INVERTING CONDITIONAL OPINIONS IN SUBJECTIVE LOGIC

Audun Jøsang [a] and Francesco Sambo [b]

(a) University of Oslo, Norway, josang@mn.uio.no
(b) University of Padova, Italy, francesco.sambo@dei.unipd.it

Abstract: *Subjective Logic has operators for conditional deduction and conditional abduction where subjective opinions are input arguments. With these operators traditional Bayesian reasoning can be generalised from taking only probabilistic arguments to also taking opinions as arguments, thereby allowing Bayesian modeling of situations affected by uncertainty and incomplete information. Conditional deduction is a Bayesian reasoning process that goes in the same direction as that of the input argument conditionals, whereas conditional abduction is a Bayesian reasoning process that goes in the direction opposite to that of the input argument conditionals. Conditional abduction is in fact a two-step process that first involves the computation of inverted conditionals, and then conditional deduction based on the inverted conditionals. This paper describes an improved generalized method for inverting opinion conditionals in order to support general Bayesian reasoning in subjective logic.*

Keywords: *Subjective logic, Conditional, Opinion, Belief, Abduction, Inversion.*

1 Introduction

Propositions can be used to express and represent states of the real world, hypothetically or practically. Our knowledge of the real world tells us that certain states are related in some way. For example, the state of rainy weather and the state of carrying an umbrella are often related, so it is meaningful to express this relationship in a sentence like *"If it rains, Bob carries an umbrella"* which is a conditional proposition in the form "IF x THEN y". Here, x denotes the parent proposition (aka. antecedent) and y the child proposition (aka. consequent). Formally, a logic conditional is typically expressed as $x \rightarrow y$.

In case of a causal conditional $x \rightarrow y$ it is assumed that the parent state x dynamically influences the child state y in time and space. Informally speaking, in case x and y initially are FALSE, and the parent x then becomes TRUE, then the child y subsequently becomes TRUE too [6].

A derivative conditional is the opposite of a causal conditional. This means that even in case of a TRUE derivative conditional, then forcing the parent proposition to become TRUE does not necessarily make the child proposition becom TRUE as well. For example, the conditional "IF *Bob carries an umbrella* THEN *it must be raining*" is a derivative conditional because forcing Bob to carry an umbrella (the parent) does not cause rain (the child).

Conditionals can also be neither causal nor derivative. For example in case two separate lamps are connected to the same electric switch, then observing one of the lamps being lit gives an indication of the other lamb being lit too, so there is clearly a conditional relationship between them. However neither lamp actually causes the other to light up, rather it is the flipping of the switch which causes both lamps to light up at the same time.

The degree of truth, or equivalently the validity of conditionals, can be expressed in different ways, e.g. as binary TRUE or FALSE, as a probability values, or as subjective opinions. Given that a conditional is a complex proposition, the parent and child propositions have their own truth values that can be different from that of the conditional at any one time.

Both binary logic and probability calculus have mechanisms for conditional reasoning. In binary logic, Modus Ponens (MP) and Modus Tollens (MT) are the classical operators for deductive and abductive reasoning respectively.

In probability calculus a binomial conditional is expressed as $p(y|x)$ where y denotes the child proposition and x the parent proposition. To be explicit, the notation is thus $p(child \mid parent)$, verbally expressed as: "the probability of *child* being TRUE given that *parent* is TRUE". Both the positive conditional $p(y|x)$ as well as the negative conditional $p(y|\bar{x})$ are required for binomial deduction expressed as:

$$p(y\|x) = p(x)p(y|x) + p(\bar{x})p(y|\bar{x}) \tag{1}$$

where the terms are interpreted as follows:

$p(y\|x)$	the deduced probability of the child y
$p(y|x)$	the conditional probability of y given x is TRUE
$p(y|\bar{x})$	the conditional probability of y given x is FALSE
$p(x)$	the probability of the parent x
$p(\bar{x})$	the complement probability of x $(= 1 - p(x))$

The term $y\|x$ denotes the (degree of) truth of the child proposition y deduced as a function of the (degree of) truth of the parent proposition x and of the conditionals. The expression $p(y\|x)$ thus represents the output result, whereas the conditional $p(y|x)$ and $p(y|\bar{x})$ represent input arguments, similarly to $p(x)$.

Probabilistic conditional reasoning is used extensively in areas where conclusions need to be derived from probabilistic input evidence, such as for making diagnoses from medical tests. A pharmaceutical company that develops a test for a particular infection will typically determine the reliability of the test by letting a group of infected and a group of non-infected people undergo the test. The result of these trials will then determine the reliability of the test in terms of its *sensitivity* $p(x|y)$ and *false positive rate* $p(x|\bar{y})$, where the propositions are expressed as x:*"Positive Test"*, y:*"Infected"* and $\bar{y}$:*"Not infected"*. The conditionals are interpreted as:

- $p(x|y)$: *"The probability of positive test given infection"*,
- $p(x|\bar{y})$: *"The probability of positive test without infection"*.

The problem with applying the measures for sensitivity and false positive rate in a practical setting is that they are causal conditionals whereas the practitioner needs derivative conditionals in order to apply the expression of Eq.(1). The derivative conditionals needed for making the diagnosis are:

- $p(y|x)$: *"The probability of infection given positive test"*,
- $p(y|\bar{x})$: *"The probability of infection given negative test"*.

These conditionals are derivative because a positive or negative test obviously does not cause the patient to be infected. Derivative conditionals are usually not directly available to the medical practitioner, but they can be obtained if the base rate, also called prior probability, of the infection is known.

