Inf2320 Chapter 1

3rd November 2004

Exercises

1.1

a) Trapezoid rule:

$$\int_0^1 \frac{1}{1+x}dx \approx (1-0)\frac{1}{2}\left[\frac{1}{1+0}+\frac{1}{1+1}\right] = \frac{3}{4}.$$

b) Exact solution:

$$\int_0^1 \frac{1}{1+x}dx = [\ln(1+x)]_0^1 = \ln 2 = 0.6931,$$

relative error:

$$\frac{|\ln 2 - \frac{3}{4}|}{|\ln 2|} = 0.08292\ (\approx 8.2\%).$$

c) Composite Trapezoid (n=2):

$$\int_0^1 \frac{1}{1+x}dx \approx \frac{1}{2}\left[\frac{1}{2}\frac{1}{1+0}+\frac{1}{1+0.5}+\frac{1}{2}\frac{1}{1+1}\right] = \frac{17}{24},$$

relative error:

$$\frac{|\ln 2 - \frac{17}{24}|}{|\ln 2|} = 0.02191\ (\approx 2.2\%).$$

d) Composite Trapezoid (n=3):

$$\int_0^1 \frac{1}{1+x}dx \approx \frac{1}{3}\left[\frac{1}{2}\frac{1}{1+0}+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{2}{3}}+\frac{1}{2}\frac{1}{1+1}\right] = \frac{7}{10},$$

relative error:

$$\frac{|\ln 2 - \frac{7}{10}|}{|\ln 2|} = 0.00989\ (\approx 1\%).$$

e) Observe that the error (E_h) divided by h^2 seems to converge to (approximately) 0.0625. We want the relative error ($E_h/\ln 2$) to be less than $\frac{1}{1000}$:

$$\begin{aligned}\frac{0.0625h^2}{\ln 2} &< \frac{1}{1000}\\ \frac{1}{n^2} &< \frac{\ln 2}{62.5}\\ n &\geq 10\end{aligned}$$

(n must be an integer.)

1.2

a) Exact values:

$$\int_0^{\frac{1}{2}} \sin x dx = [-\cos x]_0^{\frac{1}{2}} = 0.12242$$
$$\int_0^{\frac{1}{2}} \sin(5x) dx = \left[-\frac{1}{5}\cos(5x)\right]_0^{\frac{1}{2}} = 0.36023$$

b) Composite Trapezoid (n=2):

$$\int_0^{\frac{1}{2}} \sin x dx \approx \frac{1}{4}\left[\frac{1}{2}\sin 0 + \sin\frac{1}{4} + \frac{1}{2}\sin\frac{1}{2}\right] = 0.12178,$$

relative error:

$$\frac{|0.12242 - 0.12178|}{|0.12242|} = 0.00523 (\approx 0.5\%).$$

$$\int_0^{\frac{1}{2}} \sin(5x) dx \approx \frac{1}{4}\left[\frac{1}{2}\sin 0 + \sin\frac{5}{4} + \frac{1}{2}\sin\frac{5}{2}\right] = 0.31206,$$

relative error:

$$\frac{|0.36023 - 0.31206|}{|0.36023|} = 0.13372 \ (\approx 13\%).$$

c) See figure 1. This figure was produced by the program **ex12_c.m**.

1.3

a) Easy

b) Easy (*exact* results if n is an even number).

c) Easy

d) Problematic: $f(x)$ is infinite at $x = 0$. This means that the area of one of the trapezoids will also be infinite.

