

# Inf2320 Chapter 8

18th November 2003

## Exercises

### 8.1

a)

$$\begin{aligned}u_t &= -k^2\pi^2 ce^{-k^2\pi^2 t} \sin(k\pi x) \\u_x &= ce^{-k^2\pi^2 t} k\pi \cos(k\pi x) \\u_{xx} &= ce^{-k^2\pi^2 t} k\pi k\pi (-\sin(k\pi x)) = u_t\end{aligned}$$

b)

$$\begin{aligned}u(0, t) &= ce^{-k^2\pi^2 t} \sin 0 = 0 \\u(1, t) &= ce^{-k^2\pi^2 t} \sin(k\pi) = 0\end{aligned}$$

c)  $u(x, 0) = ce^{-k^2\pi^2 \cdot 0} \sin(k\pi x) = c \sin(k\pi x)$

### 8.2

a)

$$f(x) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (1 - \cos(k\pi)) \sin(k\pi x)$$

b) See the program **ex82.m**, and Figure 1.

c)

$$u(x, t) = \sum_{k=1}^{\infty} \frac{2}{k\pi} (1 - \cos(k\pi)) e^{-k^2\pi^2 t} \sin(k\pi x)$$

d) See the program **ex82d.m**, and Figure 2.

e) For initial condition  $f(x) = x$ :

$$\begin{aligned}f(x) &= \sum_{k=1}^{\infty} \frac{-2}{k\pi} \cos(k\pi) \sin(k\pi x) \\u(x, t) &= \sum_{k=1}^{\infty} \frac{-2}{k\pi} \cos(k\pi) e^{-k^2\pi^2 t} \sin(k\pi x)\end{aligned}$$

See the program **ex82e.m**, and Figure 3.

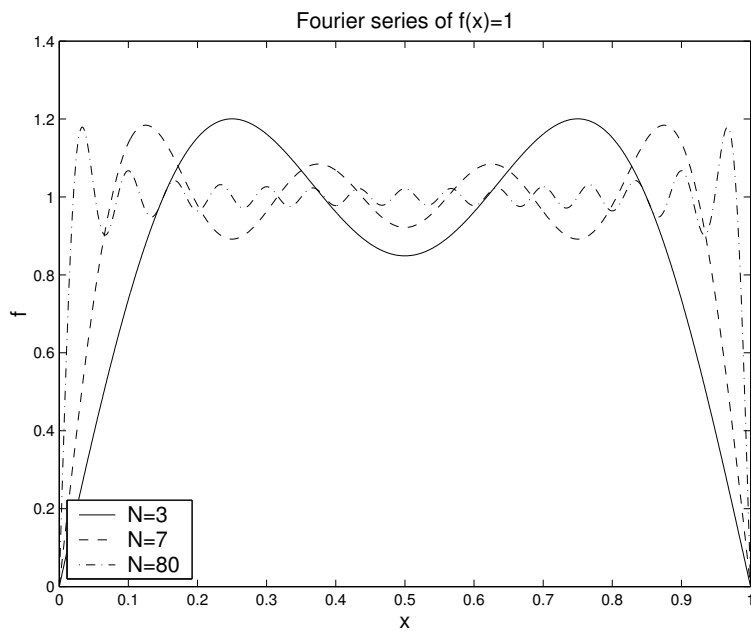


Figure 1: Exercise 8.2b. Plots of the Nth partial sum of the Fourier series of  $f(x) = 1$ , for three different values of N.