The base rate fallacy [5] in medical reasoning consists of making the erroneous assumption that $p(y|x) = p(x|y)$. While this reasoning error often can produce a relatively good approximation of the correct diagnostic probability value, it can lead to a completely wrong result and wrong diagnosis in case the base rate of the infection in the population is very low and the reliability of the test is not perfect. The required conditionals can be correctly derived by inverting the available conditionals using Bayes rule. The inverted conditionals are obtained as follows:

$$\left\{ \begin{array}{l} p(x|y) = \frac{p(x\wedge y)}{p(y)} \\ \\ p(y|x) = \frac{p(x\wedge y)}{p(x)} \end{array} \right. \quad \Rightarrow \quad p(y|x) = \frac{p(y)p(x|y)}{p(x)} \ . \tag{2}$$

On the right hand side of Eq.(2) the base rate of the infection in the population is expressed by $p(y)$, but in the following, $a(x)$ and $a(y)$ will denote the base rates of x and y respectively. By applying Eq.(1) with x and y swapped in every term, the term $p(x)$ on the right hand side in Eq.(2) can be expressed as a function of the base rate $a(y)$ and its complement $a(\bar{y}) = 1 - a(y)$. The inverted positive conditional then becomes:

$$p(y|x) = \frac{a(y)p(x|y)}{a(y)p(x|y) + a(\bar{y})p(x|\bar{y})} \ . \tag{3}$$

A medical test result is typically considered positive or negative, so when applying Eq.(1) it can be assumed that either $p(x)$ = 1 (positive test) or $p(\bar{x})$ = 1 (negative test). In case the patient tests positive, Eq.(1) can be simplified to $p(y\|x) = p(y|x)$ so that Eq.(3) will give the correct likelihood that the patient actually is infected.

Generalisation of probabilistic deduction in Eq.(1) and its inversion in Eq.(3) is explained below.

Let $X = \{x_i | i = 1 \dots k\}$ be the parent frame, and let $Y = \{y_j | j = 1 \dots l\}$ be the child frame. The conditional relationship from X to Y is then expressed with $k = |X|$ vector conditionals $\vec{p}(Y|x_i)$, each having $l = |Y|$ dimensions. The vector conditional $\vec{p}(Y|x_i)$ relates each state x_i to the frame Y, which consists of the scalar conditionals:

$$p(y_j|x_i), \qquad \text{where } \sum_{j=1}^{l} p(y_j|x_i) = 1 \ . \tag{4}$$

Probabilistic deduction from X to Y is the vector $\vec{p}(Y\|X)$ over Y where each scalar vector element $p(y_j\|X)$ is:

$$p(y_j\|X) = \sum_{i=1}^{k} p(x_i)p(y_j|x_i) \ . \tag{5}$$

Assume the conditional relations from Y to X, expressed with the l different vector conditionals $p(X|y_j)$, each being of k dimensions, and where $a(y_j)$ represents the base rate of y_j. The multinomial probabilistic inverted conditionals are:

$$p(y_j|x_i) = \frac{a(y_j)p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t)p(x_i|y_t)} \ . \tag{6}$$

By substituting the conditionals of Eq.(5) with inverted multinomial conditionals from Eq.(6), the general expression for probabilistic abduction emerges:

$$p(y_j \widetilde{\|} X) = \sum_{i=1}^{k} p(x_i) \left(\frac{a(y_j)p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t)p(x_i|y_t)} \right) . \tag{7}$$

In this paper we describe a method for inverting conditional opinions in subjective logic as a generalisation of conditional inversion in probability calculus as expressed by Eq.(6). Inversion of conditional opinions provides the basis for derivative Bayesian reasoning in subjective logic. The next section describes subjective opinions. Sec.3 presents notation and Sec.4 describes our method for inverting conditionals expressed as subjective belief opinions.

2 Subjective Opinions

A subjective opinion expresses belief about states in a *frame of discernment* or *frame* for short, which in fact is equivalent to a traditional state space. In practice, a state in a frame can be considered to be a statement or proposition, so that a frame contains a set of statements. Let X be a frame of cardinality k. An opinion distributes belief mass over the reduced powerset of the frame denoted as $\mathcal{R}(X)$ defined as:

$$\mathcal{R}(X) = 2^X \setminus \{X, \emptyset\} , \tag{8}$$

where 2^X denotes the powerset of X. All proper subsets of X are elements of $\mathcal{R}(X)$, but not $\{X\}$ nor $\{\emptyset\}$.

An opinion is a composite function that consists of a belief vector $\vec{b}$, an uncertainty parameter u and base rate vector $\vec{a}$ that take values in the interval $[0,1]$ and that satisfy the following additivity constraints.

$$\text{Belief additivity: } u_X + \sum_{x_i \in \mathcal{R}(X)} \vec{b}_X(x_i) = 1, \text{ where } x \in \mathcal{R}(X). \tag{9}$$

$$\text{Base rate additivity: } \sum_{i=1}^{k} \vec{a}_X(x_i) = 1, \quad \text{where } x \in X. \tag{10}$$

Let $\mathcal{R}(X)$ denote the reduced powerset of frame X. Let $\vec{b}_X$ be a belief vector over the elements of $\mathcal{R}(X)$, let u_X be the complementary uncertainty mass, and let $\vec{a}$ be a base rate vector over X, as seen by subject A. The composite function of Eq.(11) is then A's subjective opinion over X.

$$\text{Subjective opinion: } \quad \omega_X^A = (\vec{b}_X, u_X, \vec{a}_X) \tag{11}$$