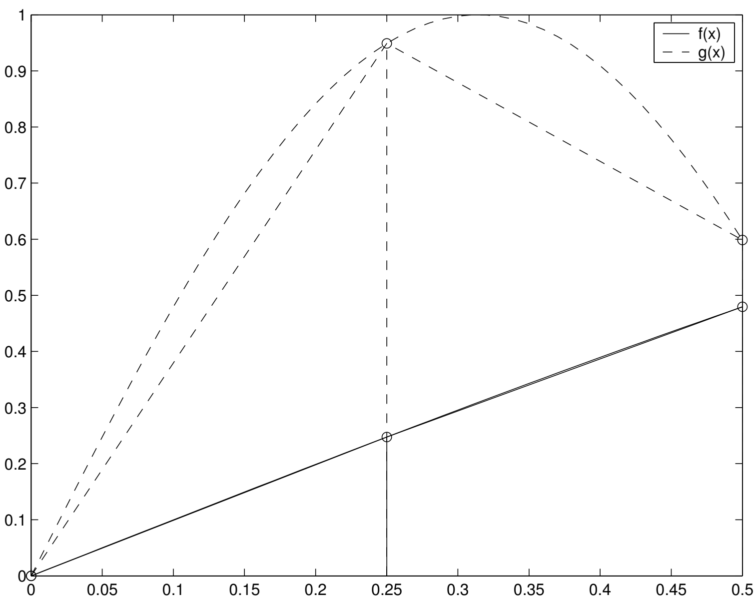

Figure 1: Exercise 1.2. For $f(x)$ the function and the trapezoids are almost indistinguishable, indicating that the error of the approximatization is low. For $g(x)$ the difference is more significant.

1.4

See the program **trapezoidal.m**. Using this program on the examples in exercise 1.3 we find that:

a) $E/h^2 = 0.25$

b) $E/h^2 = \begin{cases} 0 & n \text{ is an even number} \\ 0.25 & n \text{ is an odd number} \end{cases}$

c) $E/h^2 = 0$, note the symmetry of the function.

d) The program does not work for this problem, see **1.3 d**.

Projects

1.7

1. Using the definition of the error we find that

$$E = \left| \int_a^b f(x)\,dx - T(f,a,b) \right| = \left| \int_a^b f(x)\,dx - \int_a^b y(x)\,dx \right| = \left| \int_a^b f(x) - y(x)\,dx \right|.$$

2. Using an integral inequality it follows that

$$E = \left| \int_a^b f(x) - y(x)\,dx \right| \leq \int_a^b |f(x) - y(x)|\,dx.$$

3. From equations (1.34) and (1.35):

$$E \le \int_a^b |f(x) - y(x)|\, dx \le (b-a) \max_{a \le x \le b} |-y(x)|\,.$$

4. Taylor's Theorem inply that:

$$f(x) = f(a) + f'(a)(x-a) + f''(\xi)\frac{(x-a)^2}{2}.$$

Trapezoid:

$$y(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a).$$

That is

$$\begin{aligned} f(x) - y(x) &= f(a) + f'(a)(x-a) + f''(\xi)\frac{(x-a)^2}{2} - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right) \\ &= \left[f'(a) - \frac{f(b)-f(a)}{b-a}\right](x-a) + \frac{1}{2}(x-a)^2 f''(\xi). \end{aligned}$$

5. Taylor's Theorem:

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + f''(\eta)\frac{(b-a)^2}{2}, \\ \frac{f(b)-f(a)}{b-a} &= f'(a) + \frac{1}{2}(b-a)f''(\eta). \end{aligned}$$

Apply this to the result from part 4.

6. An upper limit for $|f(x) - y(x)|$:

$$\begin{aligned} |f(x) - y(x)| &= \left|-\tfrac{1}{2}(x-a)(b-a)f''(\eta) + \tfrac{1}{2}(x-a)^2 f''(\xi)\right| && \text{from part 5} \\ &\le \left|-\tfrac{1}{2}(x-a)(b-a)f''(\eta)\right| + \left|\tfrac{1}{2}(x-a)^2 f''(\xi)\right| && \text{triangle inequality} \\ &\le \left|-\tfrac{1}{2}(b-a)^2 f''(\eta)\right| + \left|\tfrac{1}{2}(b-a)^2 f''(\xi)\right| && \text{because } (b-a) \ge (x-a) \\ &\le \left|-\tfrac{1}{2}(b-a)^2 M\right| + \left|\tfrac{1}{2}(b-a)^2 M\right| && M \ge f''(\eta),\ M \ge f''(\xi) \\ &= M(b-a)^2. \end{aligned}$$

