|\Psi\| = 1 \Big\}$$

In the thermodynamic limit, $N, |Q| \to \infty$ with $\bar{\varrho} = N/|Q|$ fixed, for dimensional reasons,

$$\frac{E_0(N, Q, \alpha, R)}{N} \to e(\alpha, \gamma)\bar{\varrho}, \qquad \gamma := R\sqrt{\bar{\varrho}}.$$

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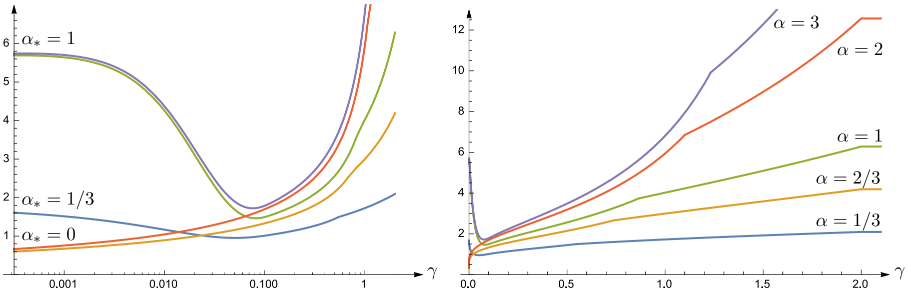
In the thermodynamic limit, $N, |Q| \to \infty$ with $\bar{\varrho} = N/|Q|$ fixed, for dimensional reasons,

$$\frac{E_0(N, Q, \alpha, R)}{N} \to e(\alpha, \gamma)\bar{\varrho}, \qquad \gamma := R\sqrt{\bar{\varrho}}.$$

We define (with Dirichlet b.c.)

$$e(\alpha, \gamma) := \liminf_{\substack{N, |Q| \to \infty \\ N/|Q| = \bar{\varrho}}} \frac{E_0(N, Q, \alpha, R)}{\bar{\varrho} N}.$$

# Universal bounds for the extended anyon gas




**Theorem ([Larson-DL'16] Bounds for the extended anyon gas)**

*Up to some universal constant $C > 0$,*

$$e(\alpha,\gamma) \gtrsim \begin{cases} \frac{2\pi}{|\ln\gamma|} + \pi(j'_{\alpha_*})^2 \geq 2\pi\alpha_*, & \gamma \to 0,\ \alpha \neq 0 \\ 2\pi|\alpha|, & \gamma \gtrsim 1. \end{cases}$$

# Ideal anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left( \frac{1}{2m}(-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2}|\mathbf{x}_j|^2 \right).$$

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Rigorous bounds for the ground-state energy $E_0(N)$:

$$\hat{H}_N\big|_{\text{ang.mom.}=L} \;\geq\; \omega \left( N + \left| L + \alpha \tfrac{N(N-1)}{2} \right| \right)$$ (Chitra, Sen, 1992)

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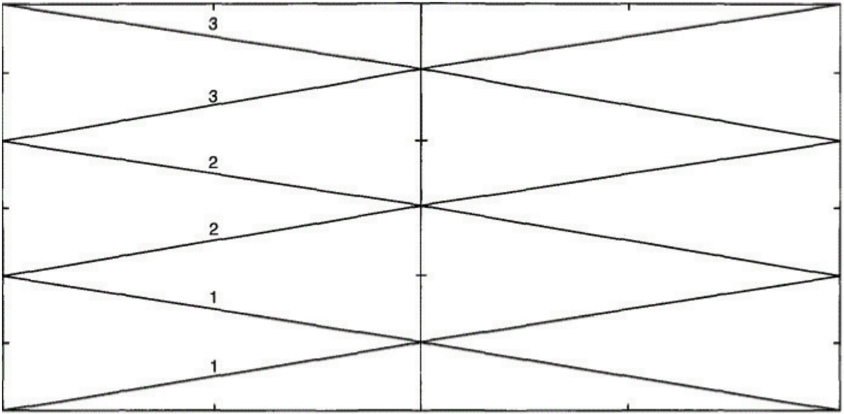
$$C_1\, j'_{\alpha_N} \ \leq \ E_0(N)/(\omega N^{\frac{3}{2}}) \ \leq \ C_2 \quad \forall \alpha, N$$ (DL, Solovej, 2013; Larson, DL, 2016)