The belief vector $\vec{b}_X$ has $(2^k - 2)$ parameters, whereas the base rate vector $\vec{a}_X$ only has k parameters. The uncertainty parameter u_X is a simple scalar. A general opinion thus contains $(2^k + k - 1)$ parameters. However, given that Eq.(9) and Eq.(10) remove one degree of freedom each, opinions over a frame of cardinality k only have $(2^k + k - 3)$ degrees of freedom. The probability projection of opinions is the vector denoted as $\vec{\mathrm{E}}_X$ in Eq.(12).

$$\vec{\mathrm{E}}_X(x_i) = \sum_{x_j \in \mathcal{R}(X)} \vec{a}_X(x_i/x_j)\, \vec{b}_X(x_j) \; + \; \vec{a}_X(x_i)\, u_X \;, \quad \forall\, x_i \in \mathcal{R}(X) \;, \tag{12}$$

where $\vec{a}_X(x_i/x_j)$ denotes relative base rate, i.e. the base rate of subset x_i relative to the base rate of (partially) overlapping subset x_j. The relative base rate is expressed as:

$$\vec{a}_X(x_i/x_j) = \frac{\vec{a}_X(x_i \cap x_j)}{\vec{a}_X(x_j)} \;, \quad \forall\, x_i, x_j \subset X \;. \tag{13}$$

Binomial opinions apply to binary frames, whereas multinomial opinions apply to arbitrary frames where belief mass is only assigned to singletons. General opinions, also called hyper opinions, also apply to arbitrary frames where belief mass can be assigned to any subset of the frame. Hyper opinion, multinomial opinions and binomial opinions are generalisations of each other

A binomial opinion is equivalent to a Beta pdf (probability density function), a multinomial opinion to a Dirichlet pdf, and a hyper opinion to a hyper-Dirichlet pdf [3]. There is no simple visualisation of hyper opinions, but visualisations for binomial and multinomial opinions are illustrated in Fig. 1 below.

Binomial opinions apply to binary frames and have a special notation as described below. Let $X = \{x, \bar{x}\}$ be a binary frame, then a binomial opinion about the truth of state x is the ordered quadruple $\omega_x = (b, d, u, a)$ where:

b,	*belief*:	belief mass in support of x being true,
d,	*disbelief*:	belief mass in support of $\bar{x}$ (NOT x),
u,	*uncertainty*:	uncertainty about probability of x,
a,	*base rate*:	non-informative prior probability of x.

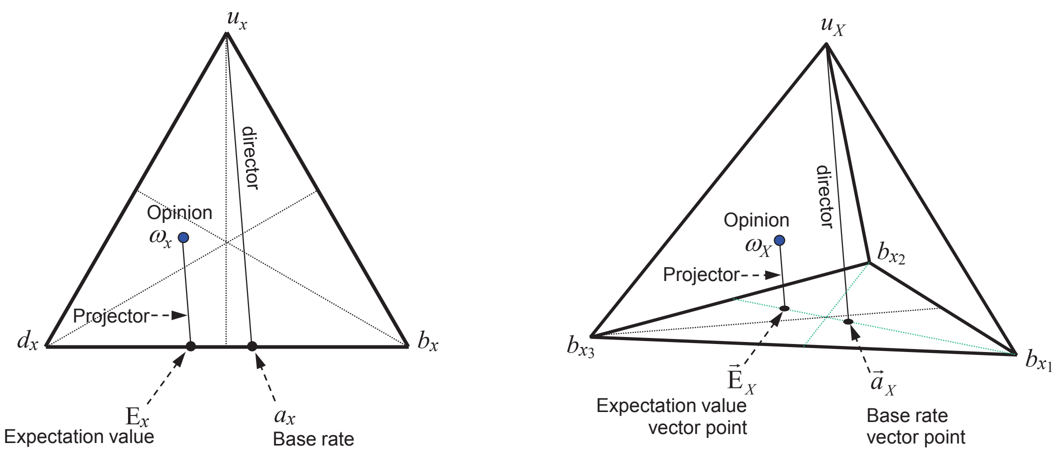
In the special case of binomial opinions Eq.(9) is simplified to Eq.(14).

$$b + d + u = 1 \; . \tag{14}$$

Similarly, in the special case of binomial opinions the probability expectation value of Eq.(12) is simplified to Eq.(15).

$$\mathrm{E}_x = b + au \; . \tag{15}$$

A binomial opinion can be visualised as a point inside an equal sided triangle as in Fig. 1.a where the belief, disbelief and uncertainty axes go perpendicularly from each edge to the opposite vertex indicated by b_x, d_x and u_x. The base rate a_x is a point on the base line, and the probability expectation value E_x is determined by projecting the opinion point to the base line in parallel with the base rate director.



(a) Binomial opinion point in triangle

(b) Trinomial opinion point in tetrahedron

Figure 1: Example opinion points in 2-dimensional and 3-dimensional simplexes

In case the opinion point is located at the left or right corner of the triangle, i.e. with $d_x = 1$ or $b_x = 1$ and $u_x = 0$, the opinion is equivalent to boolean TRUE or FALSE, then subjective logic becomes equivalent to binary logic.

Multinomial opinions can be represented as points inside a regular simplex. In particular, a trinomial opinion can be represented as a point inside a tetrahedron as in Fig. 1.b where the belief and uncertainty axes go perpendicularly from each triangular side plane to the opposite vertex indicated by the labels b_{x_i} and by u_X. The base rate vector $\vec{a}_X$ is a point on the triangular base plane, and the probability expectation vector $\vec{\mathrm{E}}_X$ is determined by projecting the opinion point onto the same base, in parallel with the base rate director. Visualisations of hyper opinions are difficult to design.

