Conditional Deduction Under Uncertainty[⋆]

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Abstract. Conditional deduction in binary logic basically consists of deriving new statements from an existing set of statements and conditional rules. Modus Ponens, which is the classical example of a conditional deduction rule, expresses a conditional relationship between an antecedent and a consequent. A generalisation of Modus Ponens to probabilities in the form of probabilistic conditional inference is also well known. This paper describes a method for conditional deduction with beliefs which is a generalisation of probabilistic conditional inference and Modus Ponens. Meaningful conditional deduction requires a degree of relevance between the antecedent and the consequent, and this relevance can be explicitly expressed and measured with our method. Our belief representation has the advantage that it is possible to represent partial ignorance regarding the truth of statements, and is therefore suitable to model typical real life situations. Conditional deduction with beliefs thereby allows partial ignorance to be included in the analysis and deduction of statements and hypothesis.

1 Introduction

A conditional is for example a statement like *"If it rains, I will carry an umbrella"*, or *"If we continue releasing more* CO_2 *into the atmosphere, we will get global warming"*, which are of the form "IF x THEN y" where x marks the antecedent and y the consequent. An equivalent way of expressing conditionals is through the concept of implication, so that *"If it rains, I will carry an umbrella"* is equivalent to *"The fact that it rains implies that I carry an umbrella"*. The statement *"It rains"* is here the antecedent, whereas *"I carry an umbrella"* is the consequent. The conditional is the statement that relates the antecedent and the consequent in a conditional fashion.

Consequents and antecedents are simple statements that in case of binary logic can be evaluated to TRUE or FALSE, in case of probability calculus be given a probability, or in case of belief calculus [7] be assigned belief values.

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Conditionals are complex statements that can be assigned binary truth, probability and belief values in the same way as for simple statements. The binary logic interpretation of conditional deduction is the Modus Ponens connector, meaning that a TRUE antecedent and a TRUE conditional necessarily will produce a TRUE consequent by deduction. Modus Ponens says nothing about the case when either the antecedent or the conditional, or both are false.

When assigning probabilities or beliefs to a conditional, the deduction process produces a probability or belief value that can be assigned to the consequent. In this case, the deduction can give a meaningful result even when the antecedent and conditional are not TRUE in a binary logic sense. The details of how this is done are described in Sec.3.

Because conditionals are not always true or relevant, it is common to hear utterings like: *"I don't usually carry an umbrella, even when it rains"* which is contradicting the truth of the first conditional expressed above, or like: *"Even if we stop releasing more* CO_2 *into the atmosphere we will still have global warming"* which says that the second conditional expressed above is irrelevant. This can be nicely expressed with conditional beliefs, as described in Sec.3.

A conditional inference operator for beliefs that in special circumstances produced too high levels of uncertainty in the consequent belief, was presented in [8]. In the present paper we describe a new operator called *conditional deduction* that produces consequent beliefs with appropriate levels of uncertainty.

The advantage of the belief representation is that it can be used to model situations where the truth or probability values of the antecedent, the consequent and the conditionals are uncertain. Notice that probability and binary logic representations are special cases of, and therefore compatible with, our belief representation.

Sec.2 details our representation of uncertain beliefs. Sec.3 describes the conditional deduction operator, and Sec.4 describes an example of how the conditional deduction operator can be applied. Sec.5 provides a brief discussion on the theory of conditionals in standard logic and probability calculus. Sec.6 summarises the contribution of this paper.

2 Representing Uncertain Beliefs

This paper uses the bipolar belief representation called *opinion* [7], characterised by the use of separate variables pertaining to a given statement, and that bear some relationship to each other. In general, bipolarity in reasoning refers to the existence of positive and negative information to support an argument or the truth of a statement [1, 4].

In simplified terms, an opinion contains a variable representing the degree of belief that a statement is true, and a variable representing the degree of disbelief that the statement is true (i.e. the belief that the statement is false). The belief and disbelief values do not necessarily add up to 1, and the remaining belief mass is attributed to uncertainty. This representation can also be mapped to beta PDFs (probability density functions) [7], which allows logic operators to be applied to beta PDFs. Subjective logic is a reasoning framework that uses the opinion representation and a set of logical connectors.

