Inf165 Chapter 5

16th September 2003

Exercises

5.1

a) Constant approximation: $y_i = 4$.

b) Linear approximation: $y_i = 28 - 4t_i$.

c) Quadratic approximation: $y_i = 28 - 4t_i + 0t_i^2$.

5.2

a) Linear model: $p(t) = 58.6 - 6t$.

c)

Years t_i	1	2	3	4	5
$z_i = \ln(y_i)$	4.01	3.86	3.63	3.56	3.40

d) $q(t) = 4.13 - 0.15t$

e) $p(t) = 62.2e^{-0.15t}$

5.3

a) $c(n) = 0.005 + 0.00043n$

b) $c(10^6) \approx 430,\ c(10^7) \approx 4300$

5.4

a) $T(n) = 9.15 \cdot 10^{-4} + 7.35 \cdot 10^{-7}n$

b) Consider the difference between not sending a vector at all $(T = 0)$, and sending a vector of length zero $(T = \alpha)$.

c) The bandwidth is 0.087 gigabits per second.

5.5

a)

$$\begin{aligned}\frac{d}{d\alpha}\int_a^b F(\alpha,t)\,dt &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^b F(\alpha+\varepsilon,t)\,dt - \int_a^b F(\alpha,t)\,dt\right]\\ &= \int_a^b \lim_{\varepsilon\to 0}\frac{F(\alpha+\varepsilon,t)-F(\alpha,t)}{\varepsilon}\\ &= \int_a^b \frac{d}{d\alpha}F(\alpha,t)\,dt\end{aligned}$$

b)

$$\begin{aligned}L(\alpha) &= \frac{d}{d\alpha}\int_a^b (\alpha - p(t)))^2\,dt\\ &= \frac{d}{d\alpha}\left[\int_a^b \alpha^2\,dt - \int_a^b 2\alpha p(t)\,dt + \int_a^b (p(t))^2\,dt\right]\\ &= 2\alpha(b-a) - 2\int_a^b p(t)\,dt\\ R(\alpha) &= \int_a^b \frac{d}{d\alpha}(\alpha - p(t)))^2\,dt\\ &= \int_a^b \frac{d}{d\alpha}(\alpha^2 - 2\alpha p(t) + (p(t))^2)\,dt\\ &= 2\alpha(b-a) - 2\int_a^b p(t)\,dt\end{aligned}$$

5.6

Constant approximations:

a)

$$\alpha = \frac{1}{\pi - 0}\int_0^\pi \left(1 + \frac{1}{100}\sin(t)\right)\,dt = 1.006$$

b)

$$\alpha = \frac{1}{e^{-1} - 0}\int_0^{e^{-1}} e^t\,dt = 1.209$$

c)

$$\alpha = \frac{1}{\pi - 0}\int_0^\pi \sqrt{t}\,dt = 1.182$$

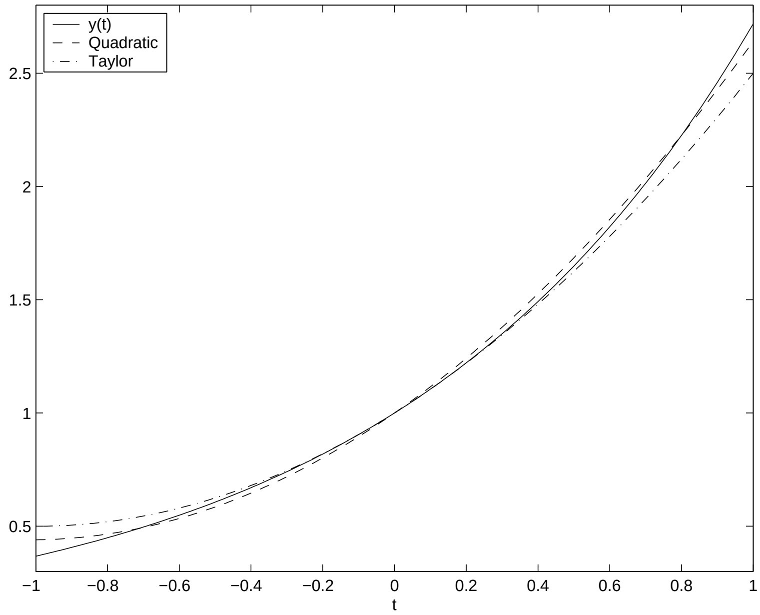

Figure 1: The function $y(t) = e^t$, approximated in two different ways.

