Inf165 Chapter 4

16th September 2003

Exercises

4.1

a) $f(0) = e^0 - 1 = 0$

b) Newton's method; ($f'(x) = e^x$):

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 - \frac{e^1 - 1}{e^1} = 0.367879 \\
x_2 &= 0.367879 - \frac{e^{0.367879} - 1}{e^{0.367879}} = 0.060080 \\
x_3 &= 0.001769 \\
x_4 &= 0.000002
\end{aligned}
$$

4.2

a) $f(0) = 0 - \sin 0 = 0$

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 - \frac{1 - \sin 1}{1 - \cos 1} = 0.655145 \\
x_2 &= 0.433590 \\
x_3 &= 0.288148 \\
x_4 &= 0.191832
\end{aligned}
$$

b) $f(0) = 1 - \cos 0 = 0$

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 - \frac{1 - \cos 1}{\sin 1} = 0.453678 \\
x_2 &= 0.222866 \\
x_3 &= 0.110969 \\
x_4 &= 0.055427
\end{aligned}
$$

4.3

a)

$$\begin{aligned} x_0 &= 3 \\ x_1 &= 3 - \frac{3^2 - 4}{2 \cdot 3} = \frac{13}{6} \\ x_2 &= \frac{313}{156} = 2.006410256 \\ x_3 &= 2.00001024 \\ x_4 &= 2.00000000 \end{aligned}$$

b)

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 - \frac{1^2}{2 \cdot 1} = \frac{1}{2} \\ x_2 &= \frac{1}{4} \\ x_3 &= \frac{1}{8} \\ x_3 &= \frac{1}{16} \end{aligned}$$

d)

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 - \frac{1^6}{5 \cdot 1^5} = \frac{5}{6} \\ x_2 &= \left(\frac{5}{6}\right)^2 \\ x_3 &= \left(\frac{5}{6}\right)^3 \\ x_4 &= \left(\frac{5}{6}\right)^4 \end{aligned}$$

4.4

a) Inserting $x = y = 0$:

$$\begin{aligned} e^y - x &= e^0 - 0 = 1 \\ x^2 - y &= 0^2 - 0 = 0 \end{aligned}$$

b) $(x_0, y_0) = (0.5, 0.5), (x_1, y_1) = (0.9061, 0.6561), (x_2, y_2) = (0.7701, 0.5746)$ (*NOTE:* The system converges towards a different solution than the one in exercise **a**.)

4.5

a) Explicit Euler:

$$u_{n+1} = \Delta t e^{-u_n} + u_n$$

b) Implicit Euler:

$$u_{n+1} - \Delta t e^{-u_{n+1}} - u_n = 0$$

c) See the program **ex45.m**.

d) Using $u(t) = \ln(1+t)$:

$$u'(t) = \frac{1}{1+t} = e^{-\ln(1+t)} = e^{-u}$$

e)+f) See the program **ex45.m**.

4.6

a) Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{c - \frac{1}{x_k}}{\frac{1}{x_k^2}} = (2 - cx_k)x_k$$

b)

$$\begin{aligned} x_0 &= 0.2 \\ x_1 &= (2 - 4 \cdot 0.2)0.2 = 0.24 \\ x_2 &= 0.2496 \\ x_3 &= 0.24999936 \\ x_4 &= 0.25000000 \end{aligned}$$

Projects

4.8; Convergence of Newton's Method

a) Newtons method for $f(x) = 0$ when $f(x) = x^2 - 4$:

$$\begin{aligned} x_0 &= 3.00000000000000 \\ x_1 &= 2.16666666666667 \\ x_2 &= 2.00641025641026 \\ x_3 &= 2.00001024002621 \\ x_4 &= 2.00000000002621 \end{aligned}$$