7. And thus an upper limit for the error is given by:

$$E \le \int_a^b |f(x) - y(x)|\, dx \le \int_a^b M(b-a)^2\, dx = M(b-a)^3.$$

8. The composite trapezoid rule:

$$\begin{aligned}\sum_{i=1}^{n} T(f,x_{i-1},x_i) &= \sum_{i=1}^{n}(x_i - x_{i-1})\frac{f(x_i)+f(x_{i-1})}{2}\\ &= \frac{h}{2}\sum_{i=1}^{n} f(x_i)+f(x_{i-1})\\ &= \frac{h}{2}\left(\sum_{i=0}^{n-1} f(x_i)+\sum_{i=1}^{n} f(x_i)\right)\\ &= h\left[\frac{1}{2}f(x_0)+\sum_{i=1}^{n-1} f(x_i)+\frac{1}{2}f(x_n)\right]\\ &= T_n.\end{aligned}$$

9. Using the triangle inequality on the definition of E_n:

$$\begin{aligned}E_n &= \left|\int_a^b f(x)\,dx - T_n\right| = \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx - T(f,x_{i-1},x_1)\right|\\ &\leq \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x^i} f(x)\,dx - T(f,x_{i-1},x_1)\right|.\end{aligned}$$

10. Using the results from **1.7.9.** on each interval we find that:

$$E_n \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1})^3 \leq M\sum_{i=1}^{n}(x_i - x_{i-1})^3.$$

11. Remember that $nh = b - a$:

$$M\sum_{i=1}^{n}(x_i - x_{i-1})^3 = M\sum_{i=1}^{n} h^3 = Mnh^3 = M(b-a)h^2.$$

12. Comparing (1.33) with the results from **1.4**:

a) $E_n/h^2 \leq 1/2$. Consistent with the result obtained in **1.4 a)**.

b) $|x|$ is not differentiable at $x = 0$, inequality (1.33) can not be used.

c) $E_n/h^2 \leq \pi^2/6$. Consistent with the result obtained in **1.4 c)**.

1.8

a) Midpoint rule:

$$\int_a^b p_0(x)dx = \int_a^b f(\frac{a-b}{2})dx = f(\frac{a-b}{2})\int_a^b dx = (b-a)f(\frac{a-b}{2}).$$

b) Midpoint rule applied to (1.45):

$$(b-a)f(\frac{a-b}{2}) = f(\frac{1}{2}) = \frac{8}{225},$$

relative error:

$$\frac{|\frac{1}{24} - \frac{8}{225}|}{|\frac{1}{24}|} = \frac{11}{75} \; (\approx 15\%).$$

c) Composite midpoint rule:

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) f(\frac{x_{i+1} + x_i}{2}) = \sum_{i=0}^{n-1} f(x_{i+1/2}).$$

Composite midpoint rule applied to (1.45), using n=2:

$$\int_a^b f(x)dx \approx \frac{1}{2}\left(f(\frac{1}{4}) + f(\frac{3}{4})\right) = 0.03980.$$

Relative error:

$$\frac{|\frac{1}{24} - 0.03980|}{|\frac{1}{24}|} = 0.04480 \; (\approx 5\%).$$

d) See the program **midp.m**, $E \approx 0.008h^2$.

e) Trapezoid rule:

$$\int_0^1 f(x)dx \approx (1-0)\frac{1}{2}(f(0) + f(1)) = \frac{1}{8},$$

relative error:

$$\frac{|\frac{1}{24} - \frac{1}{18}|}{|\frac{1}{24}|} = \frac{1}{3} \; (\approx 33\%).$$

f) Simpson's rule:

$$\int_0^1 f(x)dx \approx \frac{(1-0)}{6}[f(0) + 4f(\frac{1}{2}) + f(1)] = \frac{19}{450},$$

relative error:

$$\frac{|\frac{1}{24} - \frac{19}{450}|}{|\frac{1}{24}|} = \frac{1}{75} \; (\approx 1.3\%)$$

g) Composite Simpson's rule:

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx \approx \sum_{i=1}^{n} \frac{x_{i+1} - x_i}{6}[f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)].$$

h) See the program **simprule.m**, $E \approx 0.001h^4$.