3 Notation for Conditional Opinions

This section simply introduces the notation used for conditional deduction and abduction in subjective logic. The purpose is to define the context in which inversion of conditional opinions is useful and necessary.

3.1 Binomial Conditional Opinions

A detailed description of conditional deduction with binomial opinions is described in [4]. Furthermore, a high level description of conditional abduction with binomial opinions is described in [1]. This section simply describes the notation for binomial conditional deduction and abduction without mathematical details.

Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be two binary frames where there is a degree of relevance between X and Y. Let $\omega_x = (b_x, d_x, u_x, a_x)$, $\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_{y|x})$ and $\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_{y|\overline{x}})$ be an agent's respective opinions about x being true, about y being true given that x is true, and finally about y being true given that x is false.

The conditional deduction operator is a ternary operator (i.e. with 3 input arguments), and by using the function symbol $\circledcirc$ to designate this operator, then binomial deduction is expressed as:

$$\text{Binomial opinion deduction:} \quad \omega_{y\|x} = \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}}) \; . \tag{16}$$

Conditional abduction is a quaternary operator denoted $\overline{\circledcirc}$, where binomial abduction is expressed as:

$$\text{Binomial opinion abduction:} \quad \omega_{y\overline{\|}x} = \omega_x \; \overline{\circledcirc} \; (\omega_{x|y}, \omega_{x|\overline{y}}, a_y) \; . \tag{17}$$

The conditionally abduced opinion $\omega_{y\overline{\|}x}$ expresses the belief in y being true as a function of the beliefs in x and in the two sub-conditionals $x|y$ and $x|\overline{y}$, as well as of the base rate a_y.

In order to compute Eq.(17) it is necessary to invert the conditional opinions $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ in order to obtain the conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$, so that the final part of the abduction computation can be based on Eq.(16). Inversion of binomial opinion conditionals is described in Sec.4.1 below.

3.2 Multinomial Conditional Opinions

Let frame X have cardinality $k = |X|$ and frame Y have cardinality $l = |Y|$, where X plays the role of parent, and Y the role of child.

Assume conditional opinions of the form $\omega_{Y|x_i}$, where $i = 1 \ldots k$. There is thus one conditional for each element x_i in the parent frame. Each of these conditionals must be interpreted as the subjective opinion on Y given that x_i is TRUE. The subscript notation on each conditional opinion variable $\omega_{Y|x_i}$ specifies not only the child frame Y it applies to, but also the element x_i in the parent frame it is conditioned on.

By extending the notation for conditional deduction with binomial opinions to the case of multinomial opinions, the general expression for multinomial subjective logic conditional deduction is denoted:

$$\text{Multinomial opinion deduction:} \quad \omega_{Y\|X} = \omega_X \circledcirc \omega_{Y|X} \tag{18}$$

where $\circledcirc$ denotes the conditional deduction operator for subjective opinions, and where $\omega_{Y|X}$ is a set of $k = |X|$ different opinions conditioned on each $x_i \in X$ respectively. Similarly, the expressions for subjective logic conditional abduction can be written as:

$$\text{Multinomial opinion abduction:} \quad \omega_{Y\overline{\|}X} = \omega_X \mathbin{\overline{\circledcirc}} (\omega_{X|Y}, a_Y) \tag{19}$$

where $\overline{\circledcirc}$ denotes the general conditional abduction operator for subjective opinions, and where $\omega_{X|Y}$ is a set of $l = |Y|$ different multinomial opinions conditioned on each $y_j \in Y$ respectively. The set of base rates over Y is denoted by a_Y.

A detailed description of multinomial deduction and abduction with opinions is given in [2]. However, the inversion of conditionals described in [2] is overly conservative, meaning that it produces unnecessarily high uncertainty. A more aggressive and optimal method is described in Sec.4.2 below.

3.3 Relevance

Any relevance between a parent and child frame pair can be expressed in terms of conditional parent-child relationships. Note that the relevance is directional, meaning that in general the parent-to-child relevance is different from the child-to-parent relevance. In this section we assume that Y is the parent frame and X is the child frame.

The relevance of y to x must be interpreted as the influence that truth or falsehood of y has on the belief in x. The relevance of y to x is defined as:

$$\Psi(x|y) = |p(x|y) - p(x|\overline{y})| \; . \tag{20}$$

It can be seen that $\Psi(x|y) \in [0,1]$, where $\Psi(x|y) = 1$ expresses total relevance, and $\Psi(x|y) = 0$ expresses total irrelevance between y and x. Also notice that $\Psi(x|y) = \Psi(\overline{x}|y)$.

For conditionals expressed as binomial opinions, the same type of relevance between frame Y and state $x \in X$ can be defined as:

$$\Psi(x|Y) = |\mathrm{E}(\omega_{x|y}) - \mathrm{E}(\omega_{x|\overline{y}})| \; . \tag{21}$$

$$\begin{aligned} \Psi(\overline{x}|Y) &= |(1 - \mathrm{E}(\omega_{x|y})) - (1 - \mathrm{E}(\omega_{x|\overline{y}}))| \\ &= |\mathrm{E}(\omega_{x|\overline{y}}) - \mathrm{E}(\omega_{x|y})| \\ &= \Psi(x|Y) \; . \end{aligned} \tag{22}$$

For conditionals expressed as multinomial opinions it is possible that the parent frame cardinality $l = |Y|$ is different from the child frame cardinality $k = |X|$, which must be considered in the computation of relevance.