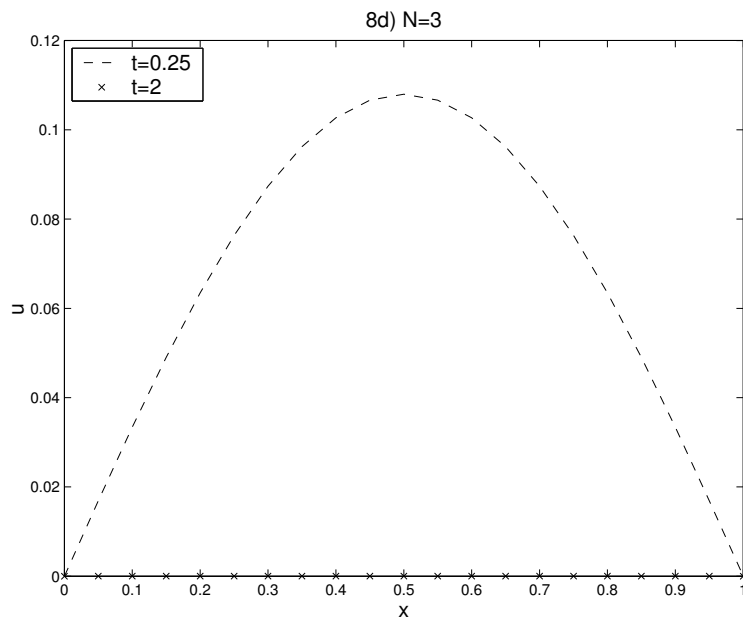


Figure 2: Exercise 8.2d. Plots of the 3rd partial sum of the formal solution of exercise c. The plots for N=7, N=80, and N=100 look exactly the same.

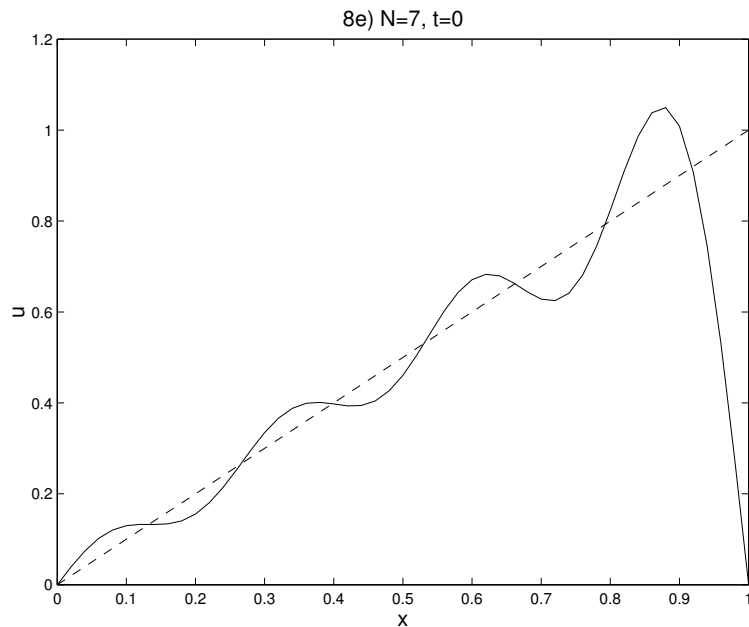


Figure 3: Exercise 8.2e. Figure made with the program **ex82e.m**. This one shows the 7th partial sum at time 0, which is just the Fourier series of  $f(x)$ . (Stapled line shows  $f(x) = x$ .)

### 8.3

a)

$$c_k = \frac{-4(1 + 2 \cos(k\pi))}{k^3 \pi^3}$$

b)

$$u(x, t) = \sum_{k=1}^{\infty} \frac{-4(1 + 2 \cos(k\pi))}{k^3 \pi^3} e^{-k^2 \pi^2 t} \sin(k\pi x)$$

c) Explicit scheme:

$$\begin{aligned} u_i^0 &= x_i^2 - x_i^3 \\ u_i^{l+1} &= \frac{\Delta t}{\Delta x^2} (u_{i-1}^l - 2u_i^l + u_{i+1}^l) + u_i^l \end{aligned}$$

d)+f) See the program **ex83.m**.

e)+g) See Figure 4.