5.7

Linear:

a) $y(t) \approx 0.88 + 1.68t$

b) $y(t) \approx 0.94 - 0.62t$

c) $y(t) \approx 1.41 - 0.48t$

5.8

Quadratic:

a) $y(t) \approx 1.01 + 0.861t + 0.829t^2$

b) $y(t) \approx 1.00 - 0.937t + 0.315t^2$

c) $y(t) \approx 2.11 + 1.45t + 0.320t^2$

5.9

a) Quadratic least squares approximation: $y(t) \approx 1 + 1.1t + 0.54t^2$

b) Taylor series expansion: $y(t) \approx e^0 + te^0 + \frac{t^2}{2}e^0 = 1 + t + \frac{t^2}{2}$

c) See figure 1.

Projects

5.5; Computing Coefficients

a) Using (5.153):

$$\int_{t_n}^{t_{n+1}} \frac{y'(t)}{y(t)}\,dt = [\ln y(t)]_{t=t_n}^{t=t_{n+1}} = \ln y(t_{n+1}) - \ln y(t_n)$$

b) Using equation (5.148) we can replace $\frac{y'(t)}{y(t)}$ with α in the equation from exercise **a**:

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \frac{y'(t)}{y(t)}\,dt = \int_{t_n}^{t_{n+1}} \alpha\,dt &= \ln y(t_{n+1}) - \ln y(t_n) \\
(t_{n+1} - t_n)\alpha &= \ln y(t_{n+1}) - \ln y(t_n) \\
\alpha &= \frac{\ln y(t_{n+1}) - \ln y(t_n)}{t_{n+1} - t_n}
\end{aligned}$$

c)

$$\begin{aligned}
\alpha_0 &= 0.01476 \\
\alpha_1 &= 0.01607 \\
\alpha_2 &= 0.01693 \\
\alpha_3 &= 0.01775 \\
\alpha_4 &= 0.01888
\end{aligned}$$

d) With these numbers we get $\alpha = 0.01688$.

e) See figure 2.

f) Using $\gamma = -\alpha/\beta$:

$$\begin{aligned}
y'(t) &= \alpha y(t)(1 - y(t)/\beta) \\
&= \alpha y(t) + \gamma y^2(t) \\
\frac{y'(t)}{y(t)} &= \alpha + \gamma y(t)
\end{aligned}$$

g) From (5.154):

$$\int_{t_n}^{t_{n+1}} \frac{y'(t)}{y(t)}\,dt = \ln y(t_{n+1}) - \ln y(t_n)$$

From (5.159):

$$\int_{t_n}^{t_{n+1}} \frac{y'(t)}{y(t)}\,dt = \int_{t_n}^{t_{n+1}} (\alpha + \gamma y(t))\,dt = \alpha(t_{n+1} - t_n) + \gamma \int_{t_n}^{t_{n+1}} y(t)\,dt$$

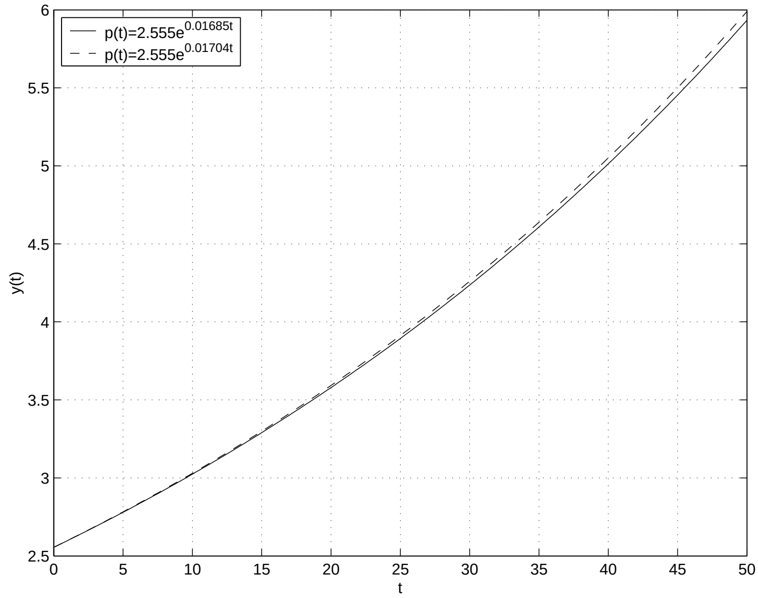

Figure 2: Two different functions $p(t)$, from (5.83) and (5.158).