The bipolar belief representation in subjective logic is based on classic belief theory[12], where the *frame of discernment* Θ defines an exhaustive set of mutually exclusive atomic states. The power set 2^{Θ} is the set of all subsets of Θ.

A belief mass assignment[3] (called BMA hereafter) is a function m_{Θ} mapping 2^{Θ} to $[0,1]$ (the real numbers between 0 and 1, inclusive) such that $\sum_{x \in 2^{\Theta}} m_{\Theta}(x) = 1$.

The BMA distributes a total belief mass of 1 amongst the subsets of Θ such that the belief mass for each subset is positive or zero. Each subset $x \subseteq \Theta$ such that $m_{\Theta}(x) > 0$ is called a focal element of m_{Θ}. In the case of total ignorance, $m_{\Theta}(\Theta) = 1$ and $m_{\Theta}(x) = 0$ for any proper subset x of Θ, and we speak about m_{Θ} being a *vacuous belief function*. If all the focal elements are atoms (i.e. one-element subsets of Θ) then we speak about *Bayesian belief functions*. A *dogmatic belief function* is defined by Smets[13] as a belief function for which $m_{\Theta}(\Theta) = 0$. Let us note that, trivially, every Bayesian belief function is dogmatic.

We are interested in expressing bipolar beliefs with respect to binary frames of discernment. In case Θ is larger than binary, this requires coarsening the original frame of discernment Θ to a binary frame of discernment. Let $x \in 2^{\Theta}$ be the element of interest for the coarsening and let $\overline{x}$ be the complement of x in Θ, then we can construct the binary frame of discernment $X = \{x, \overline{x}\}$. The coarsened belief mass assignment on 2^X can consist of maximum three belief masses, namely $m_X(x)$, $m_X(\overline{x})$ and $m_X(X)$, which we will denote by b_x, d_x and u_x because they represent *belief*, *disbelief* and *uncertainty* relative to x respectively. The *base rate*[4] of x can be determined by the relative size of the state x in the state space Θ, as defined by $a_x = \frac{|x|}{|\Theta|}$, or it can be determined on a subjective basis when no specific state space size information is known.

Coarsened belief masses can be computed e.g. with simple, normal or smooth coarsening as defined in [8, 10].

All the coarsenings have the property that b_x, d_x, u_x and a_x fall in the closed interval $[0,1]$, and that

$$b_x + d_x + u_x = 1 \ . \tag{1}$$

The expected probability of x is determined by: $\mathrm{E}(\omega_x) = \mathrm{E}(x) = b_x + a_x u_x$.

The ordered quadruple $\omega_x = (b_x,\ d_x,\ u_x,\ a_x)$, called the opinion about x, represents a bipolar belief function about x because it expresses positive belief in the form of b_x and negative belief in the form of d_x that are related by Eq.(1).

Although the coarsened frame of discernment X is binary, an opinion about $x \subset X$ carries information about the state space size of the original frame of discernment Θ through the base rate parameter a_x.

The base rate determines the probability expectation value when $u_x = 1$. In the absence of uncertainty, i.e. when $u_x = 0$, the base rate has no influence on the probability expectation value.

The opinion space can be mapped into the interior of an equal-sided triangle, where, for an opinion $\omega_x = (b_x, d_x, u_x, a_x)$, any two of the three parameters b_x, d_x and u_x determine the position of the point in the triangle representing the opinion.

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[3] Called *basic probability assignment* in [12].

[4] Called relative atomicity in [7, 8].

Fig.1 illustrates an example where the opinion about a proposition x from a binary frame of discernment has the value $\omega_x = (0.7,\ 0.1,\ 0.2,\ 0.5)$.

