Inf2320 Chapter 3

3rd November 2004

Exercises

3.1

a) See the program **ex31.m**.

b) The results will depend on the circumstances, $\Delta t = 0.005$ is often sufficiently small.

c) For $F_0 = 1$ and $S_0 = 2$ both F and S become constants.

3.2

a) If F and S solve (3.74) we can write

$$\left(1 - \frac{1}{F}\right) F' = \left(1 - \frac{1}{F}\right)(2 - S) F = \left(\frac{2}{S} - 1\right)(F - 1) S = \left(\frac{2}{S} - 1\right) S'.$$

b) Using integration by substitution:

$$\int_0^t (1 - \frac{1}{F}) F' \, d\tau = \int_{F(0)}^{F(t)} 1 - \frac{1}{u} \, du = F(t) - \ln(F(t)) - (F_0 - \ln F_0)$$

$$\int_0^t (\frac{2}{S} - 1) S' \, d\tau = \int_{S(0)}^{S(t)} \frac{2}{u} - 1 \, du = 2\ln(S(t)) - S(t) - (2\ln S_0 - S_0)$$

c) Equation (3.76) can be rewritten on the form

$$\begin{aligned}
F(t) - \ln(F(t)) - (F_0 - \ln F_0) &= 2\ln(S(t)) - S(t) - (2\ln S_0 - S_0), \\
e^{F(t) - \ln(F(t)) - (F_0 - \ln F_0)} &= e^{2\ln(S(t)) - S(t) - (2\ln S_0 - S_0)}, \\
\frac{e^F}{F}\frac{e^S}{S^2} &= \frac{e^{F_0}}{F_0}\frac{e^{S_0}}{S_0^2}.
\end{aligned}$$

d)+e) See the program **ex32.m**.

3.3

a) $r'(t) = 4u^3(-v^3) + 4v^3u^3 = 0$. The function $r(t)$ is constant, i.e. $r(t) = r(0)$ for all $t > 0$.

b) Using Euler's method:

$$u_{n+1} = u_n + \Delta t u_n' = u_n - \Delta t v_n^3$$

$$v_{n+1} = v_n + \Delta t v_n' = v_n + \Delta t u_n^3$$

c) See the program **ex33.m**.

d) Equation (3.81) holds approximately for the numerical solutions.

3.4

a) In the first equation in (3.83), divide by F and integrate on both sides:

$$\begin{aligned} \int_0^t \frac{1}{F} F' \, d\tau &= \int_0^t 2 - S \, d\tau, \\ \ln F(t) - \ln F_0 &= 2t - \int_0^t S \, d\tau \end{aligned}$$

In the second equation in (3.83), divide by S and integrate on both sides:

$$\begin{aligned} \int_0^t \frac{1}{S} S' \, d\tau &= \int_0^t F - 1 \, d\tau, \\ \ln S(t) - \ln S_0 &= \int_0^t F \, d\tau - t \end{aligned}$$

b) For $t = T$:

$$\begin{aligned} S(T) &= S_0, \\ \int_0^T F \, d\tau - T = \ln S(T) - \ln S_0 &= 0, \\ \frac{1}{T} \int_0^T F \, d\tau &= 1, \end{aligned}$$

$$\begin{aligned} F(T) &= F_0, \\ 2T - \int_0^T S \, d\tau = \ln F(T) - \ln F_0 &= 0, \\ \frac{1}{T} \int_0^T S \, d\tau &= 2, \end{aligned}$$

c) In the first equation in (3.89), divide by F and integrate on both sides:

$$\begin{aligned}\int_0^t \frac{1}{F} F' \, d\tau &= \int_0^t (2 - S - \epsilon) \, d\tau, \\ \ln F(t) - \ln F_0 &= (2 - \epsilon)t - \int_0^t S \, d\tau\end{aligned}$$

