Computable Irrational Numbers with Representations of Surprising Complexity

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Abstract

Cauchy sequences, Dedekind cuts, base-10 expansions and continued fractions are examples of well-known representations of irrational numbers. But there exist others, not so popular, which can be defined using various kinds of sum approximations and best approximations. In this paper we investigate the complexity of a number of such representations.

For any fast-growing computable function f, we define an irrational number α_f by using a series of reciprocals of powers of all primes. We prove that certain representations of α_f are of low computational complexity (which does not depend on f), whereas others, apparently similar representations, can be of arbitrarily high computational complexity (which depends on f). The existence of computable numbers like α_f allows us to prove new and non-trivial theorems on the computational complexity of representations without resorting to the standard computability-theoretic machinery involving enumerations and diagonalizations.

In the paper we also show how to construct irrational numbers γ whose representations by a Cauchy sequences are of low computational complexity, but whose base-b expansion may be of arbitrarily high computational complexity for all bases b. Moreover, for any $\mathcal{E}^2$-irrational number α, there will be an $\mathcal{E}^2$-irrational number β, such that $\alpha + \beta$ has the complexity of γ. As a consequence, two numbers which have, let us say, base-10 expansions of low computational complexity, may add up to a number whose base-10 expansion is of arbitrarily high computational complexity. The same goes for representations by base-2 expansions, base-17 expansions, Dedekind cuts, continued fractions, and so on.

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1. Introduction

1.1. Conversion of Representations and Unbounded Search

Irrational numbers are infinite objects. In computations these objects have to be represented in some way or another. We have a number of options. Some popular representations are Cauchy sequences, Dedekind cuts, decimal expansions, binary expansions and continued fractions. A number of alternative representations will be discussed in this paper. All these representations are equivalent in the sense that one representation can be converted into another by an algorithm. Our investigations are motivated by the question:

> *Do we need, or do we not need, unbounded search in order to convert one representation of an irrational number into another representation?*

Let us study a few informal examples. We can convert a decimal expansion into a binary expansion without resorting to unbounded search. Consider the infinite decimal expansion

$$\texttt{0.31270073941}\ldots \qquad (1)$$

of an irrational number β, and assume that we already have computed the first few digits of β's binary expansion, e.g., the digits `0.010`. How can we compute the next digit? Well, if `0.0101` lies below β, the next digit will be `1`, and if `0.0101` lies above β, the next digit will be `0`. So we check if a rational number, given by a finite binary expansion, lies above or below β, and then we can determine the next digit. Proceeding this way, we can can compute the digits of β's binary expansion one by one. No unbounded search will be required as *every rational with a finite binary expansion also has finite decimal expansion.* E.g., the binary expansion `0.0101` has a finite decimal expansion, namely `0.3125`, and thus we can determine if `0.0101` lies above or below β by comparing the digits `0.3125` to the first five digits `0.3127` of (1).

If we want to convert a binary expansion into a decimal expansion, we will need unbounded search. Consider an irrational β_1 that has a binary expansion of the form $\texttt{0.0(0011)}^n\texttt{0001}\ldots$ and an irrational β_2 that has a binary expansion of the form $\texttt{0.0(0011)}^n\texttt{0100}\ldots$. So both expansions start with `0.0` and proceed with the sequence `0011` repeated n times where n can be arbitrarily large. Now, the decimal expansion of β_1 starts with `0.0` whereas the decimal expansion of β_2 starts with `0.1`. This example shows that we possibly will have to examine more than $4n$ digits of a binary expansion in order to determine the first fractional digit of the decimal expansion. Since n can be arbitrarily large, we will obviously need unbounded search.

If we want to turn a (fast converging) Cauchy sequence into a decimal expansion, we will also need unbounded search. A Cauchy sequence that represents the irrational number β is a sequence $q_0, q_1, q_2, \ldots$ of rationals such that $|q_n - \beta| < 2^{-n}$. Let us say that each of the first 17 rationals of the Cauchy sequence equals 10^{-1}. Then we know that β is pretty close to 10^{-1}, but we do not know if β lies above or below 10^{-1}, and thus we do not know if the decimal expansion of β should start with `0.0` or `0.1`. Since β is irrational, there will somewhere in the sequence be a q_n which allows to determine which of the two alternatives is correct, but we obviously need unbounded search to find that q_n. By the same token we need unbounded search if we want to convert a Cauchy sequence into a Dedekind cut. (The Dedekind cut of an irrational number β is a predicate $D(q)$ that tells us if the rational number q lies above or below β. In the example just seen we have to examine an arbitrarily large initial segment of a Cauchy sequence to determine if a rational lies above or below β.) On the other hand, we do not need unbounded search in order to convert a decimal expansions into a Cauchy sequence. We can do that by chopping off initial segments of the decimal expansion. E.g., we can turn (1) into a Cauchy sequence $q_0, q_1, q_2, \ldots$ by simply letting $q_0 = \texttt{0}$, $q_1 = \texttt{0.3}$, $q_2 = \texttt{0.31}$, $q_3 = \texttt{0.312}$, and so on.

The reader might have noticed that we in some sense have kicked out the rationals. Our examples above discuss conversions between representations of *irrational numbers*, not between representations of *real numbers*. We will proceed along this line and restrict our attention to the irrational numbers. This is to a certain extent a matter of taste and convenience. It makes the presentation smooth. The interested reader might want to consider the following question:

> *Can we, or can we not,* uniformly *convert one representation of a real number into another representation?*

This question is definitely closely related to the question we ask above, but we will not discuss the nature of this relationship any further in the current paper.

1.2. Methods of Proofs and Subrecursive Classes

Let us recall our fundamental question in slightly reformulated version:

> *Do we need unbounded search to convert a representation $R1$ into a representation $R2$?*

It is rather obvious how we in principle can give a mathematical satisfying *negative* answer to this question. We can simply give an algorithm that converts an $R1$-representation into an $R2$-representation. If the algorithm does not use unbounded search, then the answer is NO. It is not all that obvious how we can give a mathematical satisfying *positive* answer. One of our examples above shows that we need unbounded search to convert a Cauchy sequence into a Dedekind cut. The example is simple and persuasive. The inevitable conclusion is that we really do need unbounded search. Still, it is an example and not a proof. Moreover, in the general case, where $R1$ and $R2$ may be unorthodox representations that we are not very familiar with, it will hardly be satisfactory to cook up an example and wave our hands.

In order to give a proper positive answer to our question, we will work with generalized subrecursive classes. A subrecursive class is a class of total computable functions. The class of (Kalmár) elementary functions is an example of a subrecursive class. So is the class of primitive recursive functions. We give a formal definition of a subrecursive class in Section 3, and we prove that for any total computable function ψ there exists a subrecursive class $\mathcal{S}$ with strong closure properties such that $\psi \in \mathcal{S}$.

Now, let us see how we can use subrecursive classes to prove that we need unbounded search to turn a Cauchy sequence into a Dedekind cut. Assume that we do not need unbounded search (so our assumption is wrong). Then, if C is a Cauchy sequence for the irrational number β, there will be a total computable function ψ_C which is the Dedekind cut of β. The function ψ_C will depend uniformly on C, and thus, for any sufficiently large subrecursive class $\mathcal{S}$ (with sufficiently strong closure properties), ψ_C will be in $\mathcal{S}$ whenever C is in $\mathcal{S}$. This entails that the set of irrational numbers that have a Cauchy sequence in $\mathcal{S}$, henceforth denoted $\mathcal{S}_C$, is a subset of the set of irrational numbers that have a Dedekind cut in $\mathcal{S}$, henceforth denoted $\mathcal{S}_D$.

So if it should be possible to avoid unbounded search when turning a Cauchy sequence into a Dedekind cut, the inclusion $\mathcal{S}_C \subseteq \mathcal{S}_D$ would hold for any sufficiently large subrecursive class $\mathcal{S}$ that has sufficiently strong closure properties. Now, if we prove that there will be arbitrarily large such $\mathcal{S}$ where $\mathcal{S}_C \nsubseteq \mathcal{S}_D$, then we can safely conclude that unbounded search really is needed in order to convert a Cauchy sequence into a Dedekind cut.

Let $\mathcal{S}_{R1}$ and $\mathcal{S}_{R2}$ denote the sets of irrational numbers that have respectively $R1$-representations and $R2$-representations in $\mathcal{S}$. The statement

> *there exists an arbitrarily large subrecursive class $\mathcal{S}$ with nice closure properties such that $\mathcal{S}_{R1} \nsubseteq \mathcal{S}_{R2}$*

should be considered as a mathematical clarification and a formalization. It is a way to state that we need unbounded search to convert an $R1$-representation into an $R2$-representation. Moreover, the statement is amenable to rigorous proof.

The reader should be aware of our main concerns. When we have proved that an algorithm can be realized by an elementary or a primitive recursive function, we will be satisfied. That is good enough for us. That implies that the algorithm does not use unbounded search. A fine-grained complexity-theoretic analysis of algorithms and conversion procedures is beyond the scope of this paper. It will be convenient to us to work with subrecursive classes that are closed under elementary, and sometimes even under primitive recursive, operations. Then we can clear the table of tedious conventions and potential problems related to coding. Uniform systems for coding finite sequences are available inside the class of elementary functions. Hence, for any reasonable coding conventions, standard operations on rational numbers and sequences of rational numbers (like addition, multiplication, exponentiation, bounded sums, bounded products) will obviously be elementary. Furthermore, we can work with straightforward and natural definitions of (fast converging) Cauchy sequences, sum approximations, best approximations, and so on. These are definitions that do not make sense below the elementary level.

1.3. References

More on computable real numbers, and on computable analysis in general, can be found in Weihrauch [23] and Aberth [1].

Subjects related to the ones we investigate have been studied over the last seven decades. In a very early paper on computable analysis, Specker [21] proves that

$$\mathcal{S}_D \;\subset\; \mathcal{S}_{10E} \;\subset\; \mathcal{S}_C$$

where $\mathcal{S}$ is the class of primitive recursive functions and $\mathcal{S}_{10E}$ is the set of irrationals that have a primitive recursive decimal expansion (Specker sequences were introduced in the same paper). In addition to Specker's paper there are works by Mostowski [14], Lehman [16], Ko [7, 8], Labhalla & Lombardi [15], Weihrauch [22], Georgiev, Skordev & Weiermann [20], Georgiev [3], Kristiansen [10, 11] and quite a few more. We will give further references throughout the paper.

Some results present in the current paper have been published in the short conference paper [4].

1.4. Our Results

We will provide irrational numbers with intriguing properties. Such a number α has a transparent definition of the form

$$\alpha \;=\; \sum_{i=1}^{\infty} \frac{1}{P_i^{f(i)}} \tag{2}$$

where P_i is the i^{th} prime and f is some fast-increasing function with certain properties (so there is one such α for each f). Take any subrecursive class $\mathcal{S}$ and pick a suitable f that is not primitive recursive and lies outside $\mathcal{S}$, that is, $f \notin \mathcal{S}$ (there will always be such an f). It turns out that for any fixed $b \geq 2$ there is a primitive recursive algorithm (indeed there is an elementary algorithm) for writing α in the form

$$\alpha \;=\; 0 \;+\; \frac{\mathtt{D}_1}{b^{k_1}} \;+\; \frac{\mathtt{D}_2}{b^{k_2}} \;+\; \frac{\mathtt{D}_3}{b^{k_3}} \;+\; \ldots \tag{3}$$

where

- $\mathtt{D}_i \in \{1, \ldots, b-1\}$ (note that $\mathtt{D}_i \neq 0$ for all i)
- $k_i \in \mathbb{N} \setminus \{0\}$ and $k_i < k_{i+1}$.

Let $\hat{A}^\alpha_b$ denote the function that gives the n^{th} summand of the series (3), that is, $\hat{A}^\alpha_b(n) = \mathtt{D}_n b^{-k_n}$. We will refer to $\hat{A}^\alpha_b$ as the *base-b sum approximation from below of* α. For each fixed $b \geq 2$, the function $\hat{A}^\alpha_b$ is elementary (and thus primitive recursive).

Still, an algorithm that generates the series (3) uniformly in b, cannot be primitive recursive. Indeed, such an algorithm cannot belong to any subrecursive class that does not contain the function f. Hence the algorithm cannot belong to $\mathcal{S}$ no matter how big $\mathcal{S}$ might be (because we have picked an f that is not in $\mathcal{S}$). To put it otherwise and more precisely: Let $\hat{G}^\alpha(b, n) = \hat{A}^\alpha_b(n)$. Then $\hat{G}^\alpha \notin \mathcal{S}$. We will refer to the function $\hat{G}^\alpha$ as the *general sum approximation from below of* α.

For any fixed $b \geq 2$, there is also a primitive recursive algorithm for writing α in the form

$$\alpha \;=\; 1 \;-\; \left(\frac{\mathtt{D}_1}{b^{k_1}} \;+\; \frac{\mathtt{D}_2}{b^{k_2}} \;+\; \frac{\mathtt{D}_3}{b^{k_3}} \;+\; \ldots \right) \tag{4}$$

where

- $\mathtt{D}_i \in \{1, \ldots, b-1\}$ (note that $\mathtt{D}_i \neq 0$ for all i)
- $k_i \in \mathbb{N} \setminus \{0\}$ and $k_i < k_{i+1}$.