The relevance of the conditionals can be computed with regard to each state in the child frame. Since there are multiple parent states in Y, the relevance to each child state x_i is considered to be the greatest difference between the expectation values on x_i among all possible pairs of conditionals, as expressed by Eq.(23).

$$\Psi(x_i|Y) = \max_{y_g \in Y}[\mathrm{E}(\omega_{X|y_g}(x_i))] - \min_{y_h \in Y}[\mathrm{E}(\omega_{X|y_h}(x_i))] \tag{23}$$

It is useful to define irrelevance denoted as $\overline{\Psi}(x_i|Y)$ to be the complement of relevance, as expresses by:

$$\overline{\Psi}(x_i|Y) = 1 - \Psi(x_i|Y) \; . \tag{24}$$

The relevance expresses the diagnostic power of the conditionals, i.e. to what degree beliefs in the truth of the parent propositions influences beliefs in the truth of the child propositions. The irrelevance $\overline{\Psi}(x_i|Y)$ therefore expresses the lack of diagnostic power, which gives rise to uncertainty.

High irrelevance leads to high uncertainty and vice versa. To see this, assume that all conditionals are equal so that $\omega_{X|y_1} = \omega_{X|y_2} = \cdots = \omega_{X|y_l}$. Obviously, when Y is totally irrelevant to X there is no evidence to support any relevance from X to Y, which translates into total uncertainty about the corresponding inverted conditionals $\omega_{Y|X}$.

4 Inversion of Conditional Opinions

4.1 Inversion of Binomial Conditional Opinions

Binomial abduction requires the inversion of binomial conditional opinions. This section describes the mathematical expressions necessary for computing the required inverted conditionals. Inversion of binomial conditional opinions generalises the inversion of probabilistic conditionals of Eq.(3).

Assume that the available conditionals are $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ which are expressed in the opposite direction to that needed for applying Eq.(16) of binomial deduction. Recall that binomial abduction simply consists of first inverting the conditionals to produce $\omega_{y|x}$ and $\omega_{y|\overline{x}}$, and subsequently to use these as input to binomial deduction specified by Eq.(16).

Deriving the inverted conditional opinions requires knowledge of the base rate a_y of the child proposition y.

First compute the probability expectation values of the available conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ using Eq.(15) to produce:

$$\begin{cases} \mathrm{E}(\omega_{x|y}) & = p(x|y) = b_{x|y} + a_x u_{x|y} \\ \\ \mathrm{E}(\omega_{x|\overline{y}}) & = p(x|\overline{y}) = b_{x|\overline{y}} + a_x u_{x|\overline{y}} \end{cases} \tag{25}$$

Following the principle of Eq.(3), compute the probability expectation values of the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ using the values of Eq.(25).

$$\begin{cases} \mathrm{E}(\omega_{y|x}) & = p(y|x) = \frac{(a_y p(x|y))}{a_y p(x|y) + a_{\overline{y}} p(x|\overline{y})} \\ \\ \mathrm{E}(\omega_{y|\overline{x}}) & = p(y|\overline{x}) = \frac{(a_y p(\overline{x}|y))}{a_y p(\overline{x}|y) + a_{\overline{y}} p(\overline{x}|\overline{y})} \end{cases} \tag{26}$$

Synthesise a pair of dogmatic conditional opinions from the expectation values of Eq.(26):

$$\begin{cases} \underline{\omega}_{y|x} = (p(y|x), p(\overline{y}|x), 0, a_y) \\ \\ \underline{\omega}_{y|\overline{x}} = (p(y|\overline{x}), p(\overline{y}|\overline{x}), 0, a_y) \end{cases} \tag{27}$$

where $p(\overline{y}|x) = (1 - p(y|x))$ and $p(\overline{y}|\overline{x}) = (1 - p(y|\overline{x}))$.

The expectation values of the dogmatic conditionals of Eq.(27) and of the inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ are equal by definition. However, the inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ do in general contain uncertainty, in contrast to the dogmatic opinions of Eq.(27) that contain no uncertainty. The inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ can be derived from the dogmatic opinions of Eq.(27) by determining their appropriate uncertainty level. This amount of uncertainty is a function of the following elements:

- the theoretical maximum uncertainty values $\widehat{u}_{Y|x}$ and $\widehat{u}_{Y|\overline{x}}$ for $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ respectively,
- the weighted uncertainty $u^{\mathrm{W}}_{X|Y}$ based on the uncertainties $u_{x|y}$ and $u_{x|\overline{y}}$ weighted by the base rates a_y and $a_{\overline{y}}$,
- the irrelevance $\overline{\Psi}(x|Y)$ and $\overline{\Psi}(\overline{x}|Y)$.

More precisely, the uncertainty $u_{y|x}$ is computed as:

$$u_{y|x} = \widehat{u}_{Y|x}\, \widetilde{u}_{Y|x} \,. \tag{28}$$

The interpretation of Eq.(28) is that the uncertainty $u_{y|x}$ of the inverted conditional $\omega_{y|x}$ is in the range $[0, \widehat{u}_{Y|x}]$ adjusted by the relative uncertainty $\widetilde{u}_{Y|x}$ defined as:

$$\begin{aligned} \widetilde{u}_{Y|x} &= u^{\mathrm{W}}_{X|Y} \sqcup \overline{\Psi}(x|Y) \\ &= u^{\mathrm{W}}_{X|Y} \ + \ \overline{\Psi}(x|Y) \ - \ u^{\mathrm{W}}_{X|Y}\, \overline{\Psi}(x|Y) \,. \end{aligned} \tag{29}$$

The interpretation of Eq.(29) is that the relative uncertainty $\widetilde{u}_{Y|x}$ is an increasing function of the weighted uncertainty $u^{\mathrm{W}}_{X|Y}$, because uncertainty in one reasoning direction must be reflected by the uncertainty in the opposite reasoning direction. A practical example is when Alice is totally uncertain about whether Bob carries an umbrella in sunny or rainy weather. Then it is natural that observing whether Bob carries an umbrella tells Alice nothing about the weather.