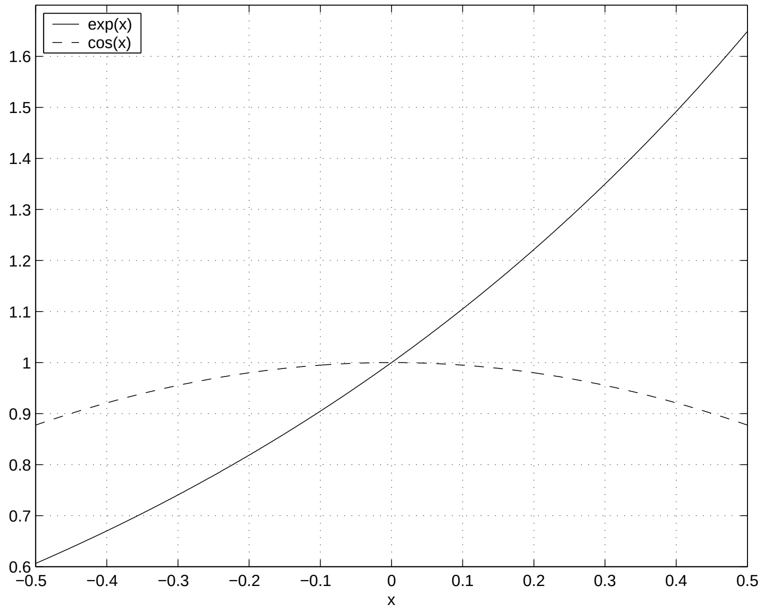

Figure 1: The function $g(x) = e^x - cos(x)$ is zero at the point where the two lines intersect, at $x = 0$.

b) The ratio $c_k = \frac{|e_{k+1}|}{e_k^2}$:

$$\begin{aligned} c_0 &= 0.16666666666667 \\ c_1 &= 0.23076923076923 \\ c_2 &= 0.24920127795669 \\ c_3 &= 0.24999625922877 \end{aligned}$$

It appears that c_k converges toward $c = 0.25$.

c) See figure 1.

$$g(0) = e^0 - \cos(0) = 1 - 1 = 0$$

d) Newtons method for $g(x) = 0$:

$$\begin{aligned} x_0 &= 0.25000000000000 \\ x_1 &= 0.04423602555409 \\ x_2 &= 0.00182264870657 \\ x_3 &= 0.00000331198793 \\ x_4 &= 0.00000000001097 \end{aligned}$$

$$c_0 = 0.70777640886547$$

$$\begin{aligned} c_1 &= 0.93143117823681 \\ c_2 &= 0.99697163563298 \\ c_3 &= 0.99999172671747 \end{aligned}$$

c converges towards 1, so the convergence is quadratic:

$$|e_{k+1}| \approx e_k^2$$

e) Newtons method for $f(x) = x^2 - 4$:

$$x_{k+1} = x_k - \frac{f(x)}{f'(x)} = x_k - \frac{x_k^2 - 4}{2x_k} = \frac{x_k^2 + 4}{2x_k}$$

f)

$$h'(x) = \frac{(x^2+4)'(2x) - (2x)'(x^2+4)}{(2x)^2} = \frac{(2x)(2x) - 2(x^2-4)}{4x^2} = \frac{x^2-4}{2x^2}$$

g) $h'(x)$ is nonnegative for all $x \geq 2$. This means that the function h is increasing for $x \geq 2$, and as $h(2) = 2$, $h(x)$ will be greater than or equal to 2 for all $x \geq 2$.

h) Using the result from **g**: If, for any k, $x_k \geq 2$ then $x_{k+1} \geq 2$, since $x_{k+1} = h(x_k)$.

i) If x_k is always 2 or more, then $(x_k - 2)$ is always positive. This means we can replace e_k with $|e_k|$.

j)

$$\begin{aligned} |e_{k+1}| &= x_{k+1} - 2 \\ &= \frac{x_k^2 + 4}{2x_k} \\ &= \frac{(x_k^2 - 2)^2}{2x_k} \\ &= \frac{e_k^2}{2x_k} \end{aligned}$$

k) We know from exercise **h** that $x_k \geq 2$. Combining this with the result from exercise **j** gives the result

$$|e_{k+1}| = \frac{e_k^2}{2x_k} \leq \frac{1}{4} e_k^2.$$

This is consistent with the ratio c_k converging towards 0.25 in exercise **b**.