The function $\check{A}^\alpha_b$ that gives the n^{th} summand of the series (4) will be called *base-b sum approximation from above of* α. It is not true in general that every irrational that has a primitive recursive base-b sum approximation from below also will have one from above (an intuitive explanation of this asymmetry is given early in Section 7). However, for each $b \geq 2$, the function $\check{A}^\alpha_b$ is primitive recursive, that is, α has a primitive recursive base-b sum approximation from above. Moreover, in contrast to the case when we were dealing with sum approximations from below, it turns out that there also is a primitive recursive algorithm that generates the sum (4) uniformly in b. Let $\breve{G}^\alpha(b,n) = \breve{A}_b^\alpha(n)$. Then $\breve{G}^\alpha$ is indeed a primitive recursive function. We will refer to the function $\breve{G}^\alpha$ as the *general sum approximation from above of* α.

A brief summary: For any subrecursive class $\mathcal{S}$, there is an irrational α such that

- for any fixed $b \geq 2$, the base-b sum approximation from below of α is primitive recursive (indeed, it is elementary)
- the general sum approximation from above of α is primitive recursive
- the general sum approximation from below of α is not in $\mathcal{S}$.

A couple of highly nontrivial theorems follow from the existence of numbers like α. Some might find it surprising (at least the authors do) that we can find the irrational numbers that are needed as witnesses in the proofs of these theorems, without resorting to the standard computability-theoretic machinery involving enumerations, universal functions, diagonalizations, and so on. The numbers we need are given by readable and transparent definitions of the form (2).

We will also prove that for any subrecursive class $\mathcal{S}$ there exist irrational numbers that cannot be expanded in any base b by an $\mathcal{S}$-algorithm, but still have elementary Cauchy sequences. Now we need to resort to classical diagonalization techniques to complete our proof. Furthermore, we use the existence of such numbers to prove some results on representations and closure under addition which allow us to draw the following conclusion: If we do not admit unbounded search, no representation of real numbers, with the exception of those equivalent to Cauchy sequences, will be closed under addition.

We will also prove some theorems on left and right best approximations. These are two possible ways to represent irrational numbers which so far have not been studied from a computability-theoretic point of view. A *left best approximant* to an irrational β is a vulgar fraction a/b, which is in its lowest terms and smaller than β, such that any vulgar fraction between a/b and β has greater denominator than a/b (if you want a better approximation that is smaller than β, you have to use a greater denominator). A *left best approximation* is a strictly increasing sequence of left best approximants. A *right best approximation* is defined symmetrically as a strictly decreasing sequence of right best approximants. Let $\mathcal{S}$ be a subrecursive class closed under primitive recursive operations, and let β be an irrational. It turns out that β has a left best approximation in $\mathcal{S}$ iff β's general sum approximation form below is in $\mathcal{S}$, and moreover, β has a right best approximation in $\mathcal{S}$ iff β's general sum approximation form above is in $\mathcal{S}$.

2. Preliminaries

2.1. Subrecursion Theory

We assume acquaintance with subrecursion theory. An introduction to this subject can be found in Péter [18], Rose [19] or Odifreddi [17]. Here we just state some important basic facts and definitions. The proofs can be found in the books cited above. We will also assume that the reader is familiar with basic concepts of computability theory, e.g., Kleene's T-predicate and computable indices. An introduction to basic computability theory can be found in, e.g., Cooper [2] or Leary & Kristiansen [13].

The *initial elementary functions* are the projection functions (I_i^n), the constants 0 and 1, addition (+) and modified subtraction ($\dot{-}$). The *elementary definition schemes* are *composition*, that is, $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ and *bounded sum* and *bounded product*, that is, respectively $f(\vec{x}, y) = \sum_{i<y} g(\vec{x}, i)$ and $f(\vec{x}, y) = \prod_{i<y} g(\vec{x}, i)$. A *function is elementary* if it can be generated from the initial elementary functions by the elementary definition schemes. A *relation* $R(\vec{x})$ *is elementary* when there exists an elementary function f with range $\{0, 1\}$ such that $f(\vec{x}) = 0$ iff $R(\vec{x})$ holds. Relations may also be called *predicates*, and we will use the two words interchangeably. A function f has *elementary graph* if the relation $f(\vec{x}) = y$ is elementary.

The definition scheme $(\mu z \leq y)[\ldots]$ is called the *bounded μ-operator*, where $(\mu z \leq y)[R(\vec{x}, z)]$ denotes the least $z \leq y$ such that the relation $R(\vec{x}, z)$ holds. Let $(\mu z \leq y)[R(\vec{x}, z)] = y + 1$ if no such z exists. The class of elementary functions is closed under the bounded μ-operator. The definition scheme

$$f(\vec{x}, 0) = g(\vec{x}) \quad \text{and} \quad f(\vec{x}, y + 1) = h(\vec{x}, y, f(\vec{x}, y))$$

is called *primitive recursion*. If f is defined by a primitive recursion over g and h and $f(\vec{x}, y) \leq j(\vec{x}, y)$, then f is defined by *bounded primitive recursion* over g, h and j. The class of elementary functions is closed under bounded primitive recursion, but not under primitive recursion. Moreover, the class of elementary relations is closed under the operations of the propositional calculus and under bounded quantification.

Let $2_0^x = x$ and $2_{n+1}^x = 2^{2_n^x}$, and let S denote the successor function. The class of elementary functions equals the closure of $\{0, S, I_i^n, 2^x, \max\}$ under composition and bounded primitive recursion. Given this characterization of the elementary functions, it is easy to see that for any elementary function f there exists k such that $f(\vec{x}) \leq 2_k^{\max(\vec{x})}$.

We will say that a class of functions is *closed under the elementary operations* when the class contains all the elementary functions and is closed under composition and bounded primitive recursion. We will say that a class of functions is *closed under the primitive recursive operations* when the class contains all the elementary functions and is closed under composition and (unbounded) primitive recursion.

Uniform systems for coding finite sequences of natural numbers are available inside the class of elementary functions. Let $\overline{f}(x)$ be the code number for the sequence $\langle f(0), f(1), \ldots, f(x)\rangle$. Then $\overline{f}$ belongs to the elementary functions if f does. We will indicate the use of coding functions with the notations $\langle \ldots \rangle$ and $(x)_i$ where $(\langle x_0, \ldots, x_i, \ldots, x_n\rangle)_i = x_i$. (So $(x, i) \mapsto (x)_i$ is an elementary function.) Our coding system is monotone, that is, $\langle x_0, \ldots, x_n\rangle < \langle x_0, \ldots, x_n, y\rangle$ holds for any y, and $\langle x_0, \ldots, x_i, \ldots, x_n\rangle < \langle x_0, \ldots, x_i + 1, \ldots, x_n\rangle$. All the closure properties of the elementary functions can be proved by using Gödel numbering and standard coding techniques.

We assume some coding of the integers ($\mathbb{Z}$) and the rational numbers ($\mathbb{Q}$) into the natural numbers. We consider such a coding to be trivial. Therefore we allow for subrecursive functions from rational numbers into natural numbers, from pairs of rational numbers into rational numbers, etc., with no further comment.

We use f^k to denote the k^{th} iterate of the function f, that is, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$.

2.2. Honest Functions

Our proofs are based on the theory of honest functions. In this subsection, we state and prove lemmas and theorems on honest functions that will be needed later. For more on honest functions, see Kristiansen et al. [12].

Definition 2.1. *A function $f : \mathbb{N} \to \mathbb{N}$ is* honest *if it is monotone ($f(x) \leq f(x+1)$), dominates 2^x ($f(x) \geq 2^x$) and has elementary graph.*

From now on, we reserve the letters $f, g, h, \ldots$ to denote honest functions. Small Greek letters like $\phi, \psi, \xi, \ldots$ will denote number-theoretic functions not necessarily being honest.

Definition 2.2. *A function ϕ is* elementary in *a function ψ, written $\phi \leq_E \psi$, if ϕ can be generated from the initial functions ψ, 2^x, max, 0, S (successor), I^n_i (projections) by composition and bounded primitive recursion.*

Lemma 2.3. *Let $\psi \leq_E f$ where f is an honest function. Then there exists $k \in \mathbb{N}$ such that*

$$\psi(x_1, \ldots, x_n) \ \leq \ f^k(\max(x_1, \ldots, x_n)) \ .$$

Proof. The function ψ can be generated from the initial functions f, 2^x, max, 0, S, I^n_i by composition and bounded primitive recursion. Use induction on such a generation of ψ to prove that the lemma holds. Use that f is monotone and that it dominates 2^x. □

Let $\mathcal{T}_n$ denote the Kleene T-predicate, and let $\mathcal{U}$ denote the decoding function known from Kleene's Normal Form Theorem. We have

$$\phi(x_1, \ldots, x_n) \ = \ \{e\}(x_1, \ldots, x_n) \ = \ \mathcal{U}(\mu t[\mathcal{T}_n(e, x_1, \ldots, x_n, t)])$$

when e is a computable index for ϕ. We will need the next theorem which is proved in Kristiansen [9].

Theorem 2.4 (Normal Form Theorem)**.** *Let f be an honest function. Let ϕ be any (Turing) computable function. Then, $\phi \leq_E f$ iff there exists a computable index e for ϕ and a fixed $k \in \mathbb{N}$ such that*

$$\phi(x_1, \ldots, x_n) \ = \ \{e\}(x_1, \ldots, x_n) \ = \ \mathcal{U}(\mu t \leq f^k(\max(x_1, \ldots, x_n))[\mathcal{T}_n(e, x_1, \ldots, x_n, t)]) \ .$$

Moreover, $\mathcal{U}$ is an elementary function, and $\mathcal{T}_n$ is an elementary predicate.

Definition 2.5. *For any honest function f, we define* the jump of *f, written f', by $f'(x) = f^{x+1}(x)$.*

Lemma 2.6. *Let f be an honest function. Then, f' is an honest function.*

Proof. It is obvious that f' is monotone and dominates 2^x. Let $\psi(x, y)$ be an elementary function that places a bound on the code number for the sequence $\langle y, y, \ldots, y\rangle$ of length $x + 1$. Then, $f'(x) = y$ is equivalent to

$$(\exists s \leq \psi(x, y))[(s)_0 = f(x) \ \wedge \ (\forall i < x)[(s)_{i+1} = f((s)_i)] \ \wedge \ (s)_x = y] \ . \tag{5}$$

Thus, the relation $f'(x) = y$ is elementary since all the functions, relations and operations involved in (5) are elementary. This proves that f' has elementary graph. □

Lemma 2.7. *Let f be an honest function, and let ψ be a unary function such that $\psi \leq_E f$. Then we have $\psi(x) < f'(x)$ for all sufficiently large x.*

Proof. By Lemma 2.3, we have $k \in \mathbb{N}$ such that $\psi(x) \leq f^k(x)$. For $x \geq k$, we have $\psi(x) \leq f^k(x) < f^{x+1}(x) = f'(x)$. □

3. Subrecursive Classes and $\mathcal{S}$-Irrational Numbers

The class of primitive recursive functions is a paradigm example of a subrecursive class. So is the class of elementary functions (these functions are also known as the Kalmár elementary, or the Csillag-Kalmár elementary, functions). The classes we find in subrecursive hierarchies, like the Grzegorczyk hierarchy and the Löb-Wainer hierarchy, are of course also called subrecursive classes. It is also reasonable to call the class of number-theoretic functions that are provably total in a formal system, e.g. Peano Arithmetic or ZFC, a subrecursive class. Many classes defined by resource-bounded machine models were often referred to as subrecursive classes in the 1960s and 1970s. Today we tend to call such classes complexity classes.

We will now give our formal definition of a subrecursive class. The definition should capture all the variants of subrecursive classes mentioned above.

Definition 3.1. *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a total function, and let*

$$[e]^{\sigma}(x) \quad = \quad \mathcal{U}(\,\mu t[\mathcal{T}_1(\sigma(e), x, t)]\,)$$

where $\mathcal{T}_1$ and $\mathcal{U}$ are the elementary functions from Kleene's Normal Form Theorem (see Theorem 2.4).

A set $\mathcal{S}$ of functions over the natural numbers is a subrecursive class *when there exists a total computable function $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

- *for each $e \in \mathbb{N}$, the function $[e]^{\sigma}$ is total*
- *for every $\phi \in \mathcal{S}$ there exists $e \in \mathbb{N}$ such that $\phi(x_1, \ldots, x_n) = [e]^{\sigma}(\langle x_1, \ldots, x_n \rangle)$.*

We say that the function σ generates *the class $\mathcal{S}$.*

So the essence of our definition is that a subrecursive class is a subset of an efficiently enumerable class of total functions. It would have made sense to also require that for any $e \in \mathbb{N}$ there should be $\phi \in \mathcal{S}$ such that $\phi(x_1, \ldots, x_n) = [e]^{\sigma}(\langle x_1, \ldots, x_n \rangle)$. However, the definition above gives us all we need to complete our proofs.