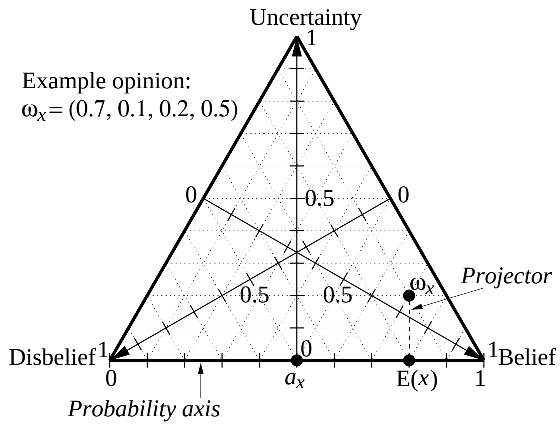

Fig. 1. Opinion triangle with example opinion

The top vertex of the triangle represents uncertainty, the bottom left vertex represents disbelief, and the bottom right vertex represents belief. The base line between the disbelief and belief vertices is the *probability axis*. The value of the base rate is indicated as a point on the probability axis.

Opinions on the probability axis have zero uncertainty and are equivalent to traditional probabilities. The distance from the probability axis to the opinion point can be interpreted as the uncertainty about the probability expectation value $\mathrm{E}(x)$.

The *projector* is defined as the line going through the opinion point parallel to the line that joins the uncertainty vertex and the base rate point. The point at which the projector meets the probability axis determines the probability expectation value of the opinion, i.e. it coincides with the point corresponding to expectation value $\mathrm{E}(x) = b_x + a_x u_x$.

Various visualisations of bipolar beliefs in the form of opinions are possible to facilitate human interpretation. For this, see http://security.dstc.edu.au/spectrum/beliefvisual/.

The next section describes a method for conditional deduction that takes bipolar beliefs in the form of opinions as input.

3 Conditional Deduction

A limitation of conditional propositions like 'IF x THEN y' is that when the antecedent is false it is impossible to assert the truth value of the consequent.

What is needed is a complementary conditional that covers the case when the antecedent is false. One that is suitable in general is the conditional 'IF NOT x THEN y'.

With this conditional it is now possible to determine the truth value of the consequent y in case the antecedent x is false.

Each conditional now provides a part of the picture and can therefore be called sub-conditionals. Together these sub-conditionals form a *complete conditional expression* that provides a complete description of the connection between the antecedent and the consequent. Complete conditional expressions have a two-dimensional truth value because they consist of two sub-conditionals that both have their own truth value.

We adopt the notation $y|x$ to express the sub-conditional 'IF x THEN y', (this in accordance with Stalnaker's [14] assumption that the probability of the proposition x implies y is equal to the probability of y given x) and $y|\overline{x}$ to express the sub-conditional 'IF NOT x THEN y' and assume that it is meaningful to assign opinions (including probabilities) to these sub-conditionals. We also assume that the belief in the truth of the antecedent x and the consequent y can be expressed as opinions. The conditional inference with probabilities, which can be found in many text books, is described below.

Definition 1 (Probabilistic Conditional Inference).

Let x and y be two statements with arbitrary dependence, and let $\overline{x} = \text{NOT } x$. Let x, $\overline{x}$ and y be related through the conditional statements $y|x$ and $y|\overline{x}$, where x and $\overline{x}$ are antecedents and y is the consequent. Let $p(x)$, $p(y|x)$ and $p(y|\overline{x})$ be probability assessments of x, $y|x$ and $y|\overline{x}$ respectively. The probability $p(y\|x)$ defined by:

$$p(y\|x) = p(x)p(y|x) + p(\overline{x})p(y|\overline{x}) \qquad = p(x)p(y|x) + (1 - p(x))p(y|\overline{x}) \ . \quad (2)$$

is then the conditional probability of y as a function of the probabilities of the antecedent and the two sub-conditionals.

The purpose of the notation $y\|x$ is to indicate that the truth or probability of the statement y is determined by the antecedent together with both the positive and the negative conditionals. The notation $y\|x$ is thus only meaningful in a probabilistic sense, i.e. so that $p(y\|x)$ denotes the consequent probability. Below, this notation will also be used for beliefs, where $\omega_{y\|x}$ denotes the consequent belief.