Cp. with fermions in 2D: $E_0(N) \sim \frac{\sqrt{8}}{3}\omega N^{\frac{3}{2}}$ as $N \to \infty$

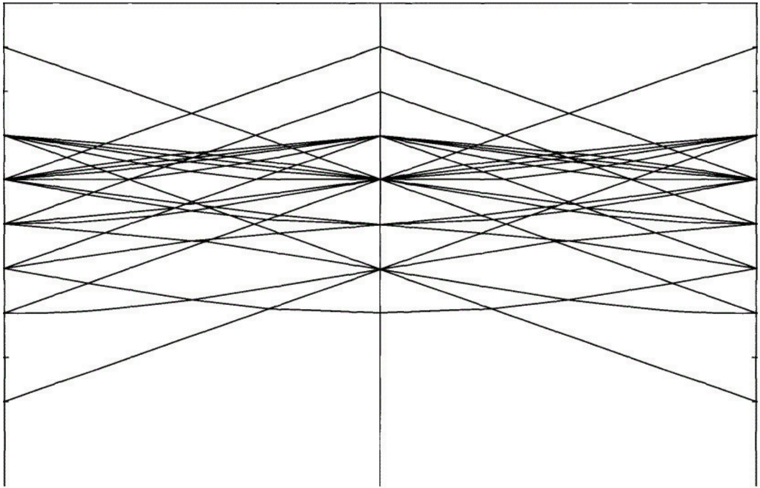
Average-field suggests: $E_0(N) \approx \frac{\sqrt{8}}{3}\sqrt{|\alpha|}\,\omega N^{\frac{3}{2}}$ as $N \to \infty$