Theorem 3.2. *For any subrecursive class $\mathcal{S}$, there exists an honest function f such that*

$$\psi \in \mathcal{S} \quad \Rightarrow \quad \psi \leq_E f\ .$$

Proof. Assume that $\mathcal{S}$ is generated by the total computable function σ. Let e_{σ} be a computable index for σ, and let

$$f(x) \quad = \quad \mu t[\ t \geq 2^x \wedge (\forall i \leq x)(\exists t_1 \leq t)[\mathcal{T}_1(e_{\sigma}, i, t_1) \wedge (\forall j \leq x)(\exists t_2 \leq t)[\mathcal{T}_1(\mathcal{U}(t_1), j, t_2)]\,]\,]\ .$$

Now, f is a total computable function as σ and $[e]^{\sigma}$ are total computable functions. The graph of f is elementary, moreover, f is monotone and dominates 2^x. Thus, f is honest. A proof of

$$x \geq e \quad \Rightarrow \quad f(x) \geq \mu t[\mathcal{T}_1(\sigma(e), x, t)] \qquad \text{(Claim)}$$

can be found in Section 8 of Kristiansen [10].

Now, let ψ be any function in $\mathcal{S}$. Then, we have e such that $\psi(\vec{x}) = [e]^{\sigma}(\langle\vec{x}\rangle)$. Let $d = \sigma(e)$. By (Claim), we have

$$\psi(\vec{x}) \;=\; [e]^{\sigma}(\langle\vec{x}\rangle) \;=\; \mathcal{U}(\mu t[\mathcal{T}_1(d, \langle\vec{x}\rangle, t)]) \;=\; \mathcal{U}(\mu t \leq f(\langle\vec{x}\rangle)[\mathcal{T}_1(d, \langle\vec{x}\rangle, t)])$$

whenever $\langle\vec{x}\rangle \geq e$. Thus, we have

$$\psi(\vec{x}) \;=\; \mathcal{U}\left((\mu t \leq f(\langle\vec{x}\rangle + e))[\mathcal{T}_1(d, \langle\vec{x}\rangle, t)] \right)$$

for all $\vec{x}$. This proves that ψ is elementary in f. $\square$

The preceding theorem implies that any total computable function belongs to some subrecursive class with strong closure properties: Let ξ be a total computable function. Then $\{\xi\}$ is a subrecursive class (the class is generated by the constant function $\sigma(x) = e$ where e is a computable index for ξ). By the theorem, we have an honest f such that $\xi \leq_E f$. Let $\mathcal{S} = \{\psi \mid \psi \leq_E f\}$. Then $\mathcal{S}$ is a subrecursive class that contains ξ, and moreover, $\mathcal{S}$ has very strong closure properties, e.g., $\mathcal{S}$ has the Ritchie-Cobham property (see Odifreddi [17]). Well, $\mathcal{S}$ will not be closed under primitive recursion, but the closure of $\mathcal{S}$ under primitive recursion, or any other constructive definition scheme that does not introduce partial functions, will still be a subrecursive class that contains ξ. However, all this is not relevant with respect to the soundness of our proofs, but the next corollary will be important in that respect.

Corollary 3.3. *For any subrecursive class $\mathcal{S}$, there exists an honest function f such that $f \notin \mathcal{S}$.*

Proof. Theorem 3.2 yields an honest g such that

$$\psi \in \mathcal{S} \quad \Rightarrow \quad \psi \leq_E g \;.$$

We will prove that $g' \notin \mathcal{S}$. Assume for the sake of contradiction that $g' \in \mathcal{S}$. Then we have $g' \leq_E g$. By Lemma 2.7, we have $g'(x) < g'(x)$ for all sufficiently large x, and that is obviously nonsense. Hence $g' \notin \mathcal{S}$. By Lemma 2.6, g' is honest. Let $f = g'$ and the lemma holds. $\square$

Let β be an irrational number. For any integer m and any positive natural number n, there will be a natural number N such that $|\beta - mn^{-1}| > N^{-1}$. If we can generate such an N in a subrecursive class $\mathcal{S}$, we will say that β is $\mathcal{S}$-irrational.

Definition 3.4. *Let $\mathcal{S}$ be a subrecursive class. An irrational number β is* $\mathcal{S}$-irrational *if there is $\nu : \mathbb{N} \to \mathbb{N}$ in $\mathcal{S}$ such that for any $m \in \mathbb{Z}$ and any $n \in \mathbb{N} \setminus \{0\}$, we have*

$$\left|\beta - \frac{m}{n}\right| \;>\; \frac{1}{\nu(n)} \;.$$

Let $\mathcal{S}$ be a sufficiently large subrecursive class with nice closure properties. The $\mathcal{S}$-irrational numbers are from a certain point of view well-behaved: For any two representations R_1 and R_2 we can subrecursively convert an R_1-representation of an $\mathcal{S}$-irrational number into a an R_2-representation. This appears to be true for all known representations of irrationals like Cauchy sequences, Dedekind cuts, continued fractions, base-2 expansions, base-17 sum approximations from above, and so on. Thus, the $\mathcal{S}$-irrationals that have a Cauchy sequence in $\mathcal{S}$ are exactly the ones that have a continued fraction in $\mathcal{S}$, which again are exactly the ones that have a base-10 expansion in $\mathcal{S}$, and so on.

Let $\mathcal{P}$ denote the class of primitive recursive functions. The concept of a number being $\mathcal{P}$-irrational was introduced by Péter [18]. Lehman [16] proved that the continued fraction of a $\mathcal{P}$-irrational will be in $\mathcal{P}$ if the number has a Cauchy sequence in $\mathcal{P}$. More on $\mathcal{S}$-irrational numbers can be found in Kristiansen [10].

4. Representations and Closure under Addition

Definition 4.1. *A* base *is a natural number strictly greater than 1, and a* base-*b* digit *is a natural number in the set* $\{0, 1, \dots, b-1\}$.

Let M be an integer, let b be a base, and let $\mathtt{D}_1, \dots, \mathtt{D}_n$ be base-b digits. We will use $(M.\mathtt{D}_1\mathtt{D}_2 \dots \mathtt{D}_n)_b$ to denote the rational number $M + \sum_{i=1}^{n} \mathtt{D}_i b^{-i}$.

Let $\mathtt{D}_1, \mathtt{D}_2, \dots$ be an infinite sequence of base-b digits. We say that $(M.\mathtt{D}_1\mathtt{D}_2 \dots)_b$ is the base-b expansion of the real number β *if for all $n \geq 1$ we have*

$$(M.\mathtt{D}_1\mathtt{D}_2 \dots \mathtt{D}_n)_b \quad \leq \quad \beta \quad < \quad (M.\mathtt{D}_1\mathtt{D}_2 \dots \mathtt{D}_n)_b + b^{-n}.$$

(Note that the second inequality is strict: every real number β has a unique base-b expansion.)

Let $(M.\mathtt{D}_1\mathtt{D}_2 \dots)_b$ be the base-b expansion of the real number β. We define the function E_b^β by $E_b^\beta(0) = M$ and $E_b^\beta(i) = \mathtt{D}_i$ (for $i \geq 1$).

For any class of functions $\mathcal{S}$, let

$$\mathcal{S}_{bE} \;=\; \{\beta \mid \beta \text{ is irrational and } E_b^\beta \in \mathcal{S}\}\ .$$

Let $\mathrm{prim}(b)$ denote the set of prime factors of the base b. Mostowski [14] proved that

$$\mathrm{prim}(a) \subseteq \mathrm{prim}(b) \quad \Rightarrow \quad \mathcal{P}_{bE} \subseteq \mathcal{P}_{aE}$$

where $\mathcal{P}$ is the class of primitive recursive functions. Kristiansen [11] proved that

$$\mathrm{prim}(a) \subseteq \mathrm{prim}(b) \quad \Leftrightarrow \quad \mathcal{S}_{bE} \subseteq \mathcal{S}_{aE} \tag{6}$$

holds for any subrecursive class $\mathcal{S}$ closed under elementary operations.

Definition 4.2. *A function $C : \mathbb{N} \to \mathbb{Q}$ is a* Cauchy sequence *for the real number β when $|\beta - C(n)| < 2^{-n}$.*

A function $D : \mathbb{Q} \to \{0, 1\}$ is a Dedekind cut *of the irrational number β when $D(q) = 0$ iff $q < \beta$. We will use D^β to denote the Dedekind cut of β.*

For any class of functions $\mathcal{S}$, let $\mathcal{S}_D$ and $\mathcal{S}_C$, respectively, denote the set of irrational numbers that have Dedekind cuts and Cauchy sequences in $\mathcal{S}$.

Let $\mathcal{S}$ be a subrecursive class closed under elementary operations. Fix a base b. It is straightforward to prove that we have $\mathcal{S}_D \subseteq \mathcal{S}_{bE} \subseteq \mathcal{S}_C$. Let a be a base such that $\mathrm{prim}(a) \nsubseteq \mathrm{prim}(b)$ and $\mathrm{prim}(b) \nsubseteq \mathrm{prim}(a)$. Then, by (6), we have $\mathcal{S}_{bE} \nsubseteq \mathcal{S}_{aE}$ and $\mathcal{S}_{aE} \nsubseteq \mathcal{S}_{bE}$. But of course we also have $\mathcal{S}_D \subseteq \mathcal{S}_{aE} \subseteq \mathcal{S}_C$. It follows that we have $\mathcal{S}_D \subset \mathcal{S}_{bE} \subset \mathcal{S}_C$ for any fixed base b. It will be a corollary of our next theorem that we even have

$$\bigcup_{b=2}^{\infty} \mathcal{S}_{bE} \;\subset\; \mathcal{S}_C\ .$$

Theorem 4.3. *Let f be any honest function. There exists an irrational number γ such that (i) γ has an elementary Cauchy sequence and (ii) for any base b, we have $E^{\gamma}_{b} \nleq_E f$.*

Proof. We will construct an elementary Cauchy sequence C and let γ be the limit of this Cauchy sequence. Our construction will guarantee that $\beta \neq \gamma$ if there is a base a such that $E^{\beta}_{a} \leq_E f$. Thus the theorem holds.

Let $d_0 = 1$ and let $d_{i+1} = f'(d_i)$ (recall that f' is the jump of f, that is, $f'(x) = f^{x+1}(x)$). Furthermore, let $C(0) = 0$ and $C(d_0) = 1/2$. Observe that $C(d_0)$ can be written in the form $(0.1)_2$.

Now, assume that we have determined $C(d_i)$ where $i = \langle k, e, a' \rangle$. Assume also that $C(d_i)$ equals a rational number q_i of the form

$$C(d_i) \;=\; q_i \;=\; (0.\mathtt{D}_1 \ldots \mathtt{D}_{d_i})_a$$

where $a = a' + 2$. We will now determine $C(n)$ when $d_i < n \leq d_{i+1}$. Assume $i + 1 = \langle k', e', b' \rangle$, and let $b = b' + 2$. If $n < d_{i+1}$, we simply let $C(n) = C(d_i)$. If $n = d_{i+1}$, let

$$C(n) \;=\; C(d_{i+1}) \;=\; q_{i+1} \;=\; (0.\dot{\mathtt{D}}_1 \ldots \dot{\mathtt{D}}_{d_{i+1}})_b + \;\varepsilon$$

where $(0.\dot{\mathtt{D}}_1\dot{\mathtt{D}}_2 \ldots)_b$ is the base-b expansion of q_i and

$$\varepsilon \;=\; \begin{cases} 2b^{-d_{i+1}} & \text{if } \mathcal{U}(\mu t \leq d_{i+1}[\mathcal{T}_1(e, d_i, t)]) < q_i \\ -b^{-d_{i+1}} & \text{otherwise.} \end{cases}$$

Observe that q_{i+1} can be written in the form $(0.\mathtt{D}_1 \ldots \mathtt{D}_{d_{i+1}})_b$.

This completes our construction of C. It is easy to see that $|C(n) - C(n+1)| \leq 2^{-n}$ for any n, and thus C will be a Cauchy sequence. Let $\gamma = \lim_{n\to\infty} C(n)$.

(Claim) We have $\beta \neq \gamma$ if there is a base a such that $E^{\beta}_{a} \leq_E f$.

We prove the claim. Since $\gamma \in (0, 1)$ we may also suppose $\beta \in (0, 1)$. Assume $E^{\beta}_{a} \leq_E f$. Then we have $\tilde{E}^{\beta}_{a} \leq_E f$ where

$$\tilde{E}^{\beta}_{a}(x) = \sum_{i=0}^{x} E^{\beta}_{a}(i) a^{-i} \;.$$

Now Theorem 2.4 yields e and k such that

$$\tilde{E}^{\beta}_{a}(x) \;=\; \mathcal{U}(\mu t \leq f^{k}(x)[\mathcal{T}_1(e, x, t)]) \;.$$

Since $f'(x) = f^{x+1}(x)$, we have

$$\tilde{E}^{\beta}_{a}(x) \;=\; \mathcal{U}(\mu t \leq f'(x)[\mathcal{T}_1(e, x, t)])$$

for all x greater than or equal to the fixed number k. Let $i = \langle k, e, a - 2 \rangle$. Then, we have $d_i > k$, and thus

$$\tilde{E}^{\beta}_{a}(d_i) \;=\; \mathcal{U}(\mu t \leq f'(d_i)[\mathcal{T}_1(e, d_i, t)]) \;=\; \mathcal{U}(\mu t \leq d_{i+1}[\mathcal{T}_1(e, d_i, t)]) \;.$$

Moreover, by our construction of C, we know that $C(d_i)$ can be written in the form $(0.\mathtt{D}_1 \ldots \mathtt{D}_{d_i})_a$. We consider the two cases (1) $\tilde{E}^{\beta}_{a}(d_i) < q_i = C(d_i)$ and (2) $\tilde{E}^{\beta}_{a}(d_i) \geq q_i = C(d_i)$.