In the second equation in (3.89), divide by S and integrate on both sides:

$$\begin{aligned}\int_0^t \frac{1}{S} S' \, d\tau &= \int_0^t F - 1 - \delta \, d\tau, \\ \ln S(t) - \ln S_0 &= \int_0^t F \, d\tau - (1 + \delta)t\end{aligned}$$

d)

$$\begin{aligned}S(T) = S_0 \quad &\Rightarrow \quad \int_0^T F \, d\tau = (1 + \delta)T, \\ F(T) = F_0 \quad &\Rightarrow \quad \int_0^T S(\tau) \, d\tau = 2 - \epsilon\end{aligned}$$

Projects

3.6; Analysis of a simple system

a) The Taylor series for u states that

$$u(t + \Delta t) = u(t) + \Delta t u'(t) + \mathcal{O}(\Delta t^2).$$

By removing the last term in the equation we get an approximation, $u_{n+1} \approx u(t + \Delta t)$;

$$u_{n+1} = u_n - \Delta t v_n$$

where $v_n \approx v(t_n)$. Similarly, for $v(t + \Delta t)$, we get:

$$v_{n+1} = v_n + \Delta t u_n$$

See the program **p36.m** for an implementation of this numerical scheme.

b)+c) See Figure 1.

d)

Δt	1/100	1/200	1/300	1/400	1/500
$c = \frac{r_N - r_0}{r_N \Delta t}$	4.8673	4.9331	4.9552	4.9664	4.9731

The value c converges to 5.

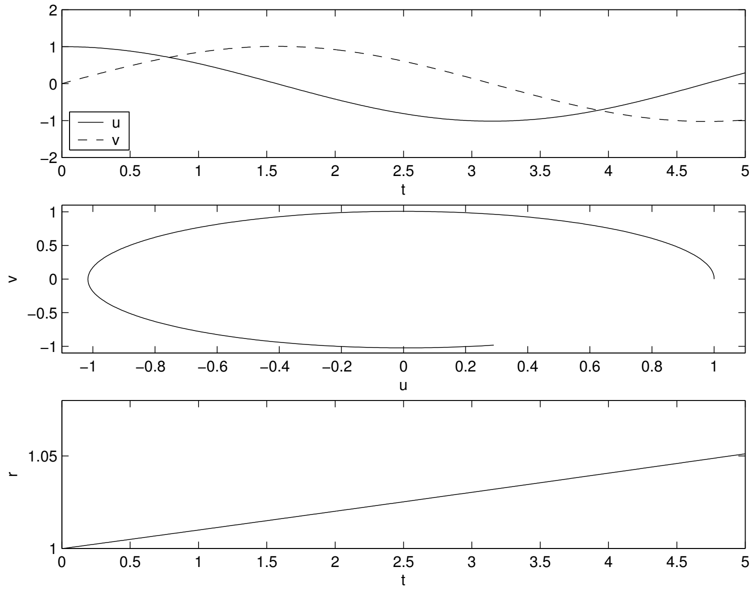

Figure 1: Solutions of u and v from (3.95), as a function of t and in the $u-v$ space. The bottom figure shows $r = u^2 + v^2$. r is almost constant, implying that both u and v are periodic functions.

e)

$$\begin{aligned} r'(t) &= \frac{d}{dt}u^2(t) + \frac{d}{dt}v^2(t) \\ &= 2u(t)u'(t) + 2v(t)v'(t) \\ &= 2u(t)(-v'(t)) + 2v(t)u'(t) \\ &= 0 \end{aligned}$$

This means that $r(t) = r_0$, i.e. $u^2(t) + v^2(t) = u_0^2 + v_0^2$.

f) From (3.95):

$$\begin{aligned} u'(t) &= -v(t), \\ u'(t)u(t) &= -v(t)u(t), \\ &= -v(t)v'(t) \end{aligned}$$

We simply take the integral on both sides of the equation:

$$\begin{aligned} \int_0^T u'(t)u(t)\,dt &= -\int_0^T v(t)v'(t)\,dt, \\ \frac{1}{2}\left(u^2(T) - u^2(0)\right) &= -\frac{1}{2}\left(v^2(T) - v^2(0)\right), \\ u^2(T) + v^2(T) &= u_0 + v_0 \end{aligned}$$

g)

$$\begin{aligned} U'(t) = u'(t) - \bar{u}'(t) &= -v(t) + \bar{v}(t) = -V(t), \\ V'(t) = v'(t) - \bar{v}'(t) &= u(t) - \bar{u}(t) = -V(t) \end{aligned}$$