h) We can use the trapezoid method to approximate the integral from **g**:

$$\begin{aligned}
\ln y(t_{n+1}) - \ln y(t_n) &= \alpha(t_{n+1} - t_n) + \gamma \int_{t_n}^{t_{n+1}} y(t)\,dt \\
\ln(y(t_{n+1})/y(t_n)) &\approx \alpha(t_{n+1} - t_n) + \gamma(t_{n+1} - t_n)\frac{1}{2}(y(t_{n+1}) + y(t_n)) \\
\frac{\ln(y(t_{n+1})/y(t_n))}{(t_{n+1} - t_n)} &\approx \alpha + \frac{\gamma}{2}(y(t_{n+1}) + y(t_n))
\end{aligned}$$

i)

year	n	$y(t_n)$	$c_n = \ln(y(t_{n+1})/y(t_n))$	$d_n = \frac{1}{2}(y(t_{n+1}) + y(t_n))$
1990	0	5.284	0.0156	5.3255
1991	1	5.367	0.0153	5.4085
1992	2	5.450	0.0148	5.4905
1993	3	5.531	0.0144	5.5710
1994	4	5.611	0.0142	5.6510
1995	5	5.691	0.0136	5.7300
1996	6	5.769	0.0134	5.8080
1997	7	5.847	0.0133	5.8860
1998	8	5.925	0.0131	5.9640
1999	9	6.003	0.0127	6.0415
2000	10	6.080		

j) By minimizing $F(\alpha, \gamma)$ we find the values of α and γ for which $\alpha + \gamma d_n$ is the linear least squares approximation to the dataset c_n.

k) $\frac{\partial F}{\partial \alpha} = 0$:

$$\begin{aligned}
\frac{\partial F}{\partial \alpha} &= 0 \\
\frac{\partial}{\partial \alpha} \sum_{n=0}^{9} (\alpha + \gamma d_n - c_n)^2 &= 0 \\
2 \sum_{n=0}^{9} (\alpha + \gamma d_n - c_n) &= 0 \\
10\alpha + \left(\sum_{n=0}^{9} d_n \right) \gamma &= \sum_{n=0}^{9} c_n
\end{aligned}$$

$\frac{\partial F}{\partial \gamma} = 0$:

$$\begin{aligned}
\frac{\partial F}{\partial \gamma} &= 0 \\
\frac{\partial}{\partial \gamma} \sum_{n=0}^{9} (\alpha + \gamma d_n - c_n)^2 &= 0 \\
2 \sum_{n=0}^{9} (\alpha + \gamma d_n - c_n) d_n &= 0 \\
\left(\sum_{n=0}^{9} d_n \right) \alpha + \left(\sum_{n=0}^{9} d_n^2 \right) \gamma &= \sum_{n=0}^{9} c_n d_n
\end{aligned}$$

l) From the table from exercise **i** we calculate $\sum_0^9 d_n = 56.8760$, $\sum_0^9 c_n = 0.1403$, $\sum_0^9 d_n^2 = 324.008$, and $\sum_0^9 c_n d_n = 0.7960$. Solving (5.166) as a system of two equations with two unknowns is trivial and gives $\alpha \approx 0.0368$, $\beta \approx -0.0040$.

m) $\beta = -\alpha/\gamma \approx -0.0368/(-0.0040) = 9.2$.

n) The solution to the logistic model with $y_0 = 6.08$ is:

$$\begin{aligned}
y(t) &= \frac{y_0 \beta}{y_0 + e^{-\alpha t}(\beta - y_0)} \\
&= \frac{6.08 \cdot 9.2}{6.08 + e^{-0.0368t}(9.2 - 6.08)} \\
&= \frac{55.94}{6.08 + 3.12 e^{-0.0368t}}
\end{aligned}$$

o) See figure 3.

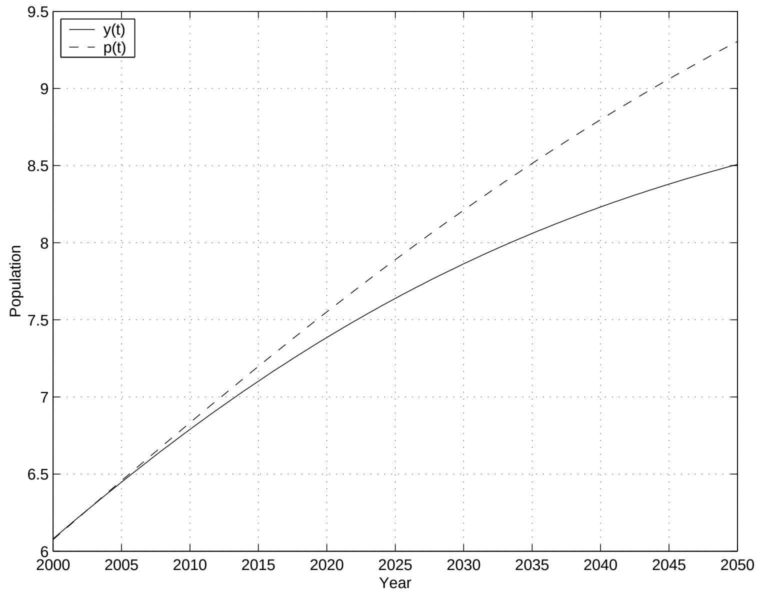

Figure 3: Population growth for 50 years, starting at year 2000, using equation 5.101 (p) and 5.169 (y).