Case (1). We have

$$(0.\tilde{\mathtt{D}}_1 \dots \tilde{\mathtt{D}}_{d_i})_a \;=\; \tilde{E}^{\beta}_a(d_i) \;<\; q_i \;=\; (0.\mathtt{D}_1 \dots \mathtt{D}_{d_i})_a \;.$$

It is easy to see that $\beta \le q_i$. Thus $\beta < q_i + b^{-d_{i+1}} \le q_{i+1}$. This implies

$$\beta \;<\; \lim_{n\to\infty} C(n) \;=\; \gamma \;.$$

Case (2). We have

$$(0.\tilde{\mathtt{D}}_1 \dots \tilde{\mathtt{D}}_{d_i})_a \;=\; \tilde{E}^{\beta}_a(d_i) \;\ge\; q_i \;=\; (0.\mathtt{D}_1 \dots \mathtt{D}_{d_i})_a \;.$$

So in this case we have $\beta \ge q_i$. Thus $\beta > q_i - b^{-d_{i+1}} \ge q_{i+1}$ which implies

$$\beta \;>\; \lim_{n\to\infty} C(n) \;=\; \gamma \;.$$

This concludes our proof of the claim.

Clause (ii) of our theorem follows obviously from the claim, as well as the fact that γ is irrational. Now we argue that C is elementary. First of all, as f' is an honest function, the function d has an elementary graph (see Lemma 2.6 and its proof). The construction of C guarantees the existence of an elementary function H, such that $q_{i+1} = H(q_i, d_{i+1})$ for all i. Moreover, (the code of) q_i is bounded above by an elementary function of d_i. Using the last two facts, the binary function $\tilde{q}$, defined by

$$\tilde{q}(i,n) \;=\; \begin{cases} q_i & \text{if } d_i \le n \\ 0 & \text{if } d_i > n, \end{cases}$$

can be shown to be elementary using bounded primitive recursion of elementary functions. Note that the relation $d_i \le n$ is elementary and we can compute d_i if it holds, because d has elementary graph. Therefore, given the input n, we can elementarily in n compute the unique i such that $d_i \le n < d_{i+1}$ and give the answer $C(n) = C(d_i) = q_i = \tilde{q}(i,n)$. This is an elementary algorithm for C, thus clause (i) of the theorem also holds. □

Corollary 4.4. *For any subrecursive class $\mathcal{S}$ closed under elementary operations, we have*

$$\bigcup_{b=2}^{\infty} \mathcal{S}_{bE} \;\subset\; \mathcal{S}_C \;.$$

Proof. Let β be any irrational number, and let b be any base. By Theorem 3.2, there exists an honest function f such that

$$E^{\beta}_b \in \mathcal{S} \;\;\Rightarrow\;\; E^{\beta}_b \le_E f \;.$$

Thus, we have $\gamma \notin \bigcup_{b=2}^{\infty} \mathcal{S}_{bE}$ when γ is the irrational given by Theorem 4.3. Moreover, γ has an elementary Cauchy sequence, therefore we have $\gamma \in \mathcal{S}_C$ as $\mathcal{S}$ is closed under elementary operations. The corollary follows since we have $\mathcal{S}_{bE} \subseteq \mathcal{S}_C$ for any fixed b. □

Definition 4.5. *The subrecursive class $\mathcal{E}^2$ is the the closure of* 0 *(zero), S (successor), I^n_i (projections), $\times$ (multiplication) and* max *under composition and bounded primitive recursion.*

The class $\mathcal{E}^2$ is know as the second Grzegorczyk class (our definition differs slightly from Grzegorczyk's original definition). Let $\mathcal{S}$ be a subrecursive class closed under elementary operations. Then we have $\mathcal{E}^2 \subset \mathcal{S}$, and thus any $\mathcal{E}^2$-irrational number will also be $\mathcal{S}$-irrational. It is proved in Georgiev [3] that a real number β is $\mathcal{E}^2$-irrational iff β is not a Liouville number, that is, iff β has a finite irrationality measure.

Lemma 4.6. *An irrational number β is $\mathcal{E}^2$-irrational iff there exists a fixed $k \in \mathbb{N}$ such that*

$$\left|\beta - \frac{x}{y}\right| > \frac{1}{y^k}$$

holds for all $x \in \mathbb{N}$ and all sufficiently large $y \in \mathbb{N}$.

Proof. Assume there is $m \in \mathbb{N}$ such that $|\beta - x/y| > y^{-k}$ holds for all $x \in \mathbb{N}$ and all $y \geq m$. Since β is irrational we can choose $\ell \in \mathbb{N}$, such that $|\beta - x/y| > \ell^{-1}$ for all $x, y \in \mathbb{N}$ with $y < m$. Let $\nu(y) = \max(y^k, \ell)$. Then, we have $\nu \in \mathcal{E}^2$. Moreover, we have $|\beta - x/y| > \nu(y)^{-1}$ for all $x, y \in \mathbb{N}$. This shows that β is $\mathcal{E}^2$-irrational.

It can be proved straightforwardly that for any $\psi \in \mathcal{E}^2$ there exists $k \in \mathbb{N}$ such that $\psi(\vec{x}) \leq \max(\vec{x}, 2)^k$ (use induction on the structure of ψ). Thus, if β is $\mathcal{E}^2$-irrational, there exists k such that we have $|\beta - x/y| > y^{-k}$ for all $x \in \mathbb{N}$ and all $y \geq 2$. □

Theorem 4.7. *Let $\mathcal{S}$ be any subrecursive class closed under elementary operations. For any $\mathcal{E}^2$-irrational number α in $\mathcal{S}_C$ there exists an $\mathcal{E}^2$-irrational number β in $\mathcal{S}_C$, such that $\alpha + \beta$ is irrational and*

$$\alpha + \beta \notin \bigcup_{b=2}^{\infty} \mathcal{S}_{bE} \ .$$

Proof. Let α be an arbitrary $\mathcal{E}^2$-irrational in $\mathcal{S}_C$. By Lemma 4.6, we have $k \in \mathbb{N}$ such that

$$\left|\alpha - \frac{x}{y}\right| > \frac{1}{y^k} \tag{7}$$

holds for all $x \in \mathbb{N}$ and all sufficiently large $y \in \mathbb{N}$.

Let f be an honest function such that $\psi \leq_E f$ for any $\psi \in \mathcal{S}$ (such an f exists by Theorem 3.2). In the proof of Theorem 4.3 we construct γ such that (i) γ has an elementary Cauchy sequence and (ii) for any base b, we have $E_b^\gamma \not\leq_E f$. Let $\beta = \gamma - \alpha$. Then, $\alpha + \beta$ is irrational and for any base b, we have $\alpha + \beta \notin \mathcal{S}_{bE}$. Obviously, $\beta \in \mathcal{S}_C$ (note that $\beta \in \mathbb{Q}$ implies that γ is $\mathcal{E}^2$-irrational, which obviously contradicts that for any base b we have $\gamma \notin \mathcal{S}_{bE}$, see the paragraph after Definition 3.4). Thus, it remains to prove that β is $\mathcal{E}^2$-irrational, and by Lemma 4.6, it suffices to prove the following claim.

(Claim) For all $x \in \mathbb{N}$ and all sufficiently large $y \in \mathbb{N}$, we have

$$\left|\beta - \frac{x}{y}\right| \ > \ \frac{1}{y^{(2k+2)^2}} \ .$$

Before we turn to the proof of the claim, let us recall the sequence $d_0, d_1, d_2, \ldots$ and some facts about γ. We have $d_0 = 1$ and $d_{i+1} = f'(d_i)$ where f' is some fast-growing function. Moreover, γ

is the limit of a sequences of rationals $q_0, q_1, q_2, \ldots$ where each q_i has finite base-a expansion of length d_i and $a = (i)_3 + 2$ (i is considered as the code of a triple). Let b denote $(i+1)_3 + 2$, and thus, q_{i+1} has finite base-b expansion of length d_{i+1}. The construction guarantees that

$$|q_i - q_{i+1}| \;\; \leq \;\; 2b^{-d_{i+1}}$$

for all $i \in \mathbb{N}$. Thus, for all $i \in \mathbb{N}$, we also have

$$|\gamma - q_i| \;\; \leq \;\; \sum_{j=i}^{\infty} \left| q_{j+1} - q_j \right| \;\; \leq \;\; 2b^{-d_{i+1}} \; + \; 2c^{-d_{i+2}} \; + \; \ldots \; < \; 3b^{-d_{i+1}} \tag{8}$$

where $c = (i+2)_3 + 2, \ldots$ (the last inequality follows easily from $d_{i+1} \geq (i+2)^{d_i+1}$ for all $i \in \mathbb{N}$).

For all natural numbers x, i and $y > 0$, we have

$$\begin{aligned} \left|\beta - \frac{x}{y}\right| \;=\; \left|\frac{x}{y} - \beta\right| \;=\; \left|\frac{x}{y} - (\gamma - \alpha) + q_i - q_i\right| \\ = \left|\alpha + \frac{x}{y} - q_i - (\gamma - q_i)\right| \;\geq\; \left|\alpha + \frac{x}{y} - q_i\right| - |\gamma - q_i| \,. \end{aligned}$$

By using (7), (8) and the fact that q_i has denominator a^{d_i} (using its finite base-a expansion of length d_i), we obtain

$$\left|\beta - \frac{x}{y}\right| \; > \; \frac{1}{a^{kd_i} y^k} - \frac{3}{b^{d_{i+1}}} \tag{9}$$

for all sufficiently large $y \in \mathbb{N}$ and all $x, i \in \mathbb{N}$.

We are now ready to prove the claim. Pick an arbitrary but sufficiently large $y \in \mathbb{N}$, and let i be the unique natural number such that $a^{d_i} \leq y^{2k+2} < b^{d_{i+1}}$. Our proof splits into two cases: (1) $a^{d_i} \leq y$ and (2) $y < a^{d_i}$.

Case (1). We have $a^{d_i} \leq y < y^{2k+2} < b^{d_{i+1}}$. By (9), we have

$$\left|\beta - \frac{x}{y}\right| \; > \; \frac{1}{y^{2k}} - \frac{3}{y^{2k+2}} \; > \; \frac{1}{y^{2k+2}} \; > \; \frac{1}{y^{(2k+2)^2}} \; .$$

Thus, the claim holds.

Case (2). This is the tricky case. We have $y < a^{d_i} \leq y^{2k+2} < b^{d_{i+1}}$. We will need the inequality

$$d_{i+1} > (i+2)^{(2k+2)d_i} \tag{10}$$

for all $i \in \mathbb{N}$.

We can without loss of generality assume that (10) holds: Recall that the sequence $d_0, d_1, d_2, \ldots$ is defined by $d_0 = 1$ and $d_{i+1} = f'(d_i)$. If it should not be the case that (10) holds, there will still be an honest function g (depending on k) such that $g \leq_E f$ and $f(x) \leq g(x)$ and (10) holds with $d_{i+1} = g'(d_i)$. When we use such a g in place of f to define the sequence $d_0, d_1, d_2, \ldots$, all our proofs will go through. A similar problem arises in the next section, see Lemma 5.4 and the comment after it.

Next, recall that $a = (i)_3 + 2 < i + 2$ and $b = (i+1)_3 + 2$. Hence, by (10), we have

$$b^{d_{i+1}} \;>\; d_{i+1} \;>\; (i+2)^{(2k+2)d_i} \;>\; a^{(2k+2)d_i} \;=\; (a^{d_i})^{2k+2} \;>\; y^{2k+2}$$

and we conclude that

$$y^{2k+2} \;<\; a^{(2k+2)d_i} \;<\; b^{d_{i+1}} \;. \tag{11}$$

Next we use (9) and (11) to obtain

$$\begin{aligned}
\left|\beta - \frac{x}{y}\right| \;&>\; \frac{1}{a^{kd_i}y^k} - \frac{3}{b^{d_{i+1}}} \;>\; \frac{1}{a^{kd_i}y^k} - \frac{3}{a^{(2k+2)d_i}} \\
&>\; \frac{1}{a^{kd_i}(a^{d_i})^k} - \frac{3}{a^{(2k+2)d_i}} \;=\; \frac{1}{a^{2kd_i}} - \frac{3}{a^{(2k+2)d_i}} \\
&\geq\; \frac{1}{a^{(2k+2)d_i}} \;=\; \frac{1}{(a^{d_i})^{2k+2}} \;\geq\; \frac{1}{y^{(2k+2)^2}}
\end{aligned}$$

for all x and all sufficiently large y. This proves that the claim holds in case (2). □

Corollary 4.8. *Let $\mathcal{S}$ be a subrecursive class closed under elementary operations, and let $\mathcal{S}_C^I$ denote the set of $\mathcal{S}$-irrational numbers that have Cauchy sequences in $\mathcal{S}$. Let X be any set of real numbers such that*

$$\mathcal{S}_C^I \;\subseteq\; X \;\subseteq\; \mathbb{Q} \cup \bigcup_{b=2}^{\infty} \mathcal{S}_{bE} \;.$$

Then X is not closed under addition.