Similarly, the relative uncertainty $\widetilde{u}_{Y|x}$ is an increasing function of the irrelevance $\overline{\Psi}(x|Y)$, because if the original conditionals $\omega_{X|Y}$ reflect total irrelevance from parent frame Y to child frame X, then there is no basis for deriving belief about the inverted conditionals $\omega_{Y|X}$, so it must be uncertainty maximized. A practical example is when Alice knows that Bob always carries an umbrella both in rain and sun. Then observing Bob carrying an umbrella tells her nothing about the weather.

The relative uncertainty $\widetilde{u}_{Y|x}$ is thus high in case the weighted uncertainty $u^{\mathrm{W}}_{X|Y}$ is high, or the irrelevance $\overline{\Psi}(x|Y)$ is high, or both are high at the same time. The correct mathematical model for this principle is to compute the relative uncertainty $\widetilde{u}_{Y|x}$ as the disjunctive combination of weighted uncertainty $u^{\mathrm{W}}_{X|Y}$ and the irrelevance $\overline{\Psi}(x|Y)$, denoted by the coproduct operator $\sqcup$ in Eq.(29). Note that in the binomial case we have $\widetilde{u}_{Y|\overline{x}} = \widetilde{u}_{Y|x}$.

The weighted uncertainty $u^{\mathrm{W}}_{X|Y}$ is expressed as:

$$u^{\mathrm{W}}_{X|Y} = a_y u_{x|y} + a_{\overline{y}} u_{x|\overline{y}} \tag{30}$$

The theoretical maximum uncertainties $\widehat{u}_{Y|x}$ for $\omega_{y|x}$ and $\widehat{u}_{Y|\overline{x}}$ for $\omega_{y|\overline{x}}$ are determined by setting either the belief or the disbelief mass to zero according to the simple IF-THEN-ELSE algorithm below. After computing the theoretical maximum uncertainty of each inverted conditional opinion, the uncertainty values of the inverted conditional opinions are computed as the product of the theoretical maximum uncertainty and the relative uncertainty. The remaining opinion parameters b and d emerge directly.

$$\begin{array}{ll} \multicolumn{2}{l}{\text{Computation of } \widehat{u}_{Y|x}} \\ \hline \text{IF} & p(y|x) < a_y \\ \text{THEN} & \widehat{u}_{Y|x} = p(y|x)/a_y \\ \text{ELSE} & \widehat{u}_{Y|x} = (1-p(y|x))/(1-a_y) \end{array} \tag{31}$$

Having computed $\widehat{u}_{Y|x}$ the opinion parameters are:

$$\begin{cases} u_{y|x} = \widehat{u}_{Y|x}\widetilde{u}_{y|x} \\ b_{y|x} = p(y|x) - a_y u_{y|x} \\ d_{y|x} = 1 - b_{y|x} - u_{y|x} \end{cases} \tag{32}$$

so that the inverted conditional opinion can be expressed as $\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_y)$.

$$\begin{array}{ll} \multicolumn{2}{l}{\text{Computation of } \widehat{u}_{Y|\overline{x}}} \\ \hline \text{IF} & p(y|\overline{x}) < a_y \\ \text{THEN} & \widehat{u}_{Y|\overline{x}} = p(y|\overline{x})/a_y \\ \text{ELSE} & \widehat{u}_{Y|\overline{x}} = (1-p(y|\overline{x}))/(1-a_y) \end{array} \tag{33}$$

Having computed $\widehat{u}_{Y|\overline{x}}$ the opinion parameters are:

$$\begin{cases} u_{y|\overline{x}} = \widehat{u}_{Y|\overline{x}}\widetilde{u}_{y|\overline{x}} \\ b_{y|\overline{x}} = p(y|\overline{x}) - a_y u_{y|\overline{x}} \\ d_{y|\overline{x}} = 1 - b_{y|\overline{x}} - u_{y|\overline{x}} \end{cases} \tag{34}$$

so that the inverted conditional opinion can be expressed as $\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_y)$.

The inverted binomial conditionals can now be used for binomial conditional deduction according to Eq.(16). Binomial abduction with the conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ according to Eq.(17) is equivalent to applying the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ in binomial conditional deduction which is described in detail in [4].

4.2 Inversion of Multinomial Conditional Opinions

Multinomial abduction requires the inversion of conditional opinions of the form $\omega_{X|y_j}$ into conditional opinions of the form $\omega_{Y|x_i}$ similarly to Eq.(6).

Fig. 2 illustrates the principle of inversion of multinomial conditional opinions. The initial conditionals project the Y-tetrahedron onto a sub-tetrahedron within the X-tetrahedron as shown in the top part of Fig. 2. The goal of the inversion is to derive conditionals that define a projection from the X-pyramid to a sub-pyramid within the Y-pyramid as shown in the bottom part of Fig. 2.