This is similar to the relationship of u and v from (3.95). We know from exercise **f** that this means that $U(t) + V(t) = U_0 + V_0$ which in turn imply that

$$(u(t) - \bar{u}(t))^2 + (v(t) - \bar{v}(t))^2 = (u_0 - \bar{u}_0)^2 + (v_0 - \bar{v}_0)^2.$$

h) Inserting $u(t) = \cos(t), v(t) = \sin(t)$ in (3.108) yields:

$$\begin{aligned} u'(t) &= \frac{d}{dt}\cos(t) = -\sin(t) = -v(t), \\ v'(t) &= \frac{d}{dt}\sin(t) = \cos(t) = u(t) \end{aligned}$$

Initial conditions:

$$\begin{aligned} u(0) &= \cos(0) = 1, \\ v(0) &= \sin(0) = 0 \end{aligned}$$

i) Using the program **p36.m**:

Δt	1/100	1/200	1/300	1/400	1/500
$e_{\Delta t}$	0.0501	0.0250	0.0167	0.125	0.0100
$\frac{e_{\Delta t}}{\Delta t}$	5.0101	5.0053	5.0036	5.0027	5.0022

These results indicate that $e_{\Delta t} \approx c\Delta t$, where $c \approx 5$.

j) The Crank-Nicolson scheme for (3.108) is:

$$\begin{aligned} u_{n+1} + \frac{\Delta t}{2} v_{n+1} &= u_n - \frac{\Delta t}{2} v_n, \\ v_{n+1} - \frac{\Delta t}{2} u_{n+1} &= v_n + \frac{\Delta t}{2} u_n \end{aligned}$$

Combining the two equations gives

$$\begin{aligned} u_{n+1} &= u_n - \frac{\Delta t}{2}\left(v_n + v_n + \frac{\Delta t}{2}(u_n + u_{n+1})\right), \\ v_{n+1} &= v_n - \frac{\Delta t}{2}\left(u_n + u_n - \frac{\Delta t}{2}(v_n + v_{n+1})\right), \end{aligned}$$

and consequently

$$\begin{aligned} u_{n+1} &= \frac{1}{1 + \frac{\Delta t^2}{4}}\left(\left(1 - \frac{\Delta t^2}{4}\right) u_n - \Delta t v_n\right), \\ v_{n+1} &= \frac{1}{1 + \frac{\Delta t^2}{4}}\left(\Delta t u_n + \left(1 - \frac{\Delta t^2}{4}\right) v_v\right). \end{aligned}$$

k) The Crank-Nicolson scheme is used in the program **p36k.m**, giving the results:

Δt	1/100	1/200	1/300	1/400	1/500
$e_{\Delta t}$	$1.5318 \cdot 10^{-4}$	$3.8295 \cdot 10^{-5}$	$1.7020 \cdot 10^{-5}$	$9.5738 \cdot 10^{-6}$	$6.1272 \cdot 10^{-6}$
$\frac{e_{\Delta t}}{\Delta t^2}$	1.5318	1.5318	1.5318	1.5318	1.5318

$e_{\Delta t} \approx d\Delta t^2$, where d is 1.5318.

Programs (MATLAB)

ex31.m

```
function f  = ex31(S0,F0,Dt)
% Implements the scheme (3.70).
% example: ex31(0.1,1.9,0.005)

     t=0:Dt:10;

     F(1)=F0;
     S(1)=S0;

     for i= 2:(10/Dt+1);
     F(i)= F(i-1)+  Dt*(2 - S(i-1))*F(i-1);
     S(i)=S(i-1) + Dt*(F(i-1)-1)*S(i-1);
     end

     subplot(2,1,1) ; plot(t,F,t,S,'--');legend('F','S');xlabel('t');
     subplot(2,1,2) ; plot(F,S); xlabel('F'); ylabel('S');
```