Proof. Theorem 4.7 yields $\mathcal{E}^2$-irrational numbers $\alpha,\beta \in \mathcal{S}_C$ such that $\alpha+\beta$ is irrational and for any base b, $\alpha+\beta \notin \mathcal{S}_{bE}$. Hence, $\alpha+\beta \notin X$. Still we have $\alpha,\beta \in X$ as any $\mathcal{E}^2$-irrational number is also $\mathcal{S}$-irrational. □

Let $\mathcal{S}$ be a subrecursive class closed under primitive recursive operations, and let $\mathcal{S}_R$ be any of the classes of irrational numbers considered in this paper with the exception of $\mathcal{S}_C$, that is, $\mathcal{S}_R$ is one of $\mathcal{S}_{bE}$, $\mathcal{S}_D$, $\mathcal{S}_{b\uparrow}$, $\mathcal{S}_{g\uparrow}$, $\mathcal{S}_{b\downarrow}$, $\mathcal{S}_{g\downarrow}$, $\mathcal{S}_<$, $\mathcal{S}_>$, $\mathcal{S}_{[\,]}$ (some of these classes will be defined below). Furthermore let $\mathcal{S}_R^I$ denote the set of $\mathcal{S}$-irrational numbers in $\mathcal{S}_R$. Then the equality $\mathcal{S}_R^I = \mathcal{S}_C^I$ will hold, and thus the inclusions

$$\mathcal{S}_C^I \;\subseteq\; \mathbb{Q} \cup \mathcal{S}_R \;\subseteq\; \mathbb{Q} \cup \bigcup_{b=2}^{\infty} \mathcal{S}_{bE}$$

will also hold. By Corollary 4.8, the real numbers in the set $\mathbb{Q} \cup \mathcal{S}_R$ will not be closed under addition. Let us restate this informally: If we do not admit unbounded search, no representation of the real numbers, with the exception of Cauchy sequences, will be closed under addition.

5. Sum Approximations

Sum approximations from below and above were discussed in the introductory section. We will now give our formal definitions. From now on we will restrict our attention to irrational numbers between 0 and 1. This entails no loss of generality.

Definition 5.1. *Let* $(0.D_1D_2\dots)_b$ *be the base-b expansion of an irrational* α.
The base-b sum approximation from below *of* α *is the function* $\hat{A}^\alpha_b : \mathbb{N} \to \mathbb{Q}$ *defined by* $\hat{A}^\alpha_b(0) = 0$ *and* $\hat{A}^\alpha_b(n+1) = E^\alpha_b(m)b^{-m}$ *where m is the least m such that*

$$\sum_{i=0}^{n} \hat{A}^\alpha_b(i) \quad < \quad (0.D_1\dots D_m)_b \ .$$

If D *is a base-b digit, then* $\overline{D}$ *denotes the complement digit of* D*, that is,* $\overline{D} = (b-1) - D$.
The base-b sum approximation from above *of* α *is the function* $\check{A}^\alpha_b : \mathbb{N} \to \mathbb{Q}$ *defined by* $\check{A}^\alpha_b(0) = 0$ *and* $\check{A}^\alpha_b(n+1) = \overline{E^\alpha_b(m)}b^{-m}$ *where m is the least m such that*

$$1 - \sum_{i=0}^{n} \check{A}^\alpha_b(i) \quad > \quad 1 - (0.\overline{D}_1\dots\overline{D}_m)_b \ .$$

The general sum approximation from below *of* α *is the function* $\hat{G}^\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{Q}$ *defined by* $\hat{G}^\alpha(b,n) = \hat{A}^\alpha_b(n)$*. The* general sum approximation from above *of* α *is the function* $\check{G}^\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{Q}$ *defined by* $\check{G}^\alpha(b,n) = \check{A}^\alpha_b(n)$*. (Let* $\hat{G}^\alpha(b,n) = \check{G}^\alpha(b,n) = 0$ *if* $b < 2$*.)*
For any class of functions $\mathcal{S}$*, let*

$$\mathcal{S}_{b\uparrow} \ = \ \{\, \alpha \mid \hat{A}^\alpha_b \in \mathcal{S} \,\} \quad \textit{and} \quad \mathcal{S}_{b\downarrow} \ = \ \{\, \alpha \mid \check{A}^\alpha_b \in \mathcal{S} \,\}$$

and let

$$\mathcal{S}_{g\uparrow} \ = \ \{\, \alpha \mid \hat{G}^\alpha \in \mathcal{S} \,\} \quad \textit{and} \quad \mathcal{S}_{g\downarrow} \ = \ \{\, \alpha \mid \check{G}^\alpha \in \mathcal{S} \,\} \ .$$

The functions $\hat{A}^\alpha_b$ and $\check{A}^\alpha_b$ are not defined if α is rational. When we use the notation it is understood that α is irrational.
It follows straightforwardly from the definitions above that

$$\alpha \ = \ \sum_{i=0}^{\infty} E^\alpha_b(i)b^{-i} \ = \ \sum_{i=0}^{\infty} \hat{A}^\alpha_b(i) \ = \ 1 - \sum_{i=0}^{\infty} \check{A}^\alpha_b(i) \ .$$

Sum approximations were introduced in Kristiansen [10] and studied further in Kristiansen [11]. It is proved in [10] that we have

$$\mathcal{S}_{b\downarrow} \not\subseteq \mathcal{S}_{b\uparrow} \quad \text{and} \quad \mathcal{S}_{b\uparrow} \not\subseteq \mathcal{S}_{b\downarrow}$$

for any $\mathcal{S}$ closed under elementary operations. Furthermore, it is proved in [11] that we have

$$\text{prim}(a) \subseteq \text{prim}(b) \quad \Leftrightarrow \quad \mathcal{S}_{b\downarrow} \subseteq \mathcal{S}_{a\downarrow} \tag{12}$$

and

$$\text{prim}(a) \subseteq \text{prim}(b) \quad \Leftrightarrow \quad \mathcal{S}_{b\uparrow} \subseteq \mathcal{S}_{a\uparrow} \tag{13}$$

for any $\mathcal{S}$ closed under primitive recursive operations. The proofs of (12) and (13) are extensions of the proof of (6).

Let $\mathcal{S}$ be any subrecursive class closed under primitive recursive operations. It is trivial that the two inclusions

$$\mathcal{S}_{g\downarrow} \subseteq \mathcal{S}_{b\downarrow} \quad \text{and} \quad \mathcal{S}_{g\uparrow} \subseteq \mathcal{S}_{b\uparrow}$$

hold for any base b, and it follows straightforwardly from (12) and (13) that these inclusions indeed are strict. In this section we prove a much stronger result, namely that the next theorem holds.

Theorem 5.2. *For any subrecursive class $\mathcal{S}$ that is closed under primitive recursive operations, we have*

$$(i)\;\; \mathcal{S}_{g\downarrow} \;\subset\; \bigcap_{b=2}^{\infty} \mathcal{S}_{b\downarrow} \quad \textit{and} \quad (ii)\;\; \mathcal{S}_{g\uparrow} \;\subset\; \bigcap_{b=2}^{\infty} \mathcal{S}_{b\uparrow} .$$

We will use the number α discussed in the introductory section to prove Theorem 5.2. The number will be in the set $\bigcap_b \mathcal{S}_{b\uparrow}$ but not in the set $\mathcal{S}_{g\uparrow}$. A number with a symmetric definition will be in the set $\bigcap_b \mathcal{S}_{b\downarrow}$ but not in the set $\mathcal{S}_{g\downarrow}$.

Definition 5.3. *Let P_i denote the i^{th} prime ($P_0 = 2, P_1 = 3, \ldots$). We define the auxiliary function g by*

$$g(0) \;=\; 1 \quad \textit{and} \quad g(j+1) \;=\; P_j^{2(j+2)(g(j)+1)^3} \; .$$

For any honest function f and any $n \in \mathbb{N}$, we define the rational number α_n^f and the number α^f by

$$\alpha_n^f \;=\; \sum_{i=0}^{n} P_i^{-h(i)} \quad \textit{and} \quad \alpha^f \;=\; \lim_{n\to\infty} \alpha_n^f$$

where $h(i) = g(f(i) + i)$ (for any $i \in \mathbb{N}$).

It is easy to see that the functions g and h in the preceding definition are strictly increasing honest functions. Only the fact that g has elementary graph requires some explanation: for all $x, y \in \mathbb{N}$ the equality $g(x) = y$ holds iff

$$\exists s[\, (s)_0 = 1 \;\wedge\; \forall j \leq x[\, (s)_{j+1} \;=\; P_j^{2(j+2)((s)_j+1)^3} \;] \;\wedge\; (s)_x \;=\; y \,] \; .$$

The formula states that s is the code of a sequence of length $x+1$ and its j^{th} element $(s)_j$ is equal to $g(j)$ for all $j \leq x$. All operations, relations and functions involved in the formula are elementary. Since g is increasing and our coding of sequences is monotone, the existential quantifier on s can be bounded by the code of the sequence $y, \ldots, y$ ($x+1$ times). This code is is elementary in x, y, and the elementary relations are closed under bounded existential quantification.

Lemma 5.4. *For any $n \in \mathbb{N}$, we have*

$$P_n^{2(n+2)(h(n)+1)^3} \;<\; h(n+1) \; .$$

Proof. Keep in mind that both g and f are increasing functions (we have $f(x) \leq f(x+1)$ since f is honest). By Definition 5.3, we have

$$h(n+1) \;=\; g(f(n+1)+n+1) \;>\; P_n^{2(n+2)(g(f(n+1)+n)+1)^3}$$
$$\geq\; P_n^{2(n+2)(g(f(n)+n)+1)^3} \;=\; P_n^{2(n+2)(h(n)+1)^3} \; .$$

□

Lemma 5.4 explains why we use the auxiliary functions g and h to define α^f. The number α^f is determined by the arbitrary honest function f, but in our proofs we need that

$$\alpha^f \;=\; \lim_{n\to\infty} \sum_{i=0}^{n} P_i^{-h(i)}$$

where h is a function that satisfies the growth property given in Lemma 5.4. An arbitrary honest function does not necessarily possess this property.

Lemma 5.5. *The number α^f is a Liouville number (and thus transcendental).*

Proof. It suffices to show that for every $n \in \mathbb{N} \setminus \{0\}$ there exist $p, q \in \mathbb{N}$ with $q > 1$, such that

$$0 \;<\; \left|\alpha^f - \frac{p}{q}\right| \;<\; \frac{1}{q^n}.$$

Fix $n \in \mathbb{N} \setminus \{0\}$, and pick relatively prime p and q such that $pq^{-1} = \alpha^f_n$. Thus $q = \prod_{i=0}^{n} P_i^{h(i)} \leq P_n^{(n+1)h(n)}$. Obviously

$$0 \;<\; \left|\alpha^f - \alpha^f_n\right| \;=\; P_{n+1}^{-h(n+1)} \;+\; P_{n+2}^{-h(n+2)} \;+\; \ldots$$

and since h is strictly increasing, the last infinite sum is bounded above by

$$P_{n+1}^{-h(n+1)} \;+\; P_{n+1}^{-h(n+1)-1} \;+\; \ldots \;\leq\; P_{n+1}^{-h(n+1)+1}.$$

Moreover, as $h(n) > n$, it follows from Lemma 5.4 that

$$h(n+1) - 1 \;\geq\; P_n^{n(n+1)h(n)}.$$

Thus

$$0 \;<\; \left|\alpha^f - \frac{p}{q}\right| \;\leq\; \frac{1}{P_{n+1}^{h(n+1)-1}} \;<\; \frac{1}{h(n+1)-1} \;\leq\; \frac{1}{P_n^{n(n+1)h(n)}} \;\leq\; \frac{1}{q^n}.$$

□

Lemma 5.6. *For any $j \in \mathbb{N}$ and any base b, we have*

(i) if P_i divides b for all $i \leq j$, then α^f_j has a finite base-b expansion of length $h(j)$, that is, α^f_j can be written in the form $(0.D_1 \ldots D_{h(j)})_b$,

(ii) if P_i does not divide b for some $i \leq j$, then α^f_j has an infinite base-b expansion of the form $(0.D_1 \ldots D_s(D_{s+1} \ldots D_t)^\omega)_b$ where $t < \prod_{i=0}^{j} P_i^{h(i)}$.

Proof. For any $j \in \mathbb{N}$ and any base b, we have

$$\alpha^f_j \;=\; \frac{m}{\prod_{i=0}^{j} P_i^{h(i)}} \tag{14}$$

where m is the sum of $j + 1$ summands. Each of these summands is divisible by all the primes in the list $P_0, P_1, \ldots, P_j$ with the exception of exactly one of them. It follows that m is relatively prime to all the primes in the list, and thus, the fraction (14) is in its lowest terms.