In case the conditionals are expressed as hyper opinion then it is required that they be projected to multinomial opinion arguments that only provide belief support for singleton statements in the frame. Eq.(35) describes the method for projecting hyper opinions onto multinomial opinions.

Let the hyper opinion be denoted as ω'_X and the resulting multinomial opinion be denoted ω_X. Evidently ω'_X and ω_X have the same expectation values. The belief vector of the multinomial opinion projection is given by Eq.(35).

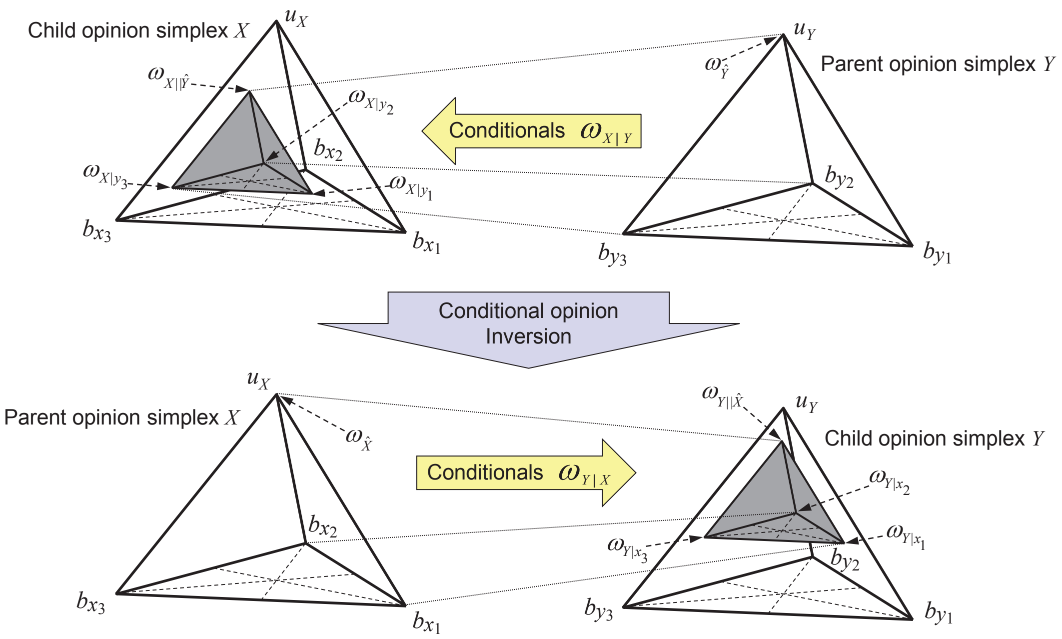


Figure 2: Inversion of multinomial conditional opinions

$$\vec{b}_X(x_i) = \sum_{x_j \in \mathcal{R}(X)} \vec{a}_X(x_i/x_j)\,\vec{b}'_X(x_j) \quad \forall\, x_i \in X \ . \tag{35}$$

Inversion of multinomial conditional opinions is based on uncertainty maximisation of opinions combined with relative uncertainty, and is a generalisation of the inversion of binomial conditionals.

Let X and Y be a frames of cardinality $k = |X|$ and $l = |Y|$ respectively, and assume the set of available conditionals:

$$\omega_{X|Y} : \left\{\omega_{X,y_j},\ \text{where } j = 1\ldots l\right\} , \tag{36}$$

and that the analyst requires the set of conditionals:

$$\omega_{Y|X} : \{\omega_{Y,x_i},\ \text{where } i = 1\ldots k\} \ . \tag{37}$$

First compute the l probability expectation values of each inverted conditional opinion $\omega_{Y|x_i}$, according to Eq.(6) as:

$$\mathrm{E}(y_j|x_i) = \frac{a(y_j)\mathrm{E}(\omega_{X|y_j}(x_i))}{\sum_{t=1}^{l} a(y_t)\mathrm{E}(\omega_{X|y_t}(x_i))} \tag{38}$$

where $a(y_j)$ denotes the base rate of y_j. Consistency dictates:

$$\mathrm{E}(\omega_{Y|x_i}(y_j)) = \mathrm{E}(y_j|x_i) \ . \tag{39}$$

The simplest opinions to satisfy Eq.(39) are the dogmatic opinions:

$$\underline{\omega}_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) & = \mathrm{E}(y_j|x_i), \quad \text{for } j = 1\ldots k\,, \\ u_{Y|x_i} & = 0\,, \\ \vec{a}_{Y|x_i} & = \vec{a}_Y \ . \end{cases} \tag{40}$$

Uncertainty maximisation of $\underline{\omega}_{Y|x_i}$ consists of converting as much belief mass as possible into uncertainty mass while preserving consistent probability expectation values according to Eq.(39). The result is the uncertainty maximised opinion denoted as $\widehat{\omega}_{Y|x_i}$. This process is illustrated in Fig. 3.

It must be noted that Fig. 3 only represents two dimensions of the multinomial opinions on Y, namely y_j and its complement. The line defined by

$$\mathrm{E}(y_j|x_i) = b_{Y|x_i}(y_j) + a_{Y|x_i}(y_j)u_{Y|x_i} \ . \tag{41}$$

that is parallel to the base rate line and that joins $\underline{\omega}_{Y|x_i}$ and $\widehat{\omega}_{Y|x_i}$ in Fig. 3, defines the opinions $\omega_{Y|x_i}$ for which the probability expectation values are consistent with Eq.(39). A opinion $\widehat{\omega}_{Y|x_i}$ is uncertainty maximised when Eq.(41) is satisfied and at least one belief mass of $\widehat{\omega}_{Y|x_i}$ is zero. In general, not all belief masses can be zero simultaneously except for vacuous opinions.