l) Combining (4.207) and (4.208):

$$\begin{aligned} e_{k+1} &= x_{k+1} - x^* \\ &= x_k - \frac{f(x_k)}{f'(x_k)} - 2 \\ &= e_k - \frac{f(x_k)}{f'(x_k)} \\ &= \frac{e_k f'(x_k) - f(x_k)}{f'(x_k)} \end{aligned}$$

m) The Taylor series for f on the interval (x^*, x_k) is:

$$\begin{aligned} f(x^*) &= f(x_k) + (x^* - x_k)f'(x_k) + \frac{(x^* - x_k)^2}{2} f''(\xi) \\ &= f(x_k) - e_k f'(x_k) + \frac{e_k^2}{2} f''(\xi) \end{aligned}$$

n) We know that x^* is a root of f so $f(x^*) = 0$. Inserting this into the result from exercise **m** gives:

$$\begin{aligned} 0 &= f(x_k) - e_k f'(x_k) + \frac{e_k^2}{2} f''(\xi) \\ e_k f'(x_k) - f(x_k) &= \frac{e_k^2}{2} f''(\xi) \end{aligned}$$

o) Combining equations (4.212) and (4.214):

$$e_{k+1} = \frac{e_k f'(x_k) - f(x_k)}{f'(x_k)} = \frac{f''(\xi)}{2f'(x_k)} e_k^2$$

p) Using (4.210) and (4.211) (And $e_k = |e_k|$):

$$|e_{k+1}| = \frac{f''(\xi)}{f'(x_k)} e_k^2 \leq \frac{\beta}{2\alpha} e_k^2$$

q) See figure 2.

r) From figure 2 we can easily see that the tangent of the funtion at $x = 1$ will come closer to the desired point for the function $\sinh(x)$ than for the funtion $(\cosh(x) - 1)$.

s)

$$\begin{aligned} f'(x) &= \sinh(x), \quad \sinh(0) = 0 \\ f''(x) &= \cosh(x), \quad \cosh(0) = 1 \\ g'(x) &= \cosh(x), \quad \cosh(0) = 1 \\ g''(x) &= \sinh(x), \quad \sinh(0) = 0 \end{aligned}$$

We know from exercise **o** that

$$e_{k+1} = \frac{f''(\xi)}{2f'(x_k)} e_k^2$$

where $\xi \in [x^*, x_k]$. For $g(x)$ this means the error will decrease rapidly, since $g''(\xi)$ goes towards zero as we approach the exact solution. For $f(x)$ however, we instead have $f'(x_k)$ becoming very small, and the error harly decreases at all.

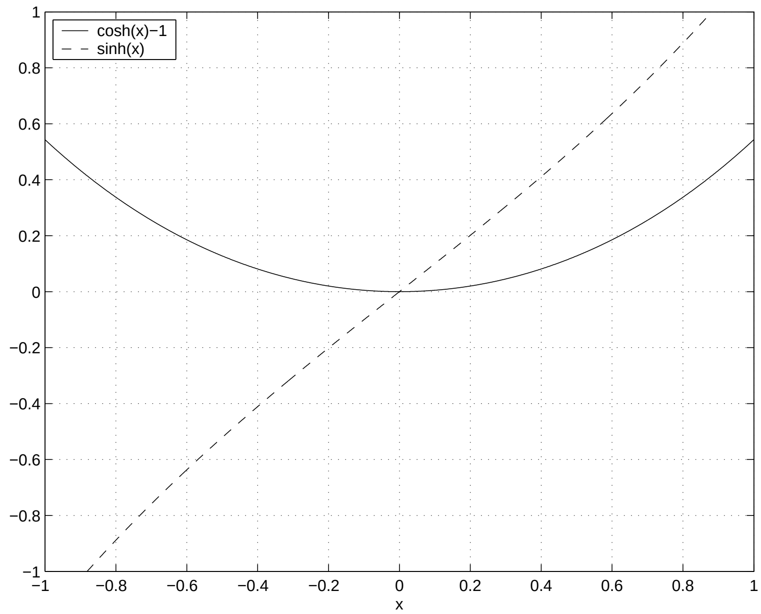

Figure 2: Funtions (4.217) and (4.218).