Now, recall that a rational number $q \in (0, 1)$ has a base-b expansion of length k iff there exists $m < b^k$, such that $q = mb^{-k}$.

Assume that P_i divides b for all $i \leq j$. Since h is increasing, there exists a natural number ℓ such that

$$\alpha^f_j \;=\; \frac{m}{\prod_{i=0}^{j} P_i^{h(i)}} \;=\; \frac{\ell \times m}{\ell \times \prod_{i=0}^{j} P_i^{h(i)}} \;=\; \frac{\ell \times m}{b^{h(j)}} \;.$$

This proves (i).

Assume that P_i does not divide b for some $i \leq j$. Furthermore, assume for the sake of contradiction that there exist $k, m' \in \mathbb{N}$ such that

$$\alpha^f_j \;=\; \frac{m}{\prod_{i=0}^{j} P_i^{h(i)}} \;=\; \frac{m'}{b^k} \;.$$

Then we have $m' \prod_{i=0}^{j} P_i^{h(i)} = mb^k$. This contradicts that mb^k has a unique prime factorization as P_i does not divide mb^k for some $i \leq j$. As α^f_j is rational, α^f_j has an infinite base-b expansion of the form $0.\mathtt{D}_1 \ldots \mathtt{D}_s(\mathtt{D}_{s+1} \ldots \mathtt{D}_t)^\omega$. We need to argue that $t < \prod_{i=0}^{j} P_i^{h(i)}$.

We have proved that the fraction (14) is in its lowest terms. Let $\prod_{i=0}^{j} P_i^{h(i)} = d_1 \times d_2$ where d_1 is relatively prime to b and as large as possible. Then s is the least natural number such that d_2 divides b^s, and the length of the period $t - s$ is the multiplicative order of b modulo d_1. It follows straightforwardly that $t < \prod_{i=0}^{j} P_i^{h(i)}$. For more details, see Chapter 9 of Hardy & Wright [5]. □

Theorem 5.7. *For any honest function f, we have $f \leq_{PR} \hat{G}^{\alpha^f}$ (f is primitive recursive in $\hat{G}^{\alpha^f}$).*

Proof. (To improve the readability, we will write α in place of α^f.) Fix $n \in \mathbb{N}$, and let b be the base $b = \prod_{i=0}^{n} P_i$. By Lemma (5.6) (i), α_n has a finite base-b expansion of length $h(n)$. By the definition of α, we have

$$\alpha \;=\; \alpha_n \;+\; P_{n+1}^{-h(n+1)} \;+\; P_{n+2}^{-h(n+2)} \;+\; \ldots \;.$$

Hence, for any $j > h(n)$, we have

$$\hat{G}^\alpha(b, j) \;\leq\; P_{n+1}^{-h(n+1)} \;+\; P_{n+2}^{-h(n+2)} \;+\; \ldots$$

which easily implies $\hat{G}^\alpha(b, j) \leq P_{n+1}^{-h(n+1)+1}$ (exactly as in the proof of Lemma 5.5). Hence we also have $(\hat{G}^\alpha(b, j))^{-1} \geq P_{n+1}^{h(n+1)-1} > h(n+1) - 1$ for any $j > h(n)$.

The considerations above show that we can compute $h(n+1)$ by the following algorithm:

- assume that $h(n)$ is computed
- compute $b = \prod_{i=0}^{n} P_i$
- search for y such that $y < (\hat{G}^\alpha(b, h(n) + 1))^{-1} + 1$ and $h(n+1) = y$
- give the output y.

This algorithm is primitive recursive in $\hat{G}^{\alpha}$: The computation of b is an elementary computation. The relation $h(x) = y$ is elementary, and thus the search for y is elementary in $h(n)$ and $\hat{G}^{\alpha}$. This proves that h is primitive recursive in $\hat{G}^{\alpha}$. But then f will also be primitive recursive in $\hat{G}^{\alpha}$ as the graph of f is elementary and $f(n) \leq h(n)$ (for any $n \in \mathbb{N}$). This proves that $f \leq_{PR} \hat{G}^{\alpha}$. $\square$

Theorem 5.8. *Let f be any honest function, and let b be any base. The function $\hat{A}_b^{\alpha^f}$ is elementary.*

Proof. First we will state and prove a few claims. Let

- $j \in \mathbb{N}$ be such that $P_j > b$
- $(0.\mathtt{D}_1\mathtt{D}_2\dots)_b$ be the base-b expansion of α^f_j
- $(0.\dot{\mathtt{D}}_1\dot{\mathtt{D}}_2\dots)_b$ be the base-b expansion of α^f_{j+1}
- $M = M(j) = P_j^{(j+1)h(j)}$ and $M' = M'(j) = h(j+1)$.

(Claim 1) There are maximum M consecutive zeros in the base-b expansion of α^f_j, that is, for any $k \in \mathbb{N} \setminus \{0\}$ there exists $i \in \mathbb{N}$ such that

$$k \leq i < k + M \quad \text{and} \quad \mathtt{D}_i \neq 0 \ .$$

(Claim 2) The first $M' - M$ digits of the base-b expansions of α^f_j and α^f_{j+1} coincide, that is

$$i \leq M' - M \quad \Rightarrow \quad \mathtt{D}_i = \dot{\mathtt{D}}_i$$

and moreover, these digits also coincide with the corresponding digits of the base-b expansion of α^f.

By Lemma 5.6 (ii), α^f_j has a base-b expansion of the form $0.\mathtt{D}_1\dots\mathtt{D}_s(\mathtt{D}_{s+1}\dots\mathtt{D}_t)^{\omega}$ with $t < \prod_{i=0}^{j} P_i^{h(i)}$. Therefore

$$t - s \quad \leq \quad t \quad < \quad \prod_{i=0}^{j} P_i^{h(i)} \quad \leq \quad P_j^{(j+1)h(j)} \quad = \quad M \ . \tag{15}$$

Thus, the first claim holds since any M consecutive digits of α^f_j contain all the digits $\mathtt{D}_{s+1}, \dots, \mathtt{D}_t$ of at least one period.

We turn to the proof of the second claim. Since $b^{M'} < P_j^{M'} = P_j^{h(j+1)} < P_{j+1}^{h(j+1)}$, we have

$$\alpha^f_j \quad < \quad \alpha^f_{j+1} \quad = \quad \alpha^f_j \ + \ P_{j+1}^{-h(j+1)} \quad \leq \quad \alpha^f_j \ + \ b^{-M'} \ . \tag{16}$$

At least one digit in the period $\mathtt{D}_{s+1}\dots\mathtt{D}_t$ is different from $b - 1$. Thus, it follows from (16) that

$$\mathtt{D}_i = \dot{\mathtt{D}}_i \text{ for any } i \leq M' - (t - s). \tag{17}$$

We obtain from (15) and (17) that the first $M' - M$ digits of the base-b expansions of α^f_j and α^f_{j+1} coincide. Moreover, since $M'(j)$ is strictly increasing in j, we have

$$\alpha^f_j \;<\; \alpha^f_{j+k} \;\leq\; \alpha^f_j \;+\; \sum_{i<k} b^{-M'(j+i)} \;\leq\; \alpha^f_j \;+\; b^{-M'(j)+1}$$

for any $k \geq 1$. When we let $k \to \infty$, we conclude as above that the first $M' - M$ digits of α^f_j and α^f coincide. This proves the second claim. We need a third claim before we can turn to the proof of the very theorem.

For all $j \in \mathbb{N}$, we have

$$M(j)^2 + M(j) + 1 < M'(j). \qquad \textbf{(Claim 3)}$$

By the definition of M, we have

$$\begin{aligned} M(j)^2 + M(j) + 1 \;&=\; P_j^{2(j+1)h(j)} \;+\; P_j^{(j+1)h(j)} \;+\; 1 \\ &<\; 3P_j^{2(j+1)h(j)} \;<\; P_j^{2(j+1)h(j)+2} \;<\; P_j^{2(j+1)(h(j)+1)^3}\,. \end{aligned}$$

Furthermore, by Lemma 5.4 and the definition of M', we have

$$M(j)^2 + M(j) + 1 \;<\; P_j^{2(j+1)(h(j)+1)^3} \;<\; h(j+1) \;=\; M'(j)\,.$$

This proves that (Claim 3) holds.

We are now prepared to prove our theorem, that is, to prove that the function $\hat{A}^{\alpha^f}_b$ is elementary. To improve the readability, we will write α in place of α^f throughout this proof.

Fix the least m such that $P_m > b$. We will argue that we can compute the rational number $\hat{A}^{\alpha}_b(n)$ elementarily in n when $n \geq M(m)$. Note that $M(m)$ is a fixed number (it does not depend on n). Thus, we can compute $\hat{A}^{\alpha}_b(n)$ by a trivial algorithm when $n < M(m)$ (use a huge table).

Assume $n \geq M(m)$. We will now give an algorithm for computing $\hat{A}^{\alpha}_b(n)$ elementarily in n.

Step 1 of the algorithm: Compute (the unique) j such that

$$M(j) \;\leq\; n \;<\; M(j+1) \tag{18}$$

(end of Step 1).

Step 1 is a computation elementary in n as M has elementary graph. So is the next step as M' also has elementary graph.

Step 2 of the algorithm: Check if the relation

$$n^2 + 1 \;<\; M'(j) \;-\; M(j) \tag{19}$$

holds. If it holds, carry out step 3A, otherwise, carry out step 3B (end of Step 2).

Step 3A of the algorithm: Compute α_j. Then compute base-b digits $\mathtt{D}_1, \ldots, \mathtt{D}_{n^2+1}$ such that

$$(0.\mathtt{D}_1\mathtt{D}_2 \ldots \mathtt{D}_{n^2+1})_b \;\leq\; \alpha_j < \; (0.\mathtt{D}_1\mathtt{D}_2 \ldots \mathtt{D}_{n^2+1})_b \;+\; b^{-(n^2+1)}\,.$$

Find k such that $\mathtt{D}_k$ is the n^{th} digit different from 0 in the sequence $\mathtt{D}_1, \dots, \mathtt{D}_{n^2+1}$. Give the output $\mathtt{D}_k b^{-k}$ (end of Step 3A).

Recall that $\alpha_j = \sum_{i=0}^{j} P_i^{-h(i)}$. We can compute α_j elementarily in n since $h(0), h(1), \dots, h(j) < M(j) \leq n$ and h is honest. Thus, we can also compute the base-b digits $\mathtt{D}_1, \mathtt{D}_2, \dots, \mathtt{D}_{n^2+1}$ elementarily in n. In order to prove that our algorithm is correct, we must verify that

(A) at least n of the digits $\mathtt{D}_1, \mathtt{D}_2, \dots, \mathtt{D}_{n^2+1}$ are different from 0, and

(B) $\mathtt{D}_1, \mathtt{D}_2, \dots, \mathtt{D}_{n^2+1}$ coincide with the first $n^2 + 1$ digits of α.

By (Claim 1) the sequence $\mathtt{D}_{kM(j)+1}, \mathtt{D}_{kM(j)+2}, \dots, \mathtt{D}_{(k+1)M(j)}$ contains at least one non-zero digit (for any $k \in \mathbb{N}$). Thus, (A) holds since $n \geq M(j)$. It follows straightforwardly from (Claim 2) and (19) that (B) holds. This proves that the output $\mathtt{D}_k b^{-k} = \hat{A}_b^{\alpha}(n)$.

Step 3B of the algorithm: Compute α_{j+1} and $M(j+1)$. Then proceed as in step 3A with α_{j+1} in place of α_j and $nM(j+1)$ in place of n^2 (end of Step 3B).

Step 3B is only executed when $M'(j) - M(j) \leq n^2 + 1$. Thus, we have $M'(j) = h(j+1) \leq n^2 + n + 1$. This entails that we can compute $h(j+1)$, and thus also α_{j+1} and $M(j+1)$, elementarily in n.

Now (Claim 3) yields

$$M(j+1)^2 \ + \ M(j+1) \ + \ 1 \ < \ M'(j+1)$$

which together with (18) implies

$$nM(j+1) \ + \ 1 \ < \ M'(j+1) \ - \ M(j+1) \, .$$

As in step 3A, there will be at least n non-zero digits among the first $nM(j+1)$ digits of α_{j+1}. Moreover, the first $nM(j+1)$ digits of α_{j+1} coincide with the corresponding digits of α. □

We are now ready to prove Theorem 5.2. The inclusion $\mathcal{S}_{g\uparrow} \subseteq \bigcap_b \mathcal{S}_{b\uparrow}$ is trivial. Let f be an honest function such that $f \notin \mathcal{S}$. Such an f exists by Corollary 3.3. By Theorem 5.8, we have $\alpha^f \in \bigcap_b \mathcal{S}_{b\uparrow}$. By Theorem 5.7, we have $\alpha^f \notin \mathcal{S}_{g\uparrow}$. This proves that $\mathcal{S}_{g\uparrow} \subset \bigcap_b \mathcal{S}_{b\uparrow}$. A symmetric argument will prove that also the inclusion $\mathcal{S}_{g\downarrow} \subset \bigcap_b \mathcal{S}_{b\downarrow}$ holds.