1.9

1. The theorem of Pythagoras applied to the triangle in figure (1.11) yields:

$$s^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x)^2 + (y(x + \Delta x) - y(x))^2.$$

2. Taylor: $y(x + \Delta x) \approx y(x) + \Delta x y'(x)$,

$$(\Delta x)^2 + (y(x + \Delta x) - y(x))^2 \approx (\Delta x)^2 + (\Delta x)^2 (y'(x))^2.$$

3. In each interval: Use the distance between the endpoints as an approximation of the lenght of the graph.

$$l(y, a, b) = \sum_{i=1}^{n} s_i \approx \sum_{i=1}^{n} \Delta x \sqrt{1 + (y'(x_i))^2}$$

4. This is a Riemann sum. When $\Delta x \to 0$ it converges towards an integral.

$$\lim_{\Delta x \to 0} \sum_i \Delta x_i \sqrt{1 + (y'(x_i))^2} = \int_a^b \sqrt{1 + (y'(x_i))^2}\, dx.$$

5.+6. 1/8 of a circle:

$$y(x) = \sqrt{1 - x^2},\ y'(x) = \frac{-x}{\sqrt{1 - x^2}},$$
$$\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1 + (y'(x))^2}\, dx = \int_0^{\frac{\sqrt{2}}{2}} \sqrt{\frac{1}{1 - x^2}}\, dx = [\arcsin x]_0^{\frac{\sqrt{2}}{2}} = \frac{\pi}{4}.$$

7. Run in MATLAB:

```
for n=10:10:100
(trapezoidal(0,sqrt(2)/2,'1/sqrt(1-x*x)',n)-pi/4)*2*n^2
end
```

8. Each patch of cable: $L = \int_0^{250} \sqrt{1 + (y'(x))^2}\, dx,\ y'(x) = 0.006x.$

9. Run in MATLAB:

```
4*trapezoidal(0,250,'sqrt(1+(0.006*x)^2)',20)
ans =
1.2999e+03
```

Programs (MATLAB)

ex2_c.m

```
% Script for making a figure of two functions
% and trapezoides with n=2.
% example:  >>ex2_c
```

```
x=0:0.01:0.5;
xt=0:0.25:0.5;

f=sin(x);
g=sin(5*x);
gt=sin(5*xt);
ft=sin(xt);

plot(x,f,x,g,'r--');legend('f(x)','g(x)');
hold on;
plot(xt,ft,xt,gt,'r--');
stem(xt,ft);stem(xt,gt,'r--');
hold off;
```

trapezoidal.m

```
function r = trapeziodal(a,b,f,n)

% Use the composite trapezoidal method, with n trapezoids,
% to compute the integral of the function f from a to b.
% example: trapezoidal(0,1,'x^4',12) .

     f= fcnchk(f);
     h=(b-a)/n;
     s=0;
     x=a;

     for i = 1:n-1
       x=x+h;
       s=s + feval(f,x);
     end

      s= 0.5*(feval(f,a) +feval(f,b)) +s;
      r = h*s;
```

midp.m

```
function r  = midp(a,b,f,n)
% Use the composite midpoint metod, with n points,
% to compute the integral of the function f from a to b.
% example: midp(0,1,'x./(4-x.*x).^2',8)

     f= fcnchk(f);
     h=(b-a)/n;

     x=(a+h*0.5):h:(b-h*0.5);

     y=feval(f,x);
     r= h*sum(y(1:n));
```

simprule.m

```
function r = simprule(a,b, f, n)

% Use the composite Simpson’s rule, with n intervals,
% to compute the integral of the function f from a to b.
% example:  simprule(0,1,’x./(4-x.*x).^2’,8)

     f= fcnchk(f);
     h=(b-a)/n;
     s=0; x=a;

     for i = 1:n-1
       x=x+h;
       s=s + 2*feval(f,x) +4*feval(f,x-h/2);
     end

      s= (feval(f,a) +feval(f,b) +4*feval(f,b-h/2) ) +s;
      r = h*s/6;
```