Quantum Fields on a Curved Background

Eline Prytz Andersen 19.08.20

Outlook

Quantum fields in flat spacetime

Quantum fields in curved spacetime

Particle creation from collapsing stars

Particle creation by accelerated observers

Current research

Quantum Fields in Flat Spacetime

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- Field equation,

$$\left(\Box + m^2\right)\phi = 0$$

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- Field equation,

$$\left(\Box + m^2\right)\phi = 0$$

- Solutions,

$$f_{\mathbf{k}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- Field equation,

$$\left(\Box + m^2\right)\phi = 0$$

- Solutions,

$$f_{\mathbf{k}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

Quantisation

- Fields are operators

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- Field equation,

$$\left(\Box + m^2\right)\phi = 0$$

- Solutions,

$$f_{\mathbf{k}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

Quantisation

- Fields are operators
- Equal time commutation relations,

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = 0$$

$$[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\delta^{n-1}(\mathbf{x} - \mathbf{x}')$$

Quantum Fields In Flat Spacetime

A Review

Classical Field

- Minkowski spacetime,

$$ds^2 = dt^2 - d\mathbf{x}^2$$

- Scalar field Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- Field equation,

$$\left(\Box + m^2\right)\phi = 0$$

- Solutions,

$$f_{\mathbf{k}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

Quantisation

- Fields are operators
- Equal time commutation relations,

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = 0$$

$$[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\delta^{n-1}(\mathbf{x}-\mathbf{x}')$$

- Expansion of field,

$$\phi = \sum_{\mathbf{k}}\left[a_{\mathbf{k}}f_{\mathbf{k}}(t,\mathbf{x}) + a^\dagger_{\mathbf{k}}f^*_{\mathbf{k}}(t,\mathbf{x})\right]$$

What is a Particle?

Part 1

- Eigenvectors of timelike Killing vector,

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}, \quad \omega > 0$$

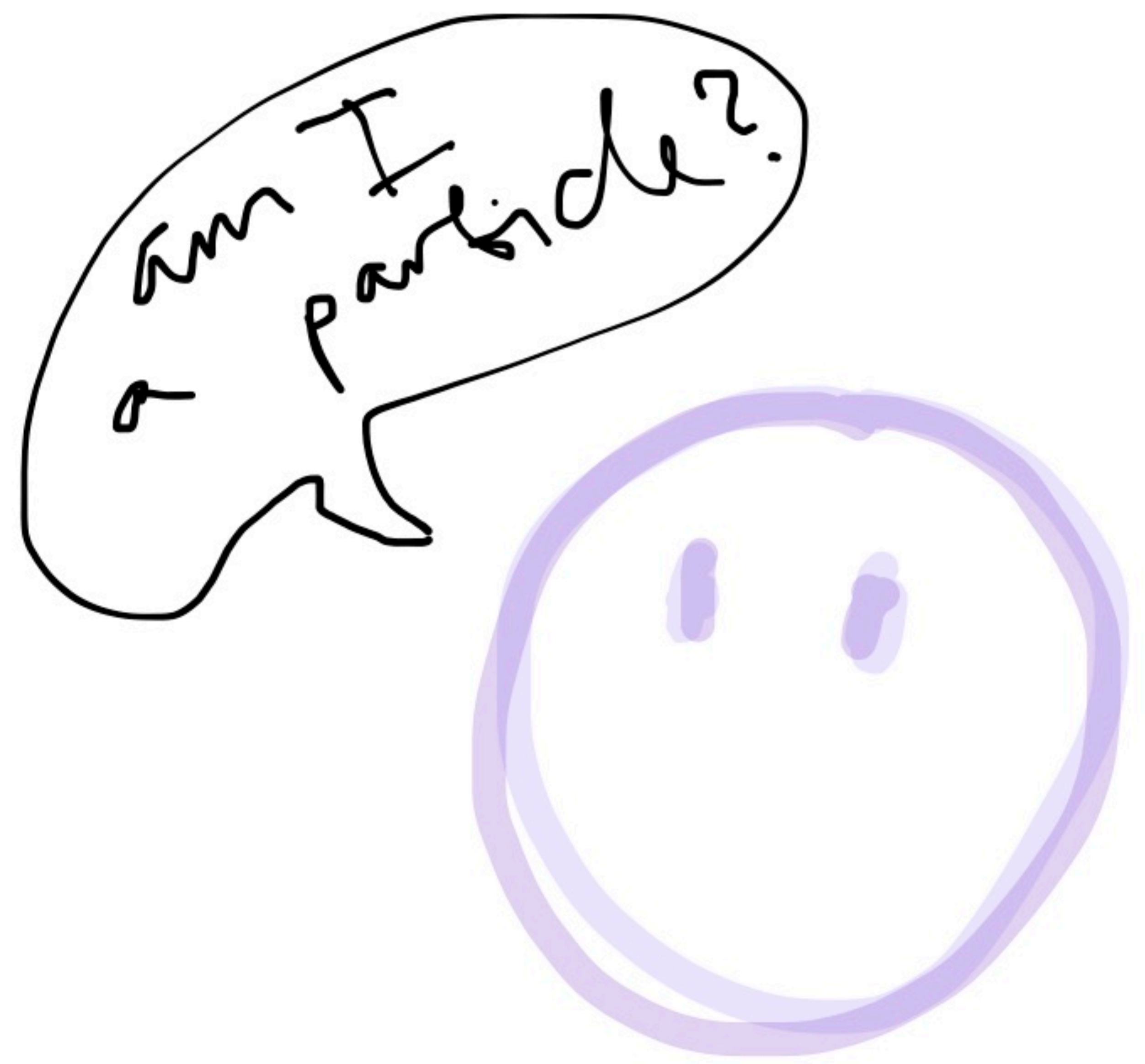
- Annihilation and creation operators, $a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}$
- Vacuum state,

$$a_{\mathbf{k}}|0\rangle = 0 \quad \forall \mathbf{k}$$

- One-particle state,

$$|1_{\mathbf{k}}\rangle = a_{\mathbf{k}}^{\dagger}|0\rangle$$

- Invariant under Poincaré group



Quantum Fields in Curved Spacetime

Scalar Field

Minimal Coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu$$

Scalar Field

Minimal Coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu$$

	Flat spacetime	**Curved spacetime**
Line element	$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

Scalar Field

Minimal Coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu$$

	Flat spacetime	Curved spacetime
Line element	$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$
Lagrangian density	$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$	$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$

Scalar Field

Minimal Coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu$$

	Flat spacetime	Curved spacetime
Line element	$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$
Lagrangian density	$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$	$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$
Equation of motion	$\left(\Box + m^2\right)\phi = 0$	$\left(\Box + m^2 + \xi R\right)\phi = 0$

Scalar Field

Minimal Coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu$$

	Flat spacetime	Curved spacetime
Line element	$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$
Lagrangian density	$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$	$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$
Equation of motion	$\left(\Box + m^2\right)\phi = 0$	$\left(\Box + m^2 + \xi R\right)\phi = 0$
Scalar product	$(\phi_1, \phi_2) = -i\int_\Sigma \phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* d\Sigma^\mu$	$(\phi_1, \phi_2) = -i\int_\Sigma \phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \sqrt{-g_\Sigma} d\Sigma^\mu$

Scalar Field Quantisation

Solutions to Wave Equation

Can always find set of solutions to field equation satisfying

$$(f_i, f_j) = \delta_{ij}, \ \left(f_i^*, f_j^*\right) = -\delta_{ij}, \ \left(f_i, f_j^*\right) = 0$$

so that

$$\phi = \sum_i \left(a_i f_i + a_i^\dagger f_i^*\right)$$

with vacuum state

$$a_i|0_f\rangle = 0 \quad \forall i$$

and commutation relations

$$[a_i, a_j] = 0; \quad \left[a_i^\dagger, a_j^\dagger\right] = 0 \quad \left[a_i, a_j^\dagger\right] = \delta_{ij}$$

Scalar Field Quantisation

Solutions to Wave Equation

Exists another basis of solutions $g_i(x^\mu)$

with field expansion

$$\phi = \sum_i \left(b_i g_i + b_i^\dagger g_i^* \right)$$

and vacuum state defined by

$$b_i|0_g\rangle = 0 \quad \forall i$$

Scalar Field Quantisation

Bogolubov Transformations

- Both sets of modes form complete basis for field
- Bogolubov transformations

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right)$$

$$f_i = \sum_j \left(\alpha_{ji}^* g_j - \beta_{ji} g_j^*\right)$$

- Relating operators

$$a_i = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger\right)$$

$$b_i = \sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger\right)$$

Scalar Field Quantisation

Particles in Vacuum

Expected number of particles in vacuum state $|0_f\rangle$ ...

Scalar Field Quantisation

Particles in Vacuum

Expected number of particles in vacuum state $|0_f\rangle$ ...

- ... described by modes f_i,

$$\langle 0_f | n_{f_i} | 0_f \rangle = \langle 0_f | a_i^\dagger a_i | 0_f \rangle = 0$$

Scalar Field Quantisation

Particles in Vacuum

Expected number of particles in vacuum state $|0_f\rangle$ …

- … described by modes f_i,

$$\langle 0_f | n_{f_i} | 0_f \rangle = \langle 0_f | a_i^\dagger a_i | 0_f \rangle = 0$$

- … described by modes g_i,

$$\langle 0_f | n_{g_i} | 0_f \rangle = \langle 0_f | b_i^\dagger b_i | 0_f \rangle = \sum_j |\beta_{ij}|^2$$

Scalar Field Quantisation

Particles in Vacuum

Expected number of particles in vacuum state $|0_f\rangle$ ...

- ... described by modes f_i,

$$\langle 0_f | n_{f_i} | 0_f \rangle = \langle 0_f | a_i^\dagger a_i | 0_f \rangle = 0$$

- ... described by modes g_i,

$$\langle 0_f | n_{g_i} | 0_f \rangle = \langle 0_f | b_i^\dagger b_i | 0_f \rangle = \sum_j |\beta_{ij}|^2$$

- Disagreeing on vacuum!

What is going on?

Flat spacetime:

- Timelike Killing vector ∂_t
- Positive frequency modes,

$$\partial_t f_i = -i\omega f_i, \quad \omega > 0$$

- Inertial observers agree on vacuum and number operator, so

$$g_i = \sum_j \alpha_{ij} f_j$$

and

$$\langle 0_f | a_{\mathbf{k}}^\dagger a_{\mathbf{k}} | 0_f \rangle = \langle 0_f | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | 0_f \rangle = 0$$

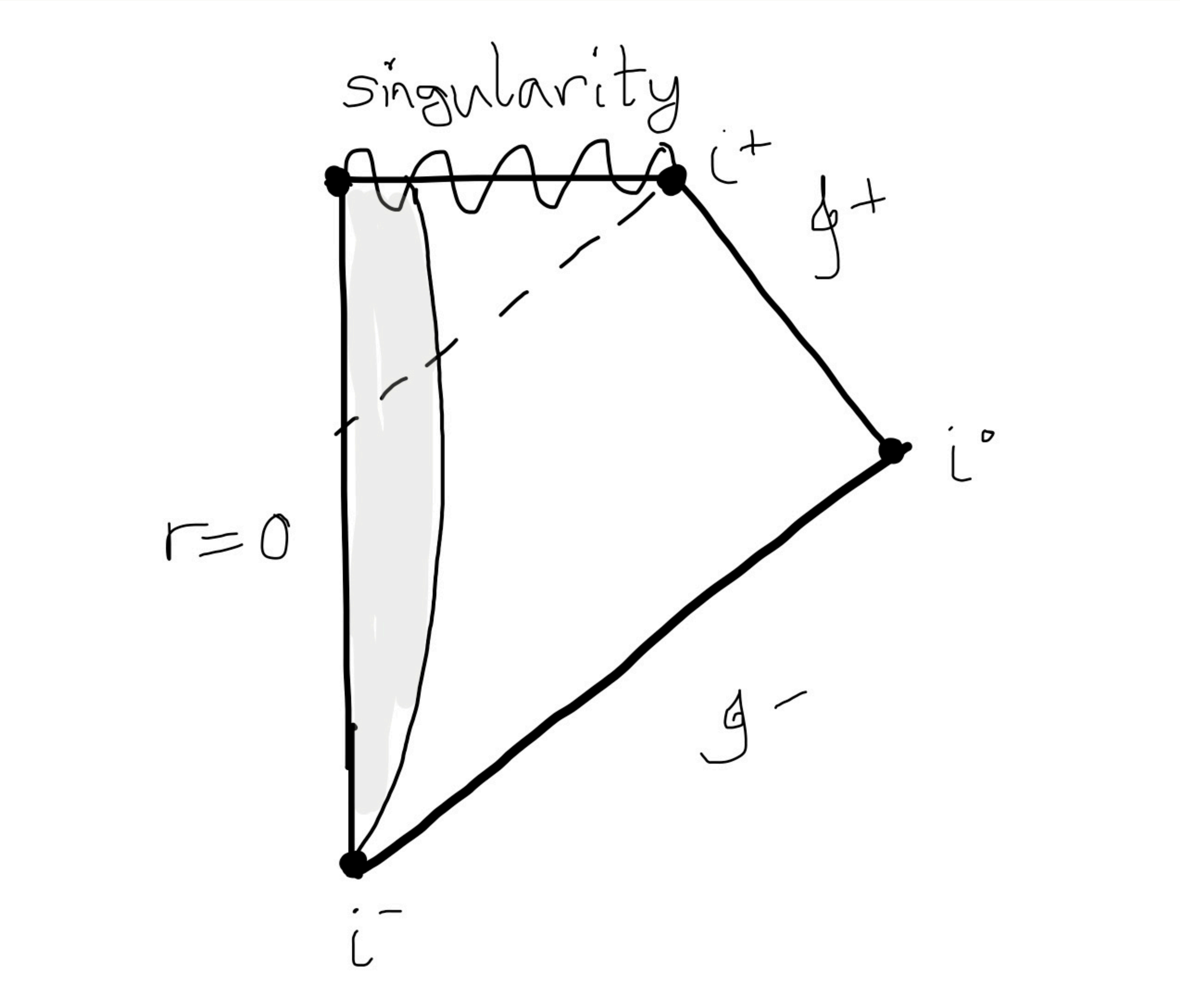
Curved spacetime:

- Poincaré group not a symmetry
- Generally no Killing vector to define positive frequency modes

Concept of particles is not universal

Particle Creation from Collapsing Stars

Penrose diagram of collapsing star



Creation of Particles

Solving

$$\Box\phi = 0$$

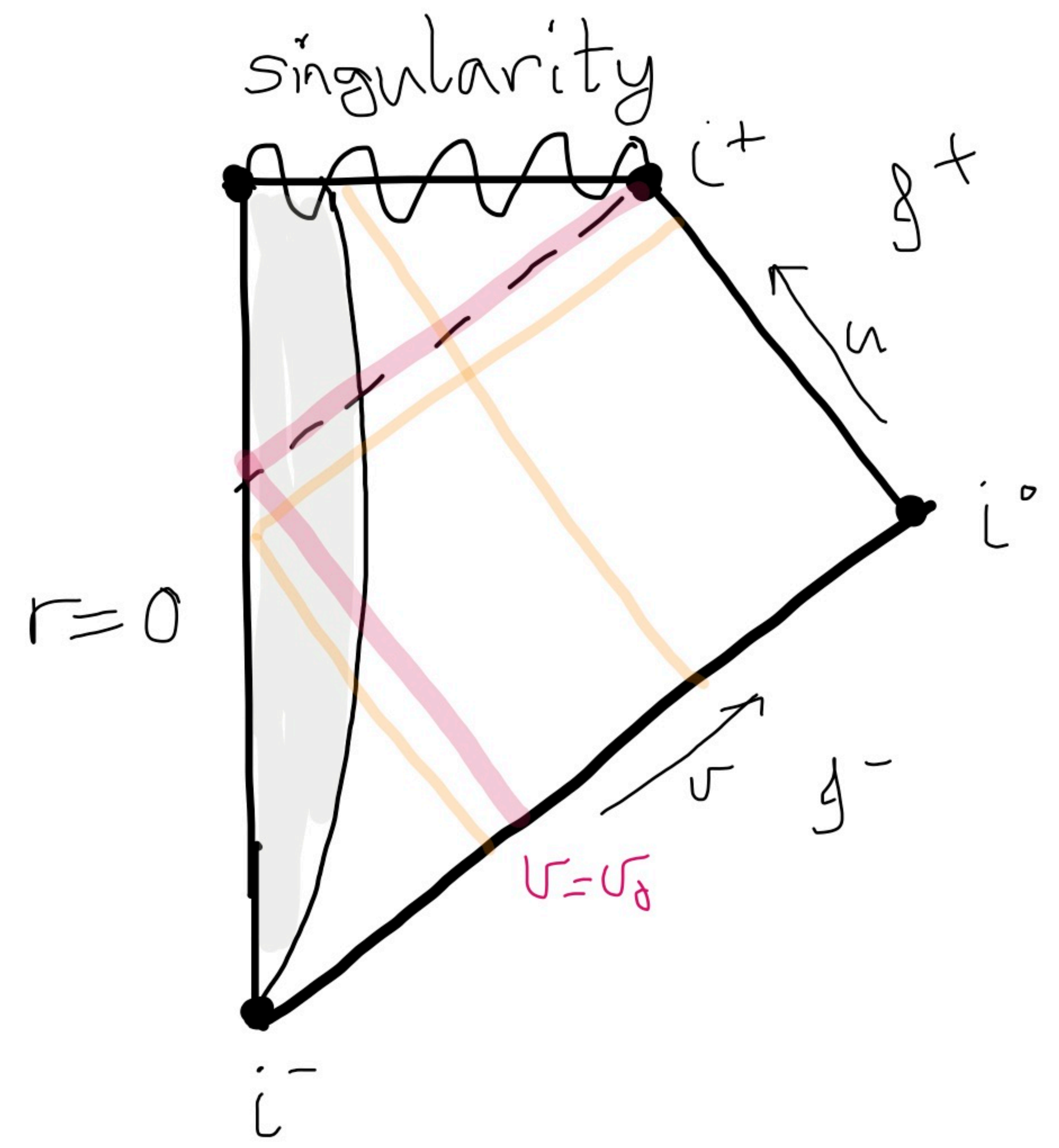
in Schwarzschild spacetime

- On past null infinity,

$$f_\omega \sim \frac{1}{r\sqrt{\omega}} Y(\theta,\phi) e^{-i\omega v}, \quad v = t + r$$

- Field expansion

$$\phi = \int d\omega \left(a_\omega f_\omega + a_\omega^\dagger f_\omega^* \right)$$



Creation of Particles

- On future null infinity,

$$p_\omega \sim \frac{1}{r\sqrt{\omega}} Y(\theta,\phi) e^{-i\omega u}, \quad u = t - r$$

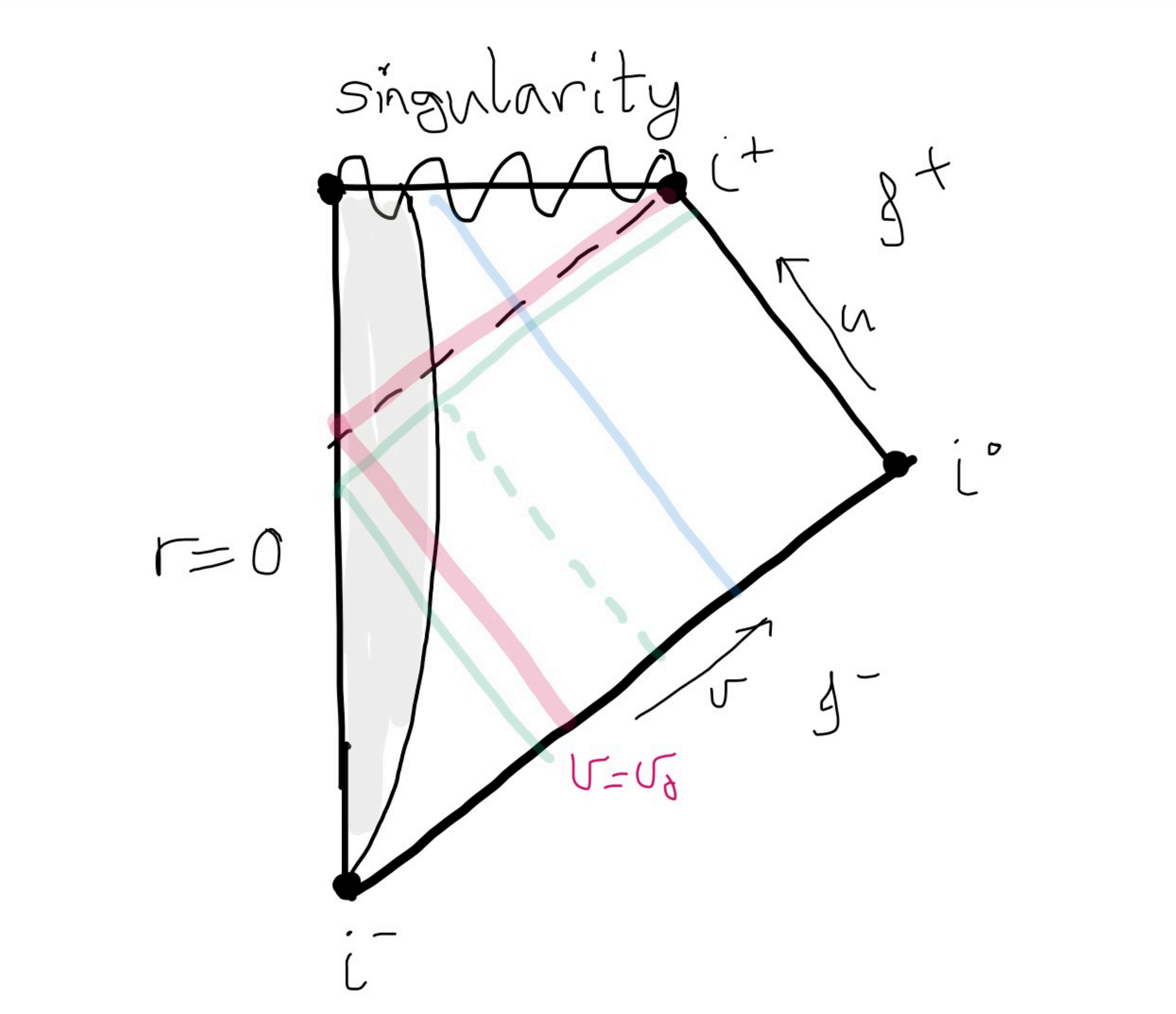
- Field expansion

$$\phi = \int d\omega \left(b_\omega p_\omega + b_\omega^\dagger p_\omega^* + c_\omega q_\omega + c_\omega^\dagger q_\omega^* \right)$$

with incoming modes q_ω on event horizon

- Bogolubov transformation yields

$$p_\omega = \int d\omega' \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^* \right)$$



Hawking Radiation

Black Holes

- Particle creation,

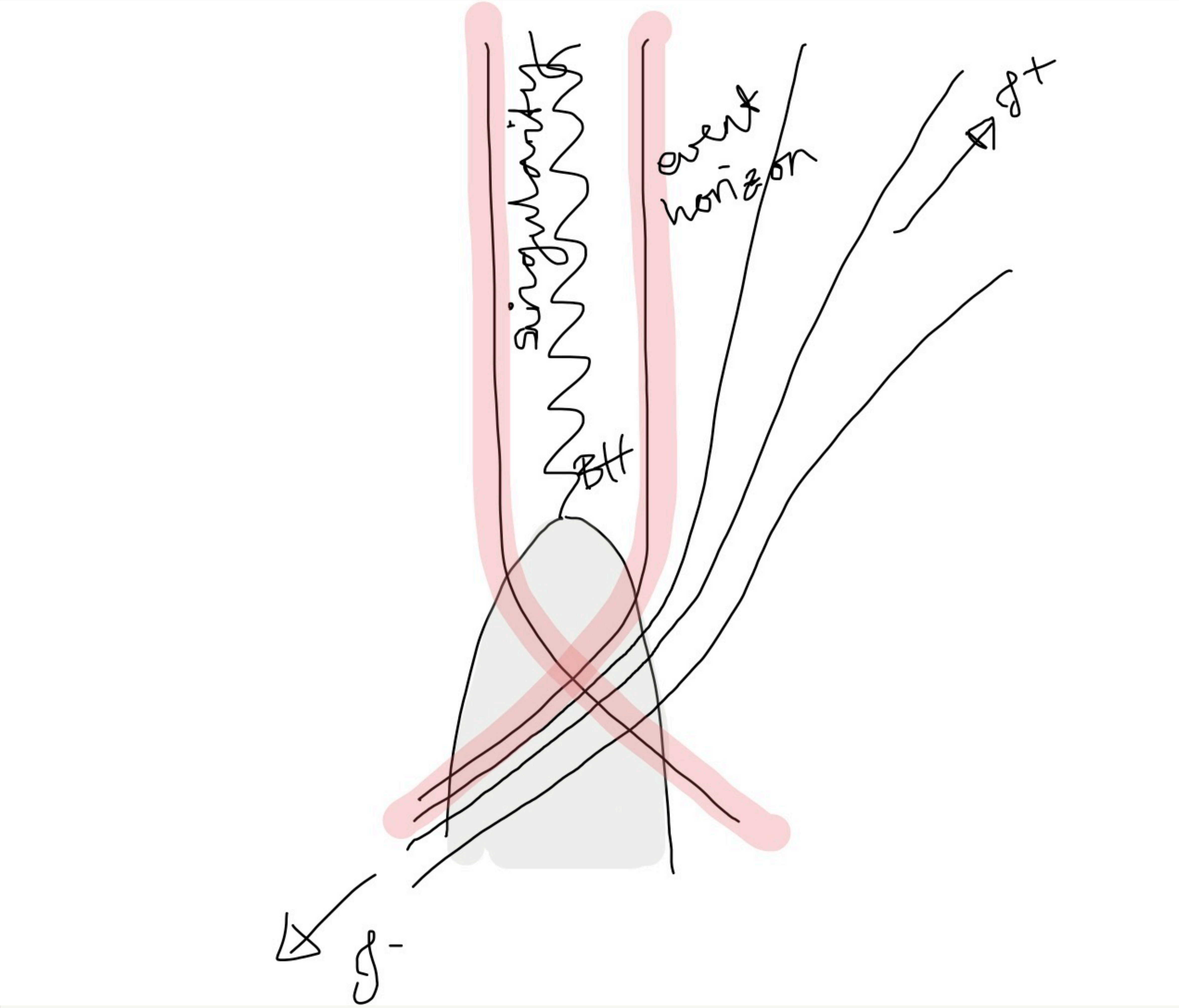
$$\langle 0_{\mathcal{I}^-} | b_\omega^\dagger b_\omega | 0_{\mathcal{I}^-} \rangle = \int d\omega' |\beta_{\omega\omega'}|^2$$

- Using

$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega)\,; \quad \beta_{\omega\omega'} = -(f^*_{\omega'}, p_\omega)$$

- Number of created particles,

$$\langle n_\omega \rangle = \frac{\Gamma(\omega)}{e^{2\pi\omega/\kappa} - 1}$$



Uniformly Accelerating Observers in Flat Spacetime

Rindler Space (2D)

- Hyperboloids describe uniform acceleration

$$x^2(\tau) - t^2(\tau) = \frac{1}{\alpha^2}$$

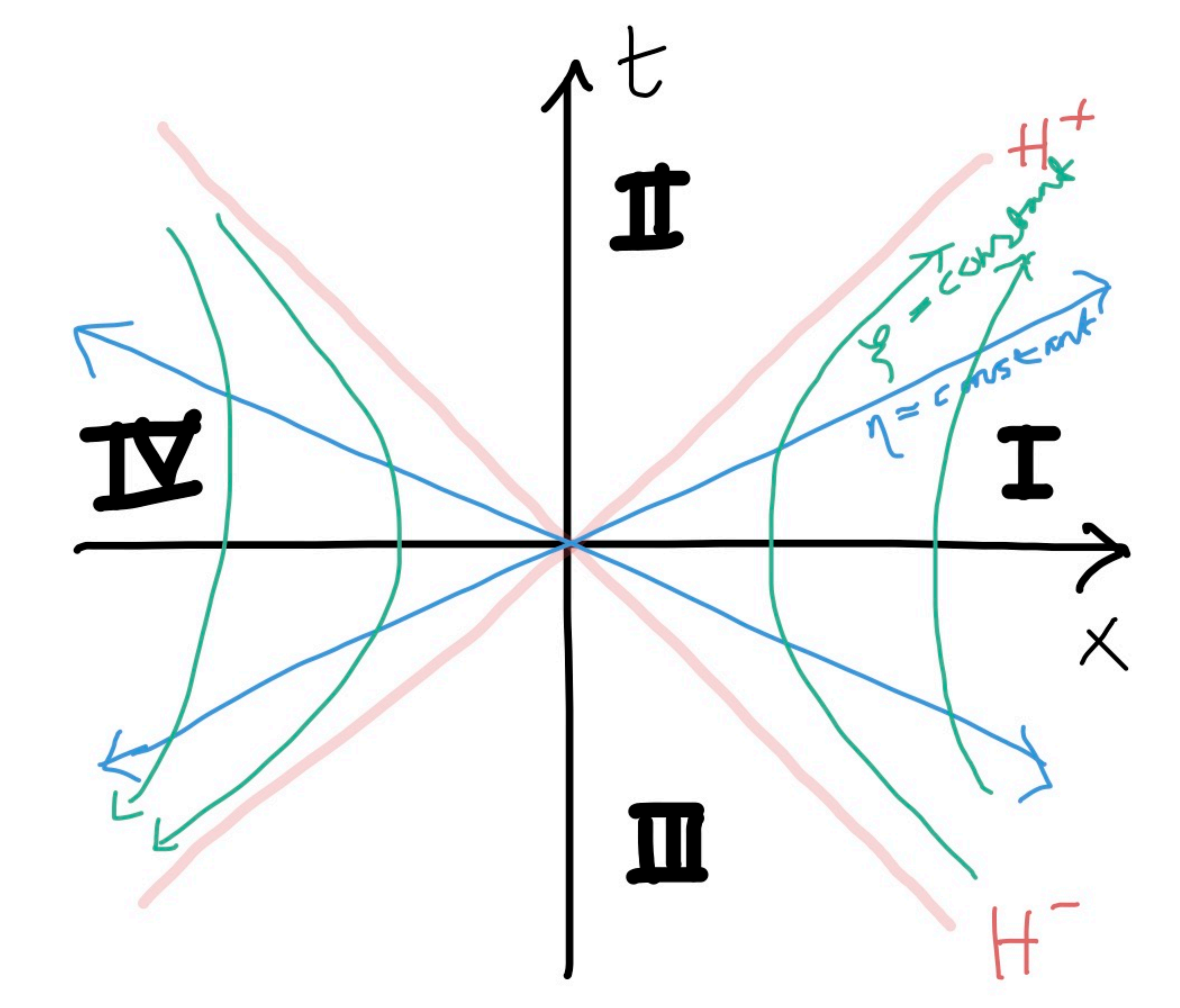
- Choose coordinates (η, ξ) such that

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta)$$

$$x = \frac{1}{a} e^{a\xi} \cosh(a\eta) \qquad (x > |t|)$$

- Then

$$ds^2 = e^{2\alpha\xi} \left(d\eta^2 - d\xi^2\right)$$



Unruh Effect

- Solve

$$\Box\phi = 0$$

  in regions I an IV

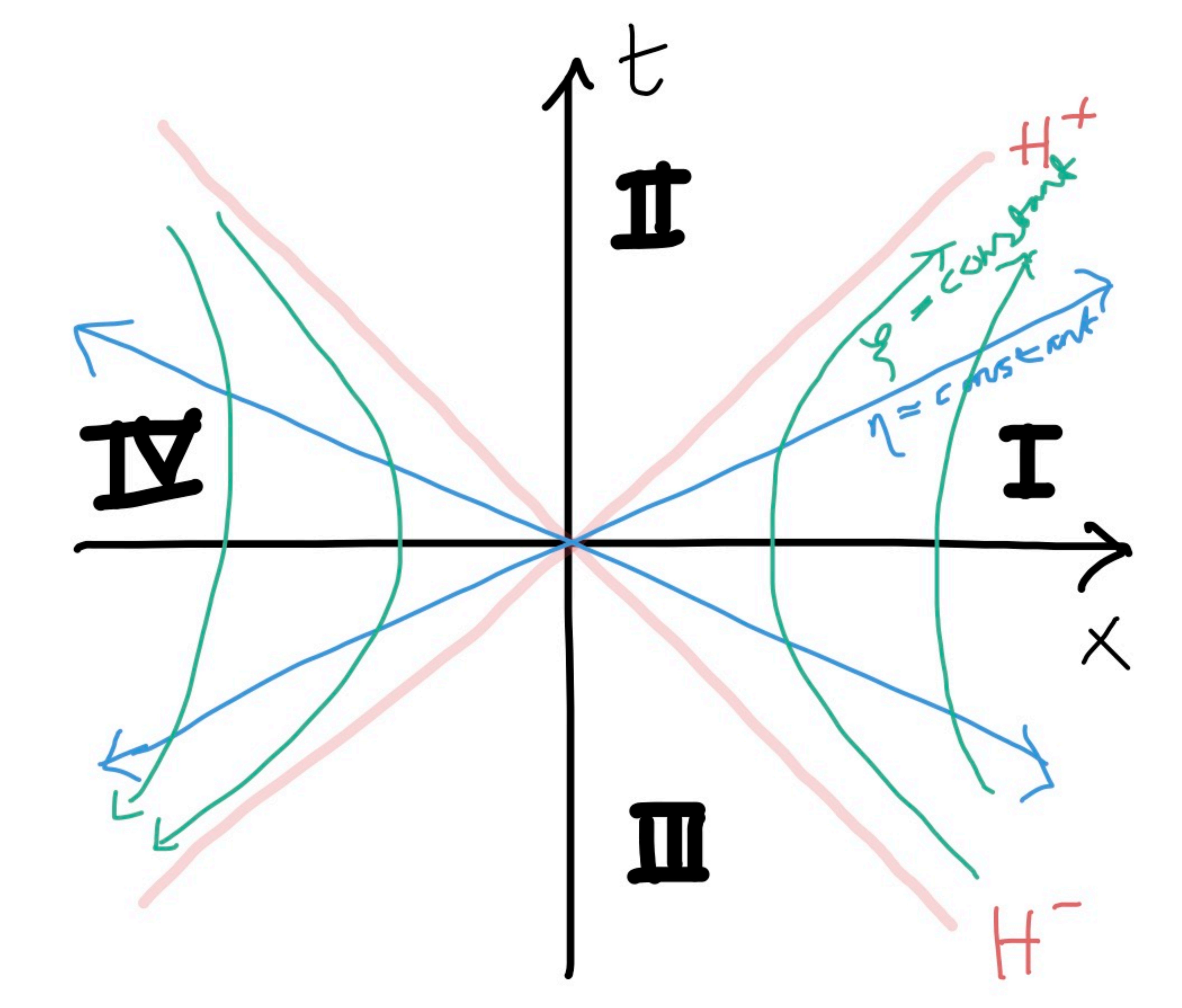
- Plane wave solutions,

$$g_k^{(1,2)} = \frac{1}{\sqrt{4\pi\omega}} e^{\mp i\omega\eta + ik\xi}, \quad \omega = |k| > 0$$

- Two equivalent field expansions

- Trick by Unruh:

$$h_k^{(1,2)} = \frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} \left(e^{\pi\omega/2a} g_k^{(1,2)} + e^{-\pi\omega/2a} g_{-k}^{(2,1)*} \right)$$



Result:

Thermal spectrum

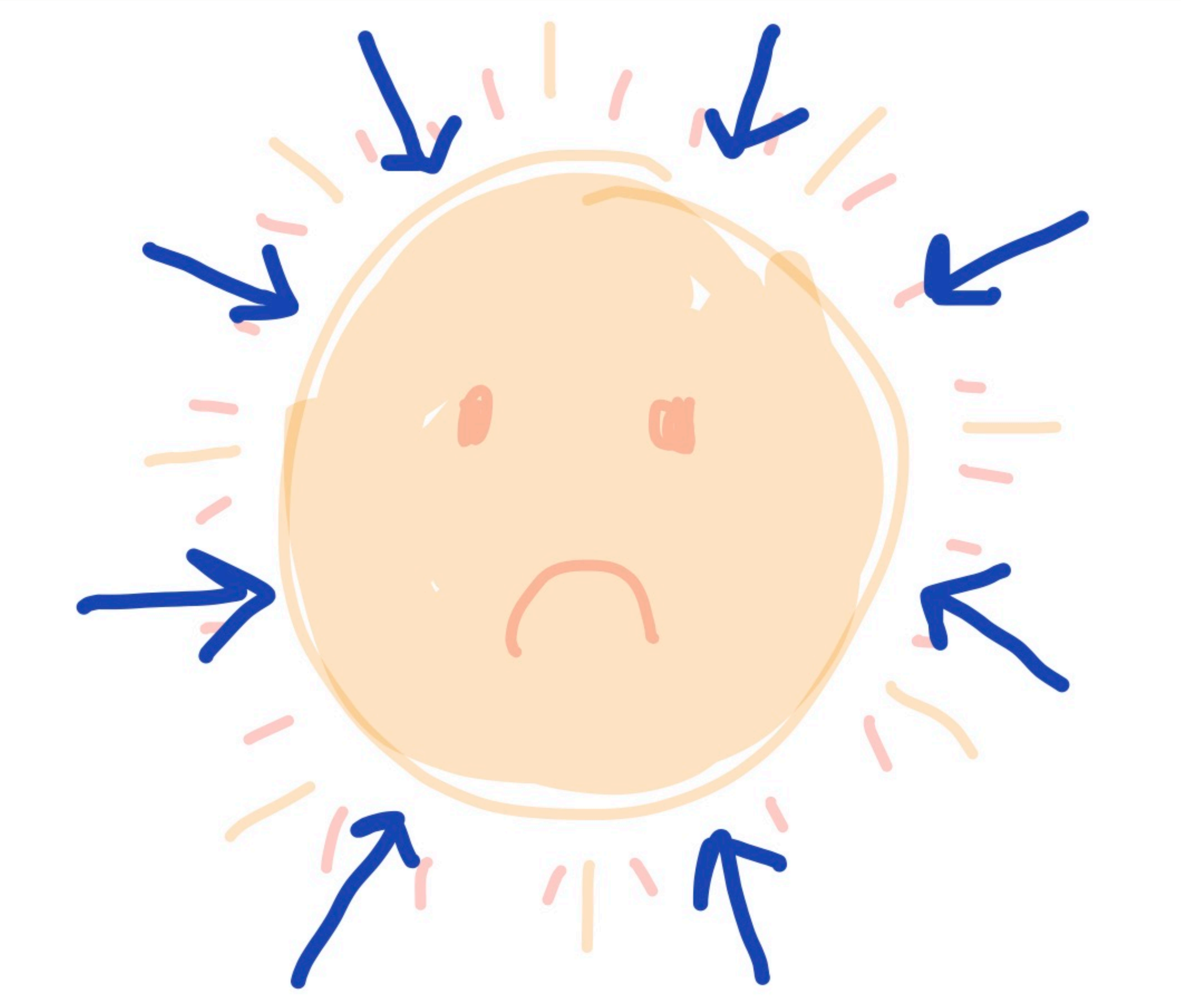
$$\langle 0_M | n_R | 0_M \rangle = \frac{1}{e^{2\pi\omega/a} - 1}\delta(0)$$

«puh»

Current Research

Fate of Collapsing Stars

- Black hole to white hole
- Temporarily trapped surface
- Baby universes
- Bounce



Backreaction of Hawking Radiation

Calculated by Mersini-Houghton for different vacuum states,

- Hartle-Hawking vacuum — Thermal bath of radiation
- Unruh vacuum — Flux of radiation

Result:

Bounce instead of collapse

Setup

- Assume star is a perfect fluid of dust (p=0),

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}$$

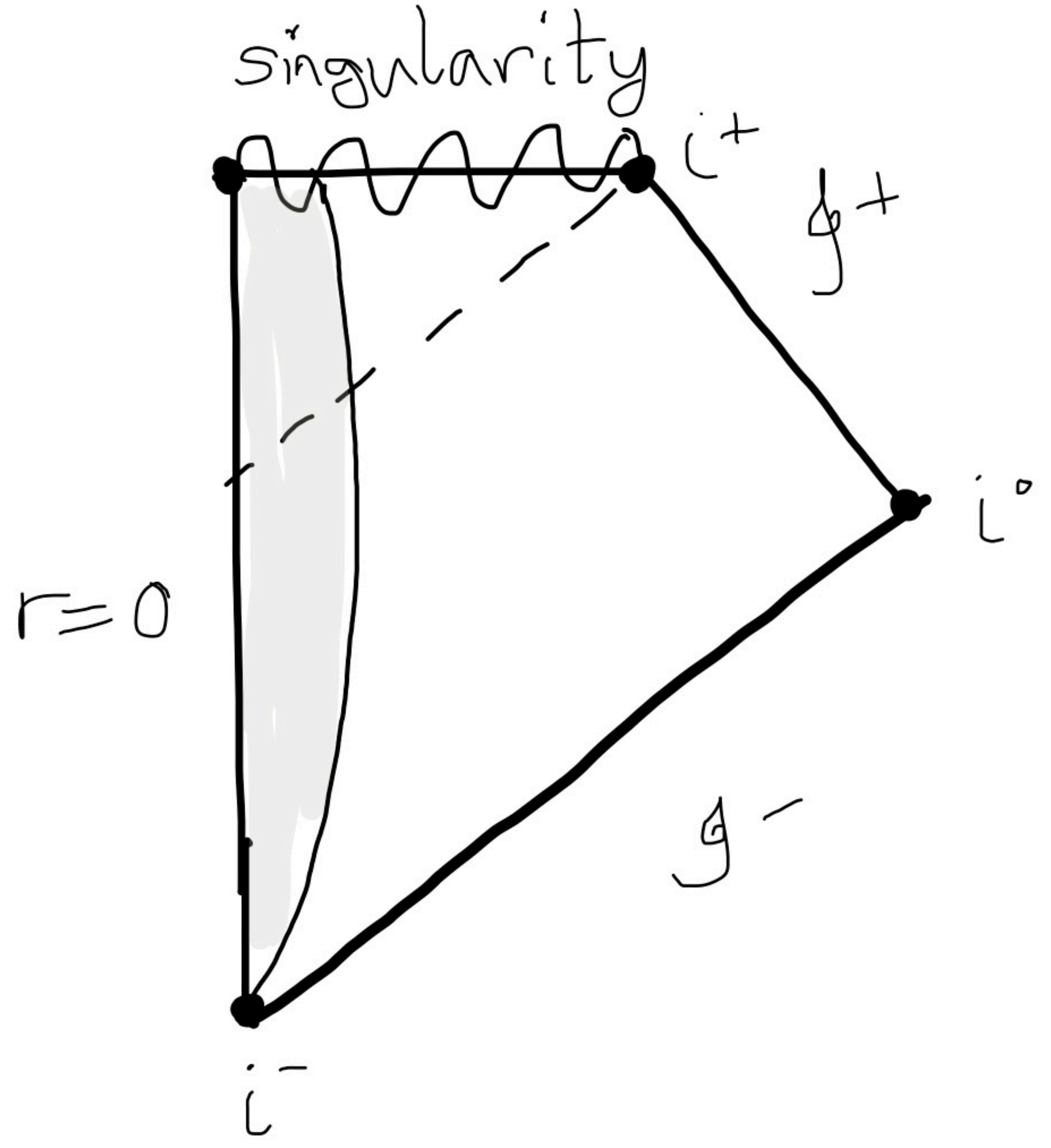
- Spherical symmetry, homogeneity and isotropy

- Exterior metric — Schwarzschild:

$$ds^2_{\text{ext}} = -\left(1 - \frac{2M}{R}\right) dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Interior metric — closed FRW universe:

$$ds^2_{\text{int}} = -dt^2 + a(t)^2 \left((1 - \kappa r^2)^{-1} dr^2 + r^2 d\Omega^2\right)$$



Include Backreaction in Collapse

- Use Hartle-Hawking vacuum
- Assume energy of Hawking radiation in interior is negative and equal in magnitude to exterior («mirror»)
- Replace energy density by

$$\rho \to \rho_0 - |\rho_{\mathrm{rad}}|$$

- Then G_{00}-component of Einsteins equations in interior is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{a^2} + \frac{\rho_0}{a^3} - \frac{|\rho_{\mathrm{rad}}|}{a^4}$$

Star collapses to finite radius

- From Friedmann equation one obtains

$$a(\eta) = \frac{\rho_0}{2} - \frac{1}{2}\sqrt{\left(\rho_0^2 - 4|\rho_{\rm rad}(\eta)|\sin^2(\eta)\right)}$$

- Solving

$$\dot{a}(t) = a'(\eta) = 0 \quad \Rightarrow \quad a_{\rm min}(\eta) \simeq \frac{\rho_{\rm rad}}{\rho_0}$$

yields minimal radius in finite time,

$$R_S^{\rm bounce} = a_{\rm min}(\eta)\sin(\chi_0) \simeq \frac{\rho_{\rm rad}}{\rho_0}\sin(\chi_0)$$

before horizon forms

Summary

Summary

- In curved spacetime concept of particles becomes ill-defined
- Particles are created by changing gravitational fields (inflation, Hawking radiation)
- Particles are created by uniformly accelerated observers (Unruh effect)
- Collapsing stars may not form black holes due to backreaction of Hawking radiation

Thank you!

References

[1] L. Mersini-Houghton, *Backreaction of Hawking radiation on a Gravitationally Collapsing Star I: Black Holes?,* arXiv:1406.1525 [hep-th], June 2014

[2] L. Mersini-Houghton, *Back-reaction of the Hawking radiation flux in Unruh's vacuum on a gravitationally collapsing star II*, arXiv:1409.1837 [hep-th], May 2015

[3] D. Malfarina, *Classical collapse to black holes and quantum bounces: A review*, arXiv:170304138 [gr-qc], May 2017

[4] S. W. Hawking, *Particle Creation by Black Holes*, Commun. math. Phys. 43, 199-220 (1975)

[5] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982

[6] L. Parker, D. Toms, *Quantum Field Theory in Curved Spacetime, Quantized Fields and Gravity*, Cambridge University Press, 2009

[7] S. M. Carroll, *An Introduction to General Relativity, Spacetime and Geometry*, Cambridge University Press, 2019