ex32.m

```
function f  = ex32(S0,F0,Dt)
% Implements the scheme (3.70).
% example: ex32(0.1,1.9,0.005)

     t=0:Dt:10;

     F(1)=F0;
     S(1)=S0;

     K(1)=exp(F0)*exp(S0)/(F0*S0*S0)

     for i= 2:(10/Dt+1);
       F(i)= F(i-1)+  Dt*(2 - S(i-1))*F(i-1);
       S(i)=S(i-1) + Dt*(F(i-1)-1)*S(i-1);
       K(i)=exp(F(i))*exp(S(i))/(F(i)*S(i)*S(i));
     end

     Ek=(K-K(1))/K(1);
     plot(t,Ek);xlabel('t'); ylabel('Error K_n');
```

```
    E = (K(i)-K(1))/(K(1)*Dt);
    disp(sprintf('N= %g, E/Dt= %g',10/Dt,E));
```

ex33.m

```
function f  = ex33(u0,v0,N,T)
% Implements the scheme (3.82)
% example: ex33(1.2,2.3,1000,2)

    Dt=T/N;
    t=0:Dt:T;

    u(1)=u0;
    v(1)=v0;
    r0=u0^4 + v0^4

    for i= 2:(N+1);
      u(i)= u(i-1) -  Dt*v(i-1)^3;
      v(i)=v(i-1) + Dt*u(i-1)^3;
    end

    rt=u(i)^4 + v(i)^4
    subplot(2,1,1) ; plot(t,u,t,v,'--');xlabel('t');legend('u','v');
    subplot(2,1,2) ; plot(u,v); xlabel('u'); ylabel('v');
```

p36.m

```
function f  = p36(u0,v0,Dt)
% Solves the system (3.95)
% using an explicit numerical scheme.
% Plots the solutions as a function of t and in the u-v space,
% and plots the value r=u^2+v^2.
% Writes error values to the screen.
% example: p36(1,0,0.01)

    t=0:Dt:5;

    u(1)=u0;  v(1)=v0;
    r(1)=u0^2 + v0^2;

    for i= 2:(5/Dt+1);;
      u(i)= u(i-1) - Dt*v(i-1);
      v(i)= v(i-1) + Dt*u(i-1);
      r(i)=u(i-1)*u(i-1) + v(i-1)*v(i-1);
    end

    Error=abs( cos(5) -u(i))/abs(cos(5)) + abs( sin(5) -v(i))/abs(sin(5));
    Err_dt = Error/Dt;

    disp(sprintf('Error: %g',Error));
```

```
     disp(sprintf('Error/Delta t: %g',Err_dt));

     subplot(3,1,1) ; plot(t,u,t,v,'r--'); xlabel('t');legend('u','v',3);
     subplot(3,1,2) ; plot(u,v); xlabel('u'); ylabel('v');
               axis(sqrt(r(1))*[-1.1 1.1 -1.1 1.1]);
     subplot(3,1,3) ; plot(t,r); xlabel('t'); ylabel('r');
```

p36k.m

```
function f  = p36k(u0,v0,Dt)
% Solves the system (3.95)
% using Crank-Nicolson scheme.
% Writes error values to the screen.
% example: p36k(1,0,0.01)

     t=0:Dt:5;

     u(1)=u0;v(1)=v0;
     hp=1+Dt*Dt/4; hm=1-Dt*Dt/4;

     for i= 2:(5/Dt+1);;
       u(i)= (hm*u(i-1)-Dt*v(i-1))/hp;
       v(i)= (Dt*u(i-1)+hm*v(i-1))/hp;
     end

    Error=abs( cos(5) -u(i))/abs(cos(5)) + abs( sin(5) -v(i))/abs(sin(5))
    Errdt = Error/(Dt*Dt)

    disp(sprintf('Error: %g',Error));
    disp(sprintf('Error/Delta t^2: %g',Err_dt));

    plot(t,u,t,v,'r--'); xlabel('t');legend('u','v',3);
```