t) Newton's method for $f(x) = \cosh(x) - 1$:

$$
\begin{array}{r|l}
x_0 = 1.0000 & \\
x_1 = 0.5379 & \frac{e_1}{e_0} = 0.5397 \\
x_2 = 0.2752 & \frac{e_2}{e_1} = 0.5117 \\
x_3 = 0.1385 & \frac{e_3}{e_2} = 0.5031 \\
x_4 = 0.0694 & \frac{e_4}{e_3} = 0.5008
\end{array}
$$

Newton's method for $g(x) = \sinh(x)$:

$$
\begin{array}{r|l}
x_0 = 1.0000 & \\
x_1 = 0.2384 & \frac{e_1}{e_0^3} = 0.2384 \\
x_2 = 4.416 \cdot 10^{-3} & \frac{e_2}{e_1^3} = 0.3259 \\
x_3 = 2.871 \cdot 10^{-8} & \frac{e_3}{e_2^3} = 0.3333 \\
x_4 = 9.926 \cdot 10^{-24} & \frac{e_4}{e_3^3} = 0.4193
\end{array}
$$

u) Newton's method does not converge if we use a value for x that results in $f'(x) = 0$. In the graphical analysis of $f(x) = 1 - x^2$ this means we are looking for the point where the tangent to the maxima of $f(x)$ intersects the x-axis. Obviously this point does not exist. If x_0 is close to zero, then the tangent will intersect the x-axis far away from the solution at $x = \pm 1$, and convergence will be slow (see fig 3).

v) When we choose $x_0 = 0$ we find that $x_1 = 2$, which is a solution. As seen in figure 4, Newton's method 'overshoots' the solutions close to x_0 and we end up with an answer that is correct, but may not be the one we are looking for.