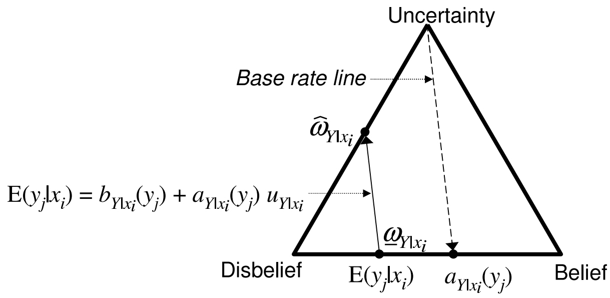


Figure 3: Uncertainty maximisation of dogmatic opinion

In order to find the dimension(s) that can have zero belief mass, the belief mass will be set to zero in Eq.(41) successively for each dimension $y_j \in Y$, resulting in l different uncertainty values defined as:

$$u^j_{Y|x_i} = \frac{\mathrm{E}(y_j|x_i)}{a_{Y|x_i}(y_j)}, \text{ where } j = 1 \dots l\ . \tag{42}$$

The minimum uncertainty of Eq.(42) is then fixed as the maximum possible uncertainty for the inverted opinion $\omega_{Y|x_i}$. This uncertainty denoted $\widehat{u}_{Y|x_i}$ is expressed as:

$$\widehat{u}_{Y|x_i} = \min_{j=1\dots l}[u^j_{Y|x_i}] \tag{43}$$

Fixing the belief mass to zero for a dimension where the uncertainty is not minimum would result in negative belief mass for other dimensions.

For multinomial opinion conditionals the weighted uncertainty $u^{\mathrm{W}}_{X|Y}$ is expressed as:

$$u^{\mathrm{W}}_{X|Y} = \sum_{j=1}^{l} a_{y_j} u_{X|y_j} \tag{44}$$

The relative uncertainty of multinomial conditional opinions can be computed using the coproduct operator $\sqcup$ as:

$$\begin{aligned} \widetilde{u}_{Y|x_i} &= u^{\mathrm{W}}_{X|Y} \sqcup \overline{\Psi}(x_i|Y) \\ &= u^{\mathrm{W}}_{X|Y} + \overline{\Psi}(x_i|Y) - u^{\mathrm{W}}_{X|Y} \cdot \overline{\Psi}(x_i|Y)\ . \end{aligned} \tag{45}$$

Given the maximum possible uncertainty $\widehat{u}_{Y|x_i}$ and the relative uncertainty $\widetilde{u}_{Y|x_i}$ the uncertainty of the inverted opinion $\omega_{Y|x_i}$ can be computed as:

$$u_{Y|x_i} = \widehat{u}_{Y|x_i}\, \widetilde{u}_{x_i|Y} \tag{46}$$

The inverted opinion can then be determined as:

$$\omega_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) &= \mathrm{E}(y_j|x_i) - a_Y(y_j) \cdot u_{Y|x_i}, \text{ for } y = 1 \dots l \\ u_{Y|x_i} &= u_{Y|x_i} \\ \vec{a}_{Y|x_i} &= \vec{a}_Y \end{cases} \tag{47}$$

With Eq.(47) the expressions for the set of inverted conditional opinions $\omega_{Y|x_i}$ (with $i = 1 \dots k$) can be computed. Conditional abduction according to Eq.(19) with the original set of multinomial conditionals $\omega_{X|Y}$ is now equivalent to multinomial conditional deduction according to Eq.(18) where the set of inverted conditionals $\omega_{Y|X}$ is used deductively. Multinomial conditional deduction is described in detail in [2].

5 Conclusions

This paper explains the need for inverting conditionals in order to apply abductive reasoning and recalls the traditional method used in probabilistic Bayesian reasoning. In order to apply abductive Bayesian reasoning in subjective logic the same principle must be applied there. We describe a method for inverting conditional opinions as a generalisation of how conditional probabilities can be inverted. This method represents and improvement over a method previously presented. The ability to invert conditional opinions provides a basis for general Bayesian reasoning in subjective logic. The advantage of applying subjective logic for Bayesian reasoning is that it takes into account degrees of uncertainty in the input arguments, in addition to being consistent with traditional probabilistic Bayesian reasoning. As a result, output conclusions reflect more realistically the situations being analysed, while at the same time being directly consistent and compatible with traditional Bayesian methods. In the same way that probabilistic logic provides the basis for more general and realistic reasoning models than binary logic, subjective logic provides the basis for more general and realistic reasoning models than probabilistic logic.

References

[1] Jøsang, A.: Abductive Reasoning with Uncertainty. In: The Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU2008) (2008)

[2] Jøsang, A.: Conditional Reasoning with Subjective Logic. Journal of Multiple-Valued Logic and Soft Computing **15**(1), 5–38 (2008)

[3] Jøsang, A., Hankin, R.: Interpretation and Fusion of Hyper Opinions in Subjective Logic. In: Proceedings of the 15th International Conference on Information Fusion (FUSION 2012) (2012)

[4] Jøsang, A., Pope, S., Daniel, M.: Conditional deduction under uncertainty. In: Proceedings of the 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005) (2005)

[5] Koehler, J.: The Base Rate Fallacy Reconsidered: Descriptive, Normative and Methodological Challenges. Behavioral and Brain Sciences **19** (1996)

[6] Pearl, J.: Causality: Models, Reasoning, and Inference. Cambridge University Press (2000)