Theorem 5.2 should be compared to a result of Lehman's [16]. He proves that

$$\mathcal{S}_D \ \subset \ \bigcap_{b=2}^{\infty} \mathcal{S}_{bE}$$

where $\mathcal{S}$ is the class of primitive recursive functions and $\mathcal{S}_D$ is the set of irrationals that have Dedekind cuts in $\mathcal{S}$. Let the function E^β be defined by $E^\beta(b, n) = E_b^\beta(n)$. It is not hard to see that E^β is primitive recursive iff the Dedekind cut of β is primitive recursive. Thus, Lehman's result on base expansions is analogous to our result on sum approximations.

6. Left and Right Best Approximations

Definition 6.1. *Let α be an irrational number in the interval* $(0, 1)$.

Let a and b be relatively prime natural numbers with $b > 0$. The fraction a/b is a left best approximant *of α if we have $c/d \leq a/b < \alpha$ or $\alpha < c/d$ for any natural numbers c, d with $0 < d \leq b$. The fraction a/b is a* right best approximant *of α if we have $\alpha < a/b \leq c/d$ or $c/d < \alpha$ for any natural numbers c, d with $0 < d \leq b$.*

A left best approximation *of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that*

$$0 \; = \; \frac{a_0}{b_0} \; < \; \frac{a_1}{b_1} \; < \; \frac{a_2}{b_2} \; < \; \ldots$$

and each a_i/b_i is a left best approximant of α. A right best approximation *of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that*

$$1 \; = \; \frac{a_0}{b_0} \; > \; \frac{a_1}{b_1} \; > \; \frac{a_2}{b_2} \; > \; \ldots$$

and each a_i/b_i is a right best approximant of α. Clearly, both sequences converge to α.

Let $\mathcal{S}_<$ denote the set of irrational numbers that have a left best approximation in the subrecursive class $\mathcal{S}$, and let $\mathcal{S}_>$ denote the set of irrational numbers that have a right best approximation in $\mathcal{S}$.

Lemma 6.2. *(i) $D^\beta \leq_E \hat{G}^\beta$ and $D^\beta \leq_E \check{G}^\beta$. (ii) Let $\{a_i/b_i\}_{i\in\mathbb{N}}$ and $\{a'_i/b'_i\}_{i\in\mathbb{N}}$ be left and right, respectively, best approximations of β. Then we have $D^\beta \leq_E \{a_i/b_i\}_{i\in\mathbb{N}}$ and $D^\beta \leq_E \{a'_i/b'_i\}_{i\in\mathbb{N}}$.*

Proof. Let c and d be natural numbers such that $0 < c/d < 1$. Observe that $D^\beta(c/d) = 0$ iff $\hat{G}^\beta(d, 1) \geq c/d$ and that $D^\beta(c/d) = 1$ iff $\check{G}^\beta(d, 1) \geq 1 - c/d$. Thus, it is easy to see that (i) holds.

We turn to the proof of (ii). Observe that we have $i < b_i$ for any i. Hence, we have $D^\beta(c/d) = 0$ iff $c/d < a_d/b_d$. This shows that the Dedekind cut of β is elementary in a left best approximation of β. A symmetric argument shows that the Dedekind cut of β is elementary in a right best approximation of β.

□

Theorem 6.3. *For any subrecursive class $\mathcal{S}$ closed under primitive recursion, we have*

$$\mathcal{S}_{g\uparrow} \; = \; \mathcal{S}_< \quad \textit{and} \quad \mathcal{S}_{g\downarrow} \; = \; \mathcal{S}_> \; .$$

Proof. We prove $\mathcal{S}_< \subseteq \mathcal{S}_{g\uparrow}$. Let $\{a_i/b_i\}_{i\in\mathbb{N}}$ be a left best approximation of β. We will argue that $\hat{G}^\beta$ can be computed primitive recursively in $\{a_i/b_i\}_{i\in\mathbb{N}}$. We assume that $\hat{G}^\beta(b, i)$ is computed for all $i \leq n$. The algorithm below shows how to compute $\hat{G}^\beta(b, n+1)$.

Step 1 of the algorithm: Compute

$$\frac{c}{d} \; = \; \sum_{i=0}^{n} \hat{G}^\beta(b, i) \; .$$

Let $c'/d' = a_d/b_d$ (end of Step 1).

We obviously have $d' > d$. Thus, since c'/d' is a left best approximant of β, we have $c/d < c'/d' < \beta$.

Step 2 of the algorithm: Search for the least $m \in \mathbb{N}$ such that $c/d + 1/b^m \leq c'/d'$. Use the Dedekind cut of β to find $k \leq m$ and non-zero base-b digit $\mathtt{D}$ such that

$$\frac{c}{d} + \frac{\mathtt{D}}{b^k} \quad \leq \quad \beta \quad < \quad \frac{c}{d} + \frac{\mathtt{D}}{b^k} + \frac{1}{b^k} \; .$$

Give the output $\mathtt{D}/b^k$ (end of Step 2).

It is easy to see that the algorithm is correct, that is, we have $\hat{G}^\beta(b, n+1) = \mathtt{D}/b^k$. It is also easily seen that the algorithm is primitive recursive in the left best approximation $\{a_i/b_i\}_{i\in\mathbb{N}}$. The algorithm uses course-of-values recursion, but it is well-known that such recursion can be reduced to primitive recursion. The algorithm uses the Dedekind cut of β. By Lemma 6.2, the Dedekind cut of β is primitive recursive in $\{a_i/b_i\}_{i\in\mathbb{N}}$. This completes the proof of $\mathcal{S}_< \subseteq \mathcal{S}_{g\uparrow}$.

Next we prove $\mathcal{S}_{g\uparrow} \subseteq \mathcal{S}_<$. We give an algorithm for computing a left best approximation $\{a_i/b_i\}_{i\in\mathbb{N}}$ of β, and we argue that our algorithm is primitive recursive in $\hat{G}^\beta$. Let $a_0/b_0 = 0/1$ and $a_1/b_1 = c/N$ where $N > 1$ and $c > 0$ are the unique natural numbers such that

$$\frac{1}{N} < \beta < \frac{1}{N-1} \quad \text{and} \quad \frac{c}{N} < \beta < \frac{c+1}{N}.$$

It is easy to see that c/N is a left best approximant of α. Assume that $n \geq 1$ and that a_n/b_n already is computed. The algorithm shows how to compute a_{n+1}/b_{n+1}.

Step 1 of the algorithm: Let

$$\frac{c'}{d'} \quad = \quad \frac{a_n}{b_n} \; + \; \hat{G}^\beta(b_n, 2)$$

with relatively prime c' and d' (end of Step 1).

We observe that $a_n/b_n = \hat{G}^\beta(b_n, 1)$. Hence, we have $a_n/b_n < c'/d' < \beta < (c'+1)/d'$, and c'/d' will be a left best approximant unless there exists a fraction c/d such that $d < d'$ and $c'/d' < c/d < \beta < (c+1)/d$.

Step 2 of the algorithm: Use the Dedekind cut of β to search for c and the smallest d such that $b_n < d \leq d'$ and $c'/d' \leq c/d < \beta < (c+1)/d$. Let $a_{n+1}/b_{n+1} = c/d$ (end of Step 2).

Given the comments above, it should be pretty clear that the sequence $\{a_i/b_i\}_{i\in\mathbb{N}}$ computed by the algorithm indeed is a left best approximation of β. Step 1 of the algorithm is obviously primitive recursive in $\hat{G}^\beta$. Step 2 uses the Dedekind cut of β. Lemma 6.2 states that this Dedekind cut is primitive recursive in $\hat{G}^\beta$, and thus it is easy to see that Step 2 is also primitive recursive in $\hat{G}^\beta$. This completes the proof of $\mathcal{S}_{g\uparrow} \subseteq \mathcal{S}_<$, and we conclude that $\mathcal{S}_{g\uparrow} = \mathcal{S}_<$. The proof of $\mathcal{S}_{g\downarrow} = \mathcal{S}_>$ is symmetric. □

7. More on Sum Approximations

Any irrational number β in the interval $(0, 1)$ can be uniquely written in the form

$$\alpha \;=\; 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where each a_i is a positive integer. The sequence $a_1, a_2, a_3, \ldots$ is called the *continued fraction* of β. For more on continued fractions see Hardy & Wright [5] or Khintchine [6].

Let $\mathcal{S}$ be a subrecursive class closed under primitive recursive operations, and let $\mathcal{S}_{[\,]}$ denote the set of irrationals that have continued fractions in $\mathcal{S}$. It was proved in Kristiansen [10] that

$$\mathcal{S}_{g\downarrow} \cap \mathcal{S}_{g\uparrow} \;=\; \mathcal{S}_{[\,]}$$

and it was conjectured in [10] that

$$\mathcal{S}_{g\downarrow} \nsubseteq \mathcal{S}_{g\uparrow} \quad \text{and} \quad \mathcal{S}_{g\uparrow} \nsubseteq \mathcal{S}_{g\downarrow} \,. \tag{20}$$

In this section we will prove that (20) indeed holds.

Intuitively, it is not very hard to see why we have $\mathcal{S}_{b\downarrow} \nsubseteq \mathcal{S}_{b\uparrow}$ and $\mathcal{S}_{b\uparrow} \nsubseteq \mathcal{S}_{b\downarrow}$ when b is a fixed base and $\mathcal{S}$ is closed under elementary operations. Let us consider the case $\mathcal{S}_{2\downarrow} \nsubseteq \mathcal{S}_{2\uparrow}$. Pick a strictly increasing honest function f such that $f \notin \mathcal{S}$, and let

$$\beta \;=\; \sum_{i=257}^{\infty} \frac{1}{2^{f(i)}} \,.$$

Then the function $\check{A}^{\beta}_{2}$ will be in $\mathcal{S}$ whereas the function $\hat{A}^{\beta}_{2}$ will not. Let us see why.

Assume for the sake of contradiction that $\hat{A}^{\beta}_{2}$ is in $\mathcal{S}$. Let $g(x) = (\hat{A}^{\beta}_{2}(x+1))^{-1}$. Then g will also be in $\mathcal{S}$ as $\mathcal{S}$ is closed under standard operations on rationals. Obviously we have $f(x) \leq g(x)$, and thus we have $f(x) = (\mu y \leq g(x))[f(x) = y]$. Now, as the relation $f(x) = y$ is elementary and $\mathcal{S}$ is closed under the bounded μ-operator, we can conclude that f is in $\mathcal{S}$. This contradicts our choice of f.

In order to see that $\check{A}^{\beta}_{2}$ is in $\mathcal{S}$, observe that the base-2 expansion of β is of the form

$$\texttt{0.}\underbrace{\texttt{000000.............0}}_{\text{lots of zeros}}\texttt{1}\underbrace{\texttt{000000.............0}}_{\text{even more zeros}}\texttt{10000....} \tag{21}$$

Now, the i^{th} fractional digit of the base-2 expansion of $1 - \beta$ will be `1` if the i^{th} fractional digit of (21) is `0`, and the other way around, `0` if the i^{th} fractional digit of (21) is `1`. This entails that $\check{A}^{\beta}_{2}(1) = 2^{-1}$ and $\check{A}^{\beta}_{2}(2) = 2^{-2}$ and so on, and thus, the function $(\check{A}^{\beta}_{2}(x))^{-1}$ will not increase too fast to be in $\mathcal{S}$. The function will be of elementary growth rate, and it is not very hard to prove that the function indeed is elementary (and thus in $\mathcal{S}$).

An easy generalization of the straightforward argument above shows that the set $\mathcal{S}_{b\downarrow} \setminus \mathcal{S}_{b\uparrow}$ is nonempty for all bases b and all $\mathcal{S}$ closed under elementary operations. A symmetric argument shows that $\mathcal{S}_{b\uparrow} \setminus \mathcal{S}_{b\downarrow}$ is nonempty. So it is rather easy to see that we have $\mathcal{S}_{b\downarrow} \nsubseteq \mathcal{S}_{b\uparrow}$ and $\mathcal{S}_{b\uparrow} \nsubseteq \mathcal{S}_{b\downarrow}$ when b is fixed. It is not all that easy to see why (20) should hold. It is not easy to come up with natural candidates that can possibly witness the nonemptyness of $\mathcal{S}_{g\uparrow} \setminus \mathcal{S}_{g\downarrow}$ and $\mathcal{S}_{g\downarrow} \setminus \mathcal{S}_{g\uparrow}$. However, it turns out that the number α^f, which is discussed in the introductory section and formally given by Definition 5.3, will be in the set $\mathcal{S}_{g\uparrow} \setminus \mathcal{S}_{g\downarrow}$ when f is a suitable honest function.