# Anyons in a harmonic trap — exact spectrum



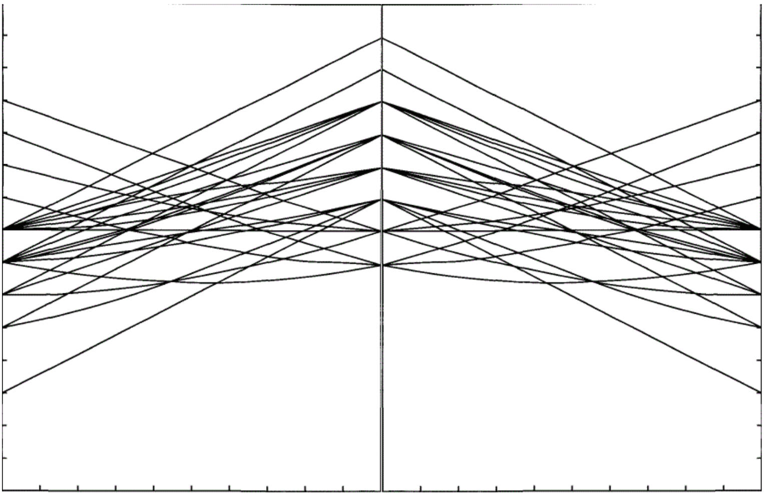

Exact $N = 2$ spectrum: Leinaas, Myrheim, 1977

# Anyons in a harmonic trap — exact spectrum



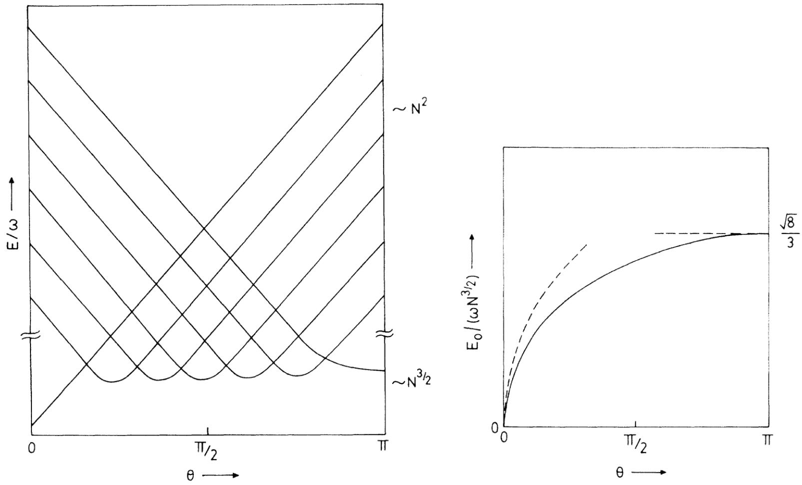
Numerical $N = 3$ spectrum: Murthy, Law, Brack, Bhaduri, 1991; Sporre, Verbaarschot, Zahed, 1991

# Anyons in a harmonic trap — exact spectrum



Numerical $N = 4$ spectrum: Sporre, Verbaarschot, Zahed, 1992

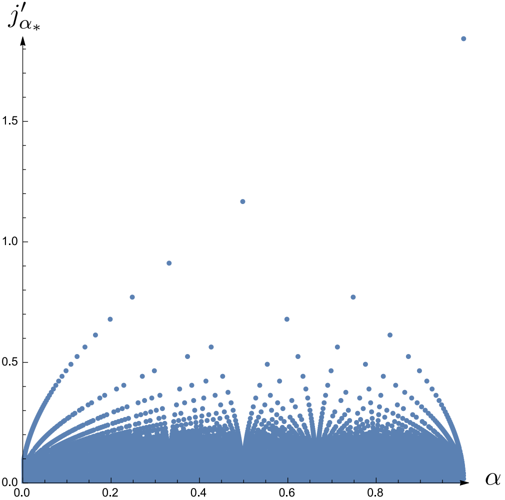
# Anyons in a harmonic trap — qualitative spectrum




Schematic $N \to \infty$ spectrum: Chitra, Sen, 1992 $(\theta = \alpha\pi)$

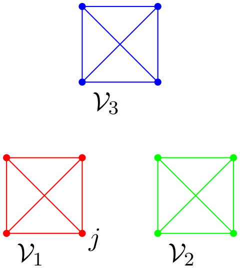
# Anyons in a harmonic trap — current lower bounds

$$\frac{E_0(N)}{\omega N^{\frac{3}{2}}} \gtrsim$$




Rigorous lower bounds: DL, Solovej, 2013/'14, improved in Larson, DL, 2016, and DL, Seiringer, 2017 ...

# Upper bounds: many-anyon trial states




$N = \nu K$ particles arranged into $\nu$ complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

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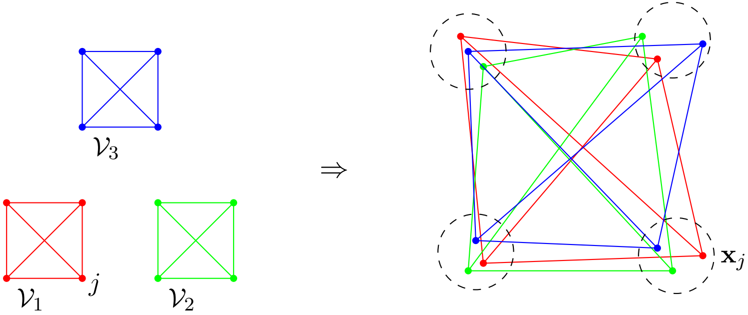