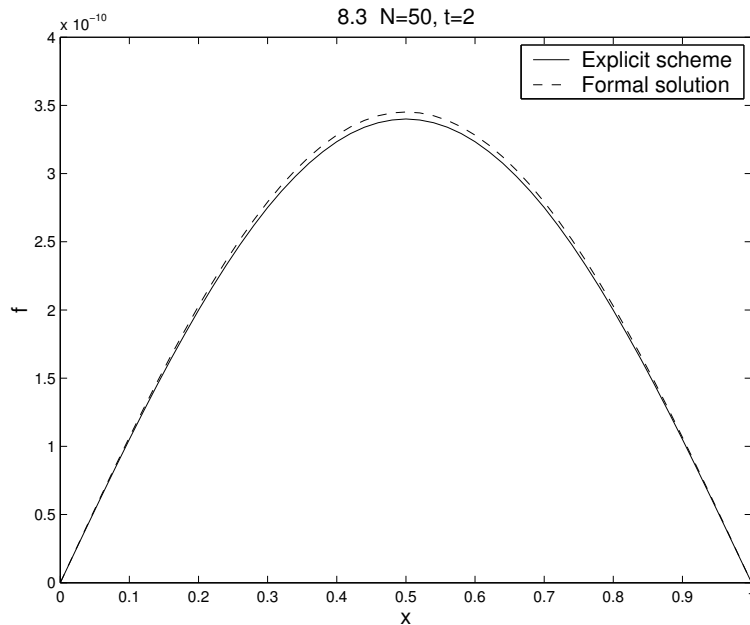


Figure 4: Exercise 8.3. Two different approximations of the solution to (8.91)-(8.93) at time 2: The explicit scheme, and the 50 first terms of the formal solution.

## 8.4

a)

$$f(x) = \sum_{k=1}^{\infty} \frac{2k\pi(1 - e \cos(k\pi)) \sin(k\pi x)}{1 + k^2\pi^2}$$

b) See the program **ex84.m**.

c) See the program **ex84c.m**, and figure 5.

## 8.5

Using (8.71):

$$\int_0^1 \sin(k\pi x) \sin(l\pi x) dx = \frac{1}{2} \int_0^1 \cos((k-l)\pi x) dx - \frac{1}{2} \int_0^1 \cos((k+l)\pi x) dx$$

If  $k = l$ :

$$\begin{aligned} \frac{1}{2} \int_0^1 \cos((k-l)\pi x) dx - \frac{1}{2} \int_0^1 \cos((k+l)\pi x) dx &= \frac{1}{2} \int_0^1 \cos 0 dx - \frac{1}{2} \left[ \frac{\sin(2k\pi x)}{2k\pi} \right]_0^1 \\ &= \frac{1}{2} \int_0^1 1 dx - \frac{1}{2} [0 - 0] = \frac{1}{2} \end{aligned}$$

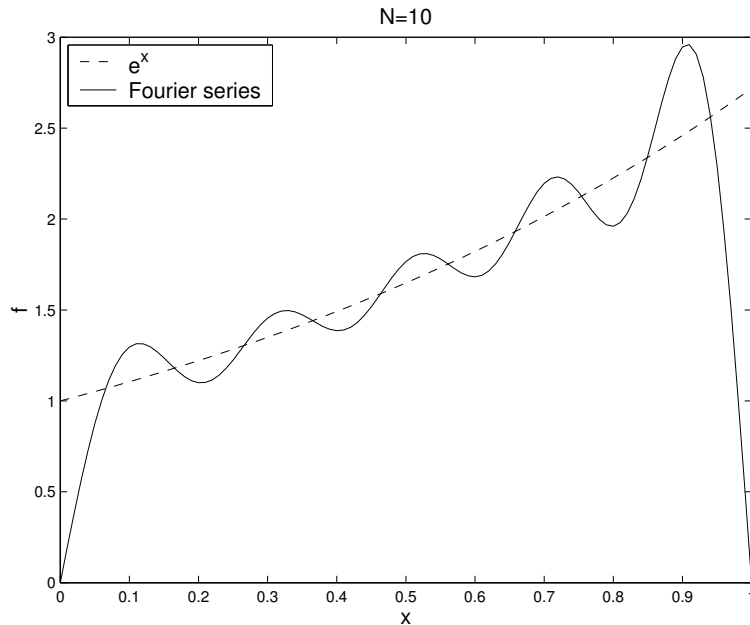


Figure 5: Exercise 8.4. Plot of the 10th partial sum of the Fourier series for  $e^x$ .

If  $k \neq l$ :

$$\begin{aligned} \frac{1}{2} \int_0^1 \cos((k-l)\pi x) dx - \frac{1}{2} \int_0^1 \cos((k+l)\pi x) dx &= \frac{1}{2} \left[ \frac{\sin((k-l)\pi x)}{(k-l)\pi} \right]_0^1 - \frac{1}{2} \left[ \frac{\sin((k+l)\pi x)}{(k+l)\pi} \right]_0^1 \\ &= \frac{1}{2}[0-0] - \frac{1}{