w) For $f(x) = x - x^3$ Newton's method can be written as

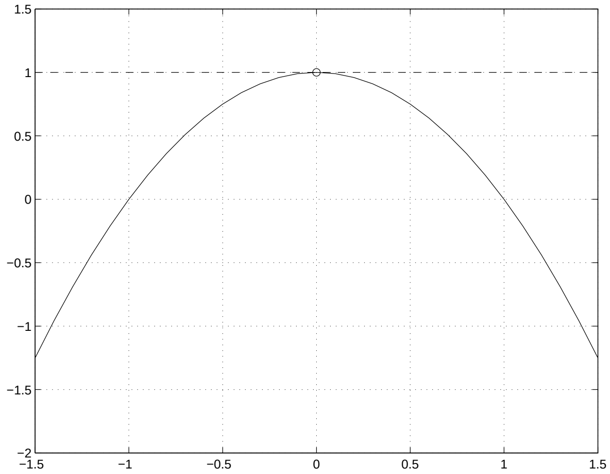

Figure 3: $f(x) = 1 - x^2$

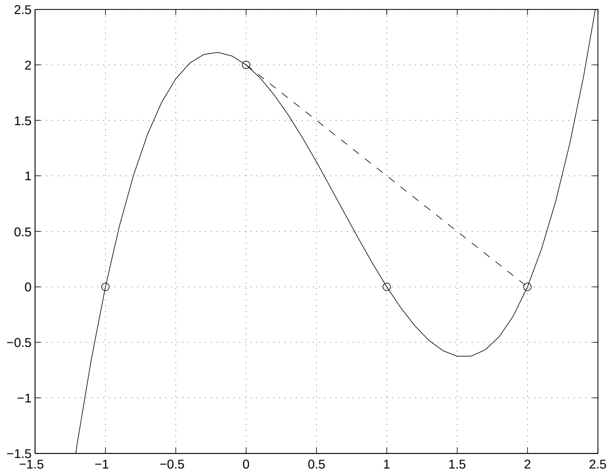

Figure 4: $f(x) = (x + 1)(x - 1)(x - 2)$

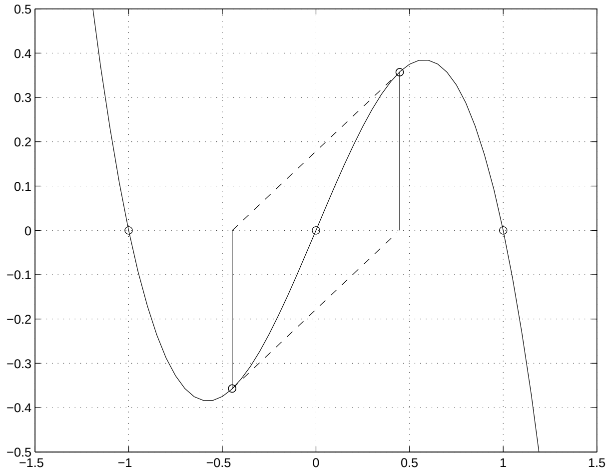

Figure 5: $f(x) = x - x^3$

$$x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)} = x_k + \frac{x_k - x_k^3}{1 - 3x_k^2} = \frac{-2x_k^3}{1 - 3x_k^2}.$$

Putting $x_0 = 1/\sqrt{5}$ we get:

$$x_1 = \frac{-2\frac{1}{5^{3/2}}}{1 - \frac{3}{5}} = -\frac{1}{\sqrt{5}},$$

and

$$x_2 = \frac{2\frac{1}{5^{3/2}}}{1 - \frac{3}{5}} = \frac{1}{\sqrt{5}}.$$

This means that instead converging towards a solution, the answers alternate between two values; $x = \pm\frac{1}{\sqrt{5}}$, as seen in figure 5.

Programs (MATLAB)

ex45.m

```
function ex45(Dt);
% Solves the expression u'=exp(-u), using four different numerical tecniques.
% Compares with the exact solution.
% example: ex45(0.05)


eE(1)=0; iE(1)=0;
t(1)=0;  S(1)=0;
Se(1)=0; Sf(1)=0;
```

```
for i= 2:(1/Dt+1);
  t(i)=(i-1)*Dt;
% exact solution
  S(i)=log(1+t(i));
% explicit Euler
  eE(i)=eE(i-1)+Dt*exp(-eE(i-1));
% implicit Euler (using Newton)
  n_1=iE(i-1);
  while(abs(f(n_1,iE(i-1),Dt))>10^(-6))
    n_1=n_1-f(n_1,iE(i-1),Dt)/df(n_1,Dt);
  end
  iE(i)=n_1;
% scheme e (using Newton)
  e_1=Se(i-1);
  while(abs(f_e(e_1,Se(i-1),Dt))>10^(-6))
     e_1=e_1-f_e(e_1,Se(i-1),Dt)/df_e(e_1,Dt);
  end
  Se(i)=e_1;
% scheme f (using Newton)
  f_1=Sf(i-1);
  while(abs(f_f(f_1,Sf(i-1),Dt))>10^(-6))
    f_1=f_1-f_f(f_1,Sf(i-1),Dt)/df_f(f_1,Sf(i-1),Dt);
  end
  Sf(i)=f_1;
end

  disp(sprintf('Error: explicit Euler: %g',abs(S(i)-eE(i))));
  disp(sprintf('Error: implicit Euler: %g',abs(S(i)-iE(i))));
  disp(sprintf('Error: scheme e:       %g',abs(S(i)-Se(i))));
  disp(sprintf('Error: scheme f:       %g',abs(S(i)-Sf(i))));

plot(t,S,t,eE,'r--',t,iE,'g:',t,Se,'kx',t,Sf,'c+');
xlabel('t');
legend('Exact solution','Explicit Euler','Implicit Euler','Scheme e','Scheme f'

function val = f(u_n1, u_n, Dt);
    val = u_n1-Dt*exp(-u_n1)-u_n;

function der = df(u_n1,Dt);
    der = 1+Dt*exp(-u_n1);

function val_e = f_e(u_n1, u_n, Dt);
    val_e=u_n1-Dt*exp(-u_n1)*0.5-u_n-Dt*exp(-u_n)*0.5;

function der_e = df_e(u_n1,Dt)
     der_e = 1+Dt*0.5*exp(-u_n1);

function val_f = f_f(u_n1, u_n, Dt);
     val_f = Dt*exp(-0.5*u_n1-0.5*u_n)-u_n1+u_n;
```

```
function der_f = df_f(u_n1, u_n, Dt)
     der_f = -1-Dt*0.5*exp(-0.5*u_n1-0.5*u_n);
```