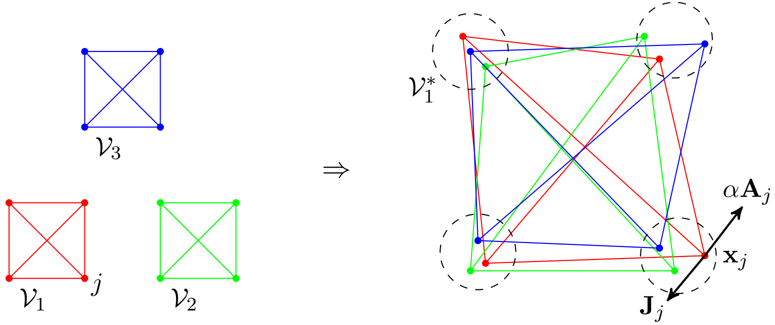

$N = \nu K$ particles arranged into $\nu$ complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **even**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[ \prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \right] \prod_{k=1}^{N} \varphi_0(z_k)$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

# Upper bounds: many-anyon trial states

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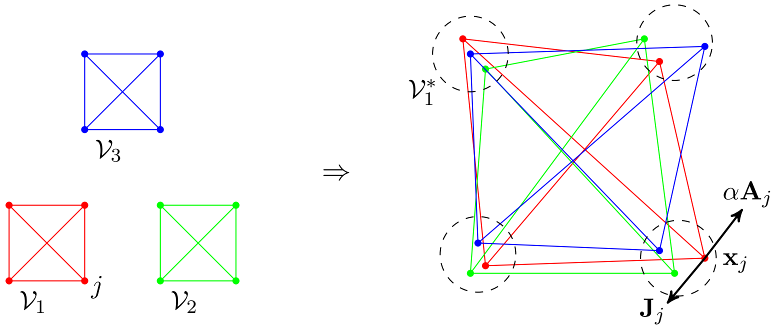


$N = \nu K$ particles arranged into $\nu$ complete graphs $(\mathcal{V}_q, \mathcal{E}_q)$

$\alpha = \frac{\mu}{\nu}$ **odd**:

$$\psi_\alpha(\mathrm{z}) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[ \prod_{q=1}^{\nu} \prod_{(j,k)\in\mathcal{E}_q} (\bar{z}_{jk})^{\mu} \bigwedge_{k=0}^{K-1} \varphi_k\left(z_{j\in\mathcal{V}_q}\right) \right]$$

(cf. Moore–Read (Pfaffian), Read–Rezayi)

# Upper bounds: many-anyon trial states

**Proposition:** For $\Psi = \Phi\psi_\alpha$, $\Phi \in H^1_{\mathrm{loc}}(\mathbb{R}^{2N};\mathbb{R})$, $\alpha = \frac{\mu}{\nu}$ **even**,

$$\langle \Psi, \hat{H}_N \Psi \rangle = \left(1 - \alpha\frac{\nu - 1}{2}\right)\omega N \int_{\mathbb{R}^{2N}} |\Psi|^2 \, d\mathrm{x} + \int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |\nabla_j \Phi|^2 |\psi_\alpha|^2 \, d\mathrm{x}.$$

# Upper bounds: many-anyon trial states

**$R$-extended case:** Replace $\prod_{j<k} |z_{jk}|^{-\alpha}$ with $e^{-\alpha \sum_{j<k} w_R(\mathbf{x}_j - \mathbf{x}_k)}$.

**Proposition:** For the free gas on a box $Q \subset \mathbb{R}^2$, $\alpha$ **even**

$$\boxed{\hat{T}_\alpha^R \psi_\alpha = \alpha W_R \psi_\alpha},$$

$$W_R(\mathrm{x}) := \sum_{j \neq k = 1}^{N} \Delta w_R(\mathbf{x}_j - \mathbf{x}_k) = 2\pi \sum_{j \neq k = 1}^{N} \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k).$$

**Proposition:** For $\Psi = \Phi \psi_\alpha$, $\Phi \in H_0^1(Q^N; \mathbb{R})$, $\alpha$ **even**

$$\langle \Psi, \hat{T}_\alpha^R \Psi \rangle = \int_{Q^N} \left( \sum_{j=1}^{N} |\nabla_j \Phi|^2 + \alpha W_R |\Phi|^2 \right) |\psi_\alpha|^2 \, d\mathrm{x}.$$