Theorem 7.1. *Let f be any honest function. There exists an elementary function $\check{T} : \mathbb{Q} \to \mathbb{Q}$ such that (i) $\check{T}(q) = 0$ if $q < \alpha^f$ and (ii) $q > \check{T}(q) > \alpha^f$ if $q > \alpha^f$.*

Proof. This proof is long and involved. In order to improve the readability, we will write α in place of α^f. Then our definition says that

$$\alpha_n \;=\; \sum_{i=0}^{n} P_i^{-h(i)} \quad \text{and} \quad \alpha \;=\; \lim_{n\to\infty} \alpha_n \; .$$

In addition to the sequence α_j we need the sequence β_j given by

$$\beta_0 \;=\; P_0^{-h(0)+1} \;=\; 2^{-h(0)+1} \quad \text{and} \quad \beta_{j+1} \;=\; \alpha_j \;+\; P_{j+1}^{-h(j+1)+1} \; .$$

Observe that we have $\alpha < \beta_j$ for all $j \in \mathbb{N}$, since

$$\alpha \;-\; \alpha_j \;=\; P_{j+1}^{-h(j+1)} \;+\; P_{j+2}^{-h(j+2)} \;+\; \ldots \;\leq\; P_{j+1}^{-h(j+1)+1}$$

for any $j \in \mathbb{N}$. Furthermore, we have

$$p \leq \beta_{\ell+1} \quad \Leftrightarrow \quad P_{\ell+1}^{h(\ell+1)} \leq (p - \alpha_\ell)^{-1} P_{\ell+1} \tag{22}$$

for any $p \in \mathbb{Q}$ and any $\ell \in \mathbb{N}$, such that $p > \alpha_\ell$. In order to see that (22) holds, observe that we have $\beta_{\ell+1} - \alpha_\ell = P_{\ell+1}^{-h(\ell+1)+1}$ by the definition of $\beta_{\ell+1}$. Thus, $p \leq \beta_{\ell+1}$ is equivalent to $p - \alpha_\ell \leq P_{\ell+1}^{-h(\ell+1)+1}$ which in turn is equivalent to $P_{\ell+1}^{h(\ell+1)} \leq (p - \alpha_\ell)^{-1} P_{\ell+1}$.

We will present an algorithm which computes a function $\check{T}$ with the properties given in the theorem. The input is the rational number q, and we will argue that the algorithm is elementary in q. We will also argue that the algorithm gives correct output, that is, if $q < \alpha$, the algorithm will output 0, and if $\alpha < q$, the algorithm will output some rational q' such that $\alpha < q' < q$. We can w.l.o.g. assume that $0 < q < 1$.

Step 1 of the algorithm: Pick any $m', n \in \mathbb{N}$ such that $q = m'n^{-1}$ and $n \geq h(0)$. Find $m \in \mathbb{N}$ such that $q = m(P_0 P_1 \ldots P_n)^{-n}$, and compute the base b such that $b = \prod_{i=0}^{n} P_i$ (end of Step 1).

The rational number q has a finite base-b expansion of length s where $s \leq n$. Moreover, the rational numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ all have finite base-b expansions. It is easy to see that Step 1 is elementary (in q).

Step 2 of the algorithm: Compute (the unique) natural number $j < n$ such that

$$h(j) \;\leq\; n \;<\; h(j+1) \; .$$

Furthermore, compute $\alpha_0, \alpha_1, \ldots, \alpha_j$ and $\beta_0, \beta_1, \ldots, \beta_j$ (end of Step 2).

All the numbers $h(0), h(1), \ldots h(j)$ are less than or equal to n, and h has elementary graph. This entails that Step 2 is elementary in n (and thus also elementary in q).

Step 3 of the algorithm: If $q \leq \alpha_k$ for some $k \leq j$, give the output 0 and terminate. If $\beta_k < q$ for some $k \leq j$, give the output β_k and terminate (end of Step 3).

Step 3 obviously gives the correct output. It is also obvious that the step is elementary.

If the algorithm has not yet terminated, we have $\alpha_j < q \leq \beta_j$. The algorithm has already determined α_j, and h is an honest function. This makes it possible to check elementarily if $q \leq \beta_{j+1}$: By (22), we have $q \leq \beta_{j+1}$ iff $P_{j+1}^{h(j+1)} \leq (q - \alpha_j)^{-1} P_{j+1}$. Thus, $q \leq \beta_{j+1}$ iff there exists y such that

$$y \leq (q - \alpha_j)^{-1} P_{j+1} \;\text{ and }\; h(j+1) = y \;\text{ and }\; P_{j+1}^{y} \leq (q - \alpha_j)^{-1} P_{j+1} \; . \tag{23}$$

If such a y exists, then an elementary computation can find its numerical value. If such a y does not exist, then an elementary computation can confirm the nonexistence.

Step 4 of the algorithm: Search for y such that (23) holds. If the search is successful, proceed with Step 5. Otherwise, that is, if the search is not successful, we have $\beta_{j+1} < q$ and the algorithm proceeds to compute q' such that $\alpha < q' < q$. We explain below how to compute such a q' (end of Step 4).

It is explained above why Step 4 is elementary in q. If the algorithm proceeds to Step 5, we have $\alpha_j < q \le \beta_{j+1}$.

Step 5 of the algorithm: Compute α_{j+1}. If $q \le \alpha_{j+1}$, give the output 0 and terminate, otherwise, proceed to Step 6 (end of Step 5).

Since we have computed $h(j+1)$ in Step 4, Step 5 will be elementary in q. It is obvious that the algorithm gives the correct output.

If the algorithm proceeds to the sixth step, we have $\alpha_{j+1} < q \le \beta_{j+1}$. Moreover, the algorithm has determined α_{j+1}. By (22), we have $q \le \beta_{j+2}$ iff there exists y such that

$$y \le (q - \alpha_{j+1})^{-1} P_{j+2} \text{ and } h(j+2) = y \text{ and } P_{j+2}^{y} \le (q - \alpha_{j+1})^{-1} P_{j+2} \,. \tag{24}$$

This makes it possible to check elementarily if $q \le \beta_{j+2}$.

Step 6 of the algorithm: Search for y such that (24) holds. If the search is successful, give output 0. Otherwise, we have $\beta_{j+2} < q$, and the algorithm will compute q' such that $\alpha < q' < q$. We explain below how to compute such a q' (end of the algorithm).

The function h is honest and the search for y is bounded. Thus, it is easy to see that Step 6 is elementary in q. We will now argue that the output is correct. If the algorithm outputs 0, we have $\alpha_{j+1} < q \le \beta_{j+2}$. We need to prove that $\alpha_{j+1} < q \le \beta_{j+2}$ implies $q < \alpha$.

It is well-known that there is a prime between x and $2x$ for any $x \ge 2$ (the Bertrand-Chebyshev Theorem). Thus, $P_y \le 2^{y+1}$ for any $y \in \mathbb{N}$. It follows that $b = P_0 P_1 \dots P_n \le 2^{(n+1)^2}$. Lemma 5.4 together with $n < h(j+1)$ yield

$$b^{h(j+1)+1} \;\le\; (2^{(n+1)^2})^{h(j+1)+1} \;<\; 2^{(h(j+1)+1)^3} \;<\; P_{j+2}^{h(j+2)-1}$$

and thus

$$\frac{1}{P_{j+2}^{h(j+2)-1}} \;<\; \frac{1}{b^{h(j+1)+1}} .$$

This entails

$$\alpha_{j+1} \;<\; \alpha \;<\; \beta_{j+2} \;=\; \alpha_{j+1} + \frac{1}{P_{j+2}^{h(j+2)-1}} \;<\; \alpha_{j+1} + \frac{1}{b^{h(j+1)+1}} \,. \tag{25}$$

Now, α_{j+1} has a finite base-b expansion of length $h(j+1)$. Thus, (25) implies the first $h(j+1)$ digits of the base-b expansions of α_{j+1}, α and β_{j+2} coincide. Moreover, $h(j+1) > n \ge s$ where s is the length of the base-b expansion of q. Thus, if we have $\alpha_{j+1} < q \le \beta_{j+2}$, we also have $q < \alpha$.

It remains to explain how the algorithm computes q' such that $\alpha < q' < q$ in Step 4 and in Step 6. We will explain how the algorithm works in Step 6. The algorithm works the same way in Step 4 (just replace $j+1$ and $j+2$ by j and $j+1$, respectively).

When the algorithm starts to compute q' in Step 6, the (numerical) value of α_{j+1} is known. It is also known that $\beta_{j+2} < q$, but the value of β_{j+2} is not known, moreover, it might not even be possible to compute the value of β_{j+2} elementarily (in q). So first our algorithm finds the least $t \in \mathbb{N}$ such that $b^t > (q - \alpha_{j+1})^{-1}$. Such a t can be computed elementarily. Then, by an elementary computation, our algorithm distinguishes between the two cases

$$\text{(A) } q - b^{-t} > \beta_{j+2} \quad \text{and} \quad \text{(B) } q - b^{-t} \leq \beta_{j+2} \ .$$

In case (A) the algorithm will not know the value of β_{j+2}, but the algorithm will give the output $q - b^{-t}$. This output is correct as $q - b^{-t} > \beta_{j+2} > \alpha$ and $b^{-t} > 0$. In case (B) the algorithm will be able to compute β_{j+2} elementarily in q. Thus, the algorithm can give obviously correct output β_{j+2}.

Thus, we need to argue that it is possible to distinguish between case (A) and case (B) by an elementary computation, and we need to argue that it is possible to elementarily compute the value of β_{j+2} when (B) holds. By (22), we have

$$q - b^{-t} \ \leq \ \beta_{j+2} \ \ \Leftrightarrow \ \ P_{j+2}^{h(j+2)} \ \leq \ (q - b^{-t} - \alpha_{j+1})^{-1} P_{j+2}. \tag{26}$$

We can elementarily decide if there exists y such that

$$y \ \leq \ (q - b^{-t} - \alpha_{j+1})^{-1} P_{j+2} \quad \text{and} \quad h(j+2) \ = \ y \quad \text{and} \quad P_{j+2}^{y} \ \leq \ (q - b^{-t} - \alpha_{j+1})^{-1} P_{j+2} \ .$$

By (26), (A) holds if such a y does not exist, and (B) holds if such a y exists. If such a y exists, we can elementarily compute the value of y, and then we can elementarily compute the value of β_{j+2} as $\beta_{j+2} = \alpha_{j+1} + P_{j+2}^{-y+1}$. □

Corollary 7.2. *Let f be any honest function. The Dedekind cut of the real number α^f is elementary.*

Proof. Let

$$D(q) \ = \ \begin{cases} 0 & \text{if } \check{T}(q) = 0 \\ 1 & \text{otherwise,} \end{cases}$$

where $\check{T}$ is the function given by Theorem 7.1. Then D is the Dedekind cut of α^f, and D is elementary since $\check{T}$ is. □

Theorem 7.3. *Let f be any honest function. There exists a primitive recursive right best approximation of the real number α^f.*

Proof. We give a primitive recursive algorithm for computing a right best approximation $\{a_i/b_i\}_{i\in\mathbb{N}}$ of α^f. Let $a_0/b_0 = 1/1$. Assume that a_i/b_i is already computed. The following algorithm shows how to compute a_{i+1}/b_{i+1}:

- Let $c/d = \check{T}(a_i/b_i)$ where $c, d \in \mathbb{N}$ and $\check{T}$ is the function given by Theorem 7.1.
- Use the Dedekind cut of α^f to search for a natural number c' such that $(c'-1)/d < \alpha^f < c'/d$ (the search is bounded as $c' \leq c$).

- Use the Dedekind cut of α^f to search for a natural number c'' and the least natural number d'' such that $d'' \leq d$ and $(c''-1)/d'' < \alpha^f < c''/d'' \leq c'/d$ (the search is bounded as $c'' \leq d''$).
- Let $a_{i+1}/b_{i+1} = c''/d''$.

It is obvious that $\{a_i/b_i\}_{i\in\mathbb{N}}$ is a right best approximation of α^f. We invite the reader to check that the algorithm indeed is primitive recursive. □

The next corollary follows straightaway form Theorem 6.3 and Theorem 7.3.

Corollary 7.4. *Let f be any honest function. The general sum approximation from above of α^f, that is the function $\check{G}^{\alpha_f}$, is primitive recursive.*

We are now ready to state and prove the last of our main results.

Theorem 7.5. *For any subrecursive class $\mathcal{S}$ closed under primitive recursive operations, there exist irrational numbers α and β such that*

$$(i)\ \ \alpha \in \mathcal{S}_{g\downarrow} \setminus \mathcal{S}_{g\uparrow} \quad \textit{and} \quad (ii)\ \ \beta \in \mathcal{S}_{g\uparrow} \setminus \mathcal{S}_{g\downarrow}\ .$$

Proof. Pick an honest function f such that $f \notin \mathcal{S}$. Such an f exists by Corollary 3.3. By Corollary 7.4, we have $\alpha^f \in \mathcal{S}_{g\downarrow}$. By Theorem 5.7, we have $\alpha^f \notin \mathcal{S}_{g\uparrow}$. This proves (i). The proof of (ii) is symmetric. □

The next corollary follows from Theorem 6.3 and the preceding theorem.

Corollary 7.6. *For any subrecursive class $\mathcal{S}$ that is closed under primitive recursive operations, we have*

$$\mathcal{S}_{g\uparrow} \ = \ \mathcal{S}_{<} \ \neq \ \mathcal{S}_{>} \ = \ \mathcal{S}_{g\downarrow}\ .$$

Declaration of competing interest

The authors hereby declare that the current paper does not affect any competing interests.

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