It can easily be seen that this definition of probabilistic deduction is a generalisation of Modus Ponens. Let for example x be TRUE (i.e. $p(x) = 1$) and $x \rightarrow y$ be TRUE (i.e. $p(y|x) = 1$), then it can be deduced that y is TRUE (i.e. $p(y\|x) = 1$). In the case $p(x) = 1$, only the positive conditional counts, and in case $p(x) = 0$, only the negative conditional counts. In all other cases, both the positive and the negative conditionals are needed to to determine the probability of y.

Conditional deduction with bipolar beliefs will be defined next. It is a generalisation of probabilistic conditional inference with probabilities. The definition is different from that of the conditional inference operator defined in [8], and the difference is explained in Sec.4.

Definition 2 (Conditional Deduction with Bipolar Beliefs). *Let $\Theta_X = \{x, \overline{x}\}$ and $\Theta_Y = \{y, \overline{y}\}$ be two frames of discernment with arbitrary mutual dependence. Let $\omega_x = (b_x, d_x, u_x, a_x)$, $\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_{y|x})$ and $\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_{y|\overline{x}})$ be an agent's respective opinions about x being true, about y being true given that x is* *true and about y being true given that x is false. Let $\omega_{y\|x} = (b_{y\|x}, d_{y\|x}, u_{y\|x}, a_{y\|x})$ be the opinion about y such that:*

$$\omega_{y\|x} \quad \textit{is defined by:} \quad \begin{cases} b_{y\|x} = b^{\mathrm{I}}_{y} - a_y K \\ d_{y\|x} = d^{\mathrm{I}}_{y} - (1-a_y)K \\ u_{y\|x} = u^{\mathrm{I}}_{y} + K \\ a_{y\|x} = a_y \ . \end{cases}$$

$$\textit{where} \begin{cases} b^{\mathrm{I}}_{y} = b_x b_{y|x} + d_x b_{y|\overline{x}} + u_x(b_{y|x}a_x + b_{y|\overline{x}}(1-a_x)) \\ d^{\mathrm{I}}_{y} = b_x d_{y|x} + d_x d_{y|\overline{x}} + u_x(d_{y|x}a_x + d_{y|\overline{x}}(1-a_x)) \\ u^{\mathrm{I}}_{y} = b_x u_{y|x} + d_x u_{y|\overline{x}} + u_x(u_{y|x}a_x + u_{y|\overline{x}}(1-a_x)) \end{cases}$$

and K can be determined according to the following selection criteria:

Case I: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}})) \vee ((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$
$\Longrightarrow K = 0.$

Case II.A.1: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) \leq (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))$
$\wedge\ (\mathrm{E}(\omega_x) \leq a_x)$
$\Longrightarrow K = \frac{a_x u_x (b^{\mathrm{I}}_{y} - b_{y|\overline{x}})}{(b_x + a_x u_x) a_y}.$

Case II.A.2: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) \leq (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))$
$\wedge\ (\mathrm{E}(\omega_x) > a_x)$
$\Longrightarrow K = \frac{a_x u_x (d^{\mathrm{I}}_{y} - d_{y|x})(b_{y|x} - b_{y|\overline{x}})}{(d_x + (1-a_x) u_x) a_y (d_{y|\overline{x}} - d_{y|x})}.$

Case II.B.1: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) > (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))$
$\wedge\ (\mathrm{E}(\omega_x) \leq a_x)$
$\Longrightarrow K = \frac{(1-a_x) u_x (b^{\mathrm{I}}_{y} - b_{y|\overline{x}})(d_{y|\overline{x}} - d_{y|x})}{(b_x + a_x u_x)(1-a_y)(b_{y|x} - b_{y|\overline{x}})}.$

Case II.B.2: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) > (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))$
$\wedge\ (\mathrm{E}(\omega_x) > a_x)$
$\Longrightarrow K = \frac{(1-a_x) u_x (d^{\mathrm{I}}_{y} - d_{y|x})}{(d_x + (1-a_x) u_x)(1-a_y)}.$

Case III.A.1: $((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) \leq (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))$
$\wedge\ (\mathrm{E}(\omega_x) \leq a_x)$
$\Longrightarrow K = \frac{(1-a_x) u_x (d^{\mathrm{I}}_{y} - d_{y|\overline{x}})(b_{y|\overline{x}} - b_{y|x})}{(b_x + a_x u_x) a_y (d_{y|x} - d_{y|\overline{x}})}.$

Case III.A.2: $((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) \leq (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))$
$\wedge\ (\mathrm{E}(\omega_x) > a_x)$
$\Longrightarrow\ K = \frac{(1-a_x)u_x(b^{\mathrm{I}}_y - b_{y|x})}{(d_x+(1-a_x)u_x)a_y}$.

Case III.B.1: $((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) > (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))$
$\wedge\ (\mathrm{E}(\omega_x) \leq a_x)$
$\Longrightarrow\ K = \frac{a_x u_x(d^{\mathrm{I}}_y - d_{y|\overline{x}})}{(b_x+a_x u_x)(1-a_y)}$.

Case III.B.2: $((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))$
$\wedge\ (\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) > (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))$
$\wedge\ (\mathrm{E}(\omega_x) > a_x)$
$\Longrightarrow\ K = \frac{a_x u_x(b^{\mathrm{I}}_y - b_{y|x})(d_{y|x} - d_{y|\overline{x}})}{(d_x+(1-a_x)u_x)(1-a_y)(b_{y|\overline{x}} - b_{y|x})}$.

where $\mathrm{E}(\omega_{y|\mathrm{vac}(x)}) = b_{y|x}a_x + b_{y|\overline{x}}(1 - a_x) + a_y(u_{y|x}a_x + u_{y|\overline{x}}(1 - a_x))$
and $\mathrm{E}(\omega_x) = b_x + a_x u_x$.

Then $\omega_{y\|x}$ is called the conditionally deduced opinion of ω_x by $\omega_{y|x}$ and $\omega_{y|\overline{x}}$. The opinion $\omega_{y\|x}$ expresses the belief in y being true as a function of the beliefs in x and the two sub-conditionals $y|x$ and $y|\overline{x}$. The conditional deduction operator is a ternary operator, and by using the function symbol '$\circledcirc$' to designate this operator, we define $\omega_{y\|x} = \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}})$.

3.1 Justification

The expressions for conditional inference is relatively complex, and the best justification can be found in its geometrical interpretation.

The image space of the consequent opinion is a subtriangle where the two sub-conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ form the two bottom vertices. The third vertex of the sub-triangle is the consequent opinion resulting from a vacuous antecedent. This particular consequent opinion, denoted by $\omega_{y|\mathrm{vac}(x)}$, is determined by the base rates of x and y as well as the horizontal distance between the sub-conditionals. The antecedent opinion then determines the actual position of the consequent within that subtriangle.

For example, when the antecedent is believed to be TRUE, i.e. $\omega_x = (1, 0, 0, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y|x}$, when the antecedent is believed to be FALSE, i.e. $\omega_x = (0, 1, 0, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y|\overline{x}}$, and when the antecedent opinion is vacuous, i.e. $\omega_x = (0, 0, 1, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y|\mathrm{vac}(x)}$. For all other opinion values of the antecedent, the consequent opinion is determined by linear mapping from a point in the antecedent triangle to a point in the consequent subtriangle according to Def.2.

It can be noticed that when $\omega_{y|x} = \omega_{y|\overline{x}}$, the consequent subtriangle is reduced to a point, so that it is necessary that $\omega_{y\|x} = \omega_{y|x} = \omega_{y|\overline{x}} = \omega_{y|\mathrm{vac}(x)}$ in this case. This means that there is no relevance relationship between antecedent and consequent, as will be explained in Sec.5.

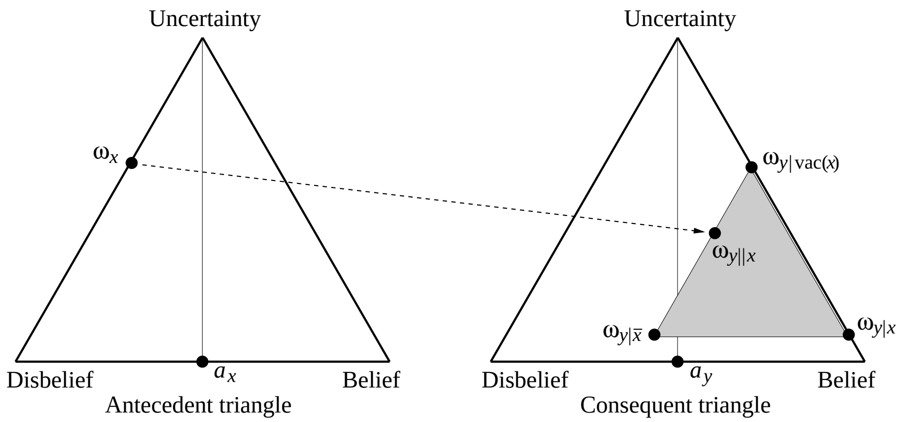

Fig. 2. Mapping from antecedent triangle to consequent subtriangle

Fig.2 illustrates an example of a consequent image defined by the subtriangle with vertices $\omega_{y|x} = (0.90,\ 0.02,\ 0.08,\ 0.50)$, $\omega_{y|\overline{x}} = (0.40,\ 0.52,\ 0.08,\ 0.50)$ and $\omega_{y|\mathrm{vac}(x)} = (0.40,\ 0.02,\ 0.58,\ 0.50)$.

Let for example the opinion about the antecedent be $\omega_x = (0.00,\ 0.38,\ 0.62,\ 0.50)$. The opinion of the consequent $\omega_{y\|x} = (0.40,\ 0.21,\ 0.39,\ 0.50)$ can then be obtained by mapping the position of the antecedent ω_x in the antecedent triangle onto a position that relatively seen has the same belief, disbelief and uncertainty components in the subtriangle (shaded area) of the consequent triangle.

In the general case, the consequent image subtriangle is not equal sided as in this example. By setting base rates of x and y different from 0.5, and by defining subconditionals with different uncertainty, the consequent image subtriangle will be skewed, and it is even possible that the uncertainty of $\omega_{y|\mathrm{vac}(x)}$ is less that that of $\omega_{x|y}$ or $\omega_{x|\overline{y}}$.

4 Example

Let us divide the weather into 3 the exclusive types *"sunny"*, *"overcast"* and *"rainy"*, and assume that we are interested in knowing whether I carry umbrella when it rains. To define the conditionals, we need the beliefs in the statement y: *"I carry an umbrella"* in case the antecedent x: *"It rains"* is TRUE, as well in case it is FALSE. Let the opinion values of the antecedent and the two sub-conditionals, as well as their rough fuzzy verbal descriptions be defined as:

$$\begin{aligned}
\omega_{y|x} &= (0.72,\ 0.18,\ 0.10,\ 0.50) \ : \ \text{quite likely but somewhat uncertain,} \\
\omega_{y|\overline{x}} &= (0.13,\ 0.57,\ 0.30,\ 0.50) \ : \ \text{quite unlikely but rather uncertain,} \\
\omega_{x} &= (0.70,\ 0.00,\ 0.30,\ 0.33) \ : \ \text{quite likely but rather uncertain.}
\end{aligned} \tag{3}$$

The opinion about the consequent $y\|x$ can be deduced with the conditional deduction operator expressed by $\omega_{y\|x} = \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}})$. Case II.A.2 of Def.2 is invoked in this case. This produces:

$$\omega_{y\|x} = (0.54,\ 0.20,\ 0.26,\ 0.50) \ : \quad \text{somewhat likely but rather uncertain.} \qquad (4)$$

This example is visualised in Fig.3, where the dots represent the opinion values. The dot in the left triangle represents the opinion about the antecedent x. The middle triangle shows the conditionals, where the dot labelled "T" (TRUE) represents the opinion of $y|x$, and the dot labelled "F" (FALSE) represents the opinion of $y|\overline{x}$. The dot in the right hand triangle represents the opinion about the consequent $y\|x$.

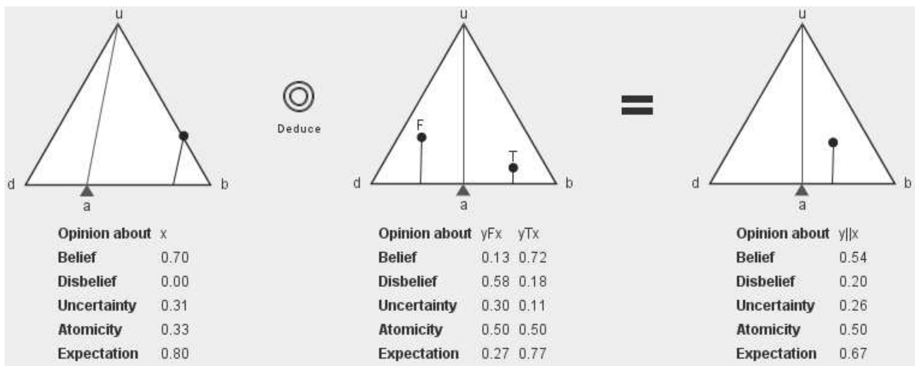

Fig. 3. Conditional deduction example

The consequent opinion value produced by the conditional deduction operator in this example contains slightly less uncertainty than the conditional inference operator defined in [8] would have produced. The simple conditional inference operator would typically produce too high uncertainty in case of state spaces different from $\frac{1}{2}$. More specifically, $\omega_{y|\text{vac}(x)}$ would not necessarily be a vertex in the consequent subtriangle in case of the simple conditional inference operator, whereas this is always the case for the deduction operator defined here. In the example of Fig.3, the antecedent state space was deliberately set to $\frac{1}{3}$ to illustrate that $\omega_{y|\text{vac}(x)}$ is the third vertex in the subtriangle. The conditional deduction operator defined here behaves well with any state space size, and it can be mentioned that $\omega_{y|\text{vac}(x)} = (0.13, 0.25, 0.62, 0.50)$, which is determined by Case II.A of Def.2 in this example.

The influence that the base rate has on the result increases as a function of the uncertainty. In the extreme case of a dogmatic antecedent opinion ($u_x = 0$), the base rate a_x has no influence on the result, and in the case of a vacuous antecedent opinion ($u_x = 1$), the consequent belief is fully conditioned by the base rate.

An online interactive demonstration of the conditional deduction operator can be accessed at http://security.dstc.edu.au/spectrum/trustengine/ . Fig.3 is a screen shot taken from that demonstrator.

5 Discussion

The idea of having a conditional connection between the antecedent and the consequent can be traced back to Ramsey [11] who articulated what has become known as Ramsey's Test: *To decide whether you believe a conditional, provisionally or hypothetically add the antecedent to your stock of beliefs, and consider whether to believe the consequent*. By introducing Ramsey's test there has been a switch from truth and truth-functions to belief and whether to believe which can also be expressed in terms of probability and conditional probability. This idea was articulated by Stalnaker [14] and expressed by the so-called Stalnaker's Hypothesis as: $p(\text{IF } x \text{ THEN } y) = p(y|x)$.

However, Lewis [9] argues that conditionals do not have truth-values and that they do not express propositions. In mathematical terms this means that given any propositions x and y, there is no proposition z for which $p(z) = p(y|x)$, so the conditional probability can not be the same as the probability of conditionals. Without going into detail we believe in Stalnaker's Hypothesis, and would argue against Lewis by simply saying that it is meaningful to assign a probability to a sub-conditional statement like "$y|x$", which is defined in case x is true, and undefined in case x is false.

A meaningful conditional deduction requires that the antecedent is relevant to the consequent, or in other words that the consequent depends on the antecedent, as explicitly expressed in relevance logics [5]. Conditionals that are based on the dependence between consequent and antecedent are considered to be universally valid (and not truth functional), and are called *logical conditionals* [3]. Deduction with logical conditionals reflect human intuitive conditional reasoning, and do not lead to any of the paradoxes of material implication.

Material implication, defined as $(x \rightarrow y) = (\overline{x} \vee y)$, is counterintuitive and riddled with paradoxes. Material implication, which is purely truth functional, ignores any relevance connection between antecedent x and the consequent y, and attempts to determine the truth value of the conditional as a function of the truth values of the antecedent and consequent alone. Material implication does not lend itself to any meaningful interpretation, and should never have been introduced into the theory of logic in the first place.

We will now show that it is possible to express the relevance between the antecedent and the consequent as a function of the conditionals.

For probabilistic conditional deduction, the relevance denoted as $\mathrm{R}(x, y)$ can be defined as:

$$\mathrm{R}(x, y) = |p(y|x) - p(y|\overline{x})| \ . \tag{5}$$

It can be seen that $\mathrm{R}(x, y) \in [0, 1]$, where $\mathrm{R}(x, y) = 0$ expresses total irrelevance/independence, and $\mathrm{R}(x, y) = 1$ expresses total relevance/dependence between x and y. For belief conditionals, the same type of relevance can be defined as:

$$\mathrm{R}(x, y) = |\mathrm{E}(\omega_{y|x}) - \mathrm{E}(\omega_{y|\overline{x}})| \ . \tag{6}$$

For belief conditionals, a second order *uncertainty relevance*, denoted as $\mathrm{R}_u(x, y)$, can be defined:

$$\mathrm{R}_u(x, y) = |u_{y|x} - u_{y|\overline{x}}| \ . \tag{7}$$

In case $\mathrm{R}(x, y) = 0$, there can thus still exist a relevance which can make conditional deduction meaningful regarding the certainty in the consequent belief.

In the example of Fig.3, the relevance $\mathrm{R}(x, y)$ is visualised as the horizontal distance between the probability expectations of the conditionals (i.e. where the projectors intersect with the base line) in the middle triangle. The uncertainty relevance $\mathrm{R}_u(x, y)$ is visualised as the vertical distance between the two dots representing the conditionals in the middle triangle.

Our approach to conditional deduction can be compared to that of conditional event algebras[6] where the set of events e.g. x, y in the probability space is augmented to include so-called class conditional events denoted by $y|x$. The primary objective in doing this is to define the conditional events in such a way that $p((y|x)) = p(y|x)$, that is so that the probability of the conditional event $y|x$ agrees with the conditional probability of y given x. There are a number of established conditional event algebras, each with their own advantages and disadvantages. In particular, one approach[2] used to construct them has been to employ a ternary truth system with values true, false and undefined, which corresponds well with the belief, disbelief and uncertainty components of bipolar beliefs.

Modus Ponens and probabilistic conditional inference are sub-cases of conditional deduction with bipolar beliefs. It can easily be seen that Def.2 collapses to Def.1 when the argument opinions are all dogmatic, i.e. when the opinions contain zero uncertainty. It can further be seen that Def.1 collapses to Modus Ponens when the arguments can only take probability values 0 or 1. It can also be seen that the probability expectation value of the deduced opinions of Def.2 is equal to the deduced probabilities of Def.1 when the input values are the probability expectation values of the original opinion arguments. This is formally expressed below:

$$\mathrm{E}(\omega_{y\|x}) = \mathrm{E}(\omega_x)\mathrm{E}(\omega_{y|x}) + \mathrm{E}(\omega_{\overline{x}})\mathrm{E}(\omega_{y|\overline{x}}) \; . \tag{8}$$

By using the mapping between opinions and beta PDFs described in [7], it is also possible to perform conditional deduction when antecedents and conditionals are expressed in the form of beta PDFs. It would be impossible to do conditional deduction with beta PDFs algebraically, although numerical methods are probably possible. This conditional deduction operator defined here is therefore an approximation to an ideal case. The advantages are simple expressions and fast computation.

6 Conclusion

The subjective logic operator for conditional deduction with beliefs described here represents a generalisation of the binary logic Modus Ponens rule and of probabilistic conditional inference. The advantage of our approach is that it is possible to perform conditional deduction under uncertainty and see the effect it has on the result. When considering that subjective logic opinions can be interpreted as probability density functions, this operator allows conditional deduction to be performed on conditionals and antecedents represented in the form of probability density functions. Purely analytical conditional inference with probability density functions would normally be too complex to be practical. Our approach can be seen as a good approximation in this regard, and provides a bridge between belief theory on the one hand, and binary logic and probability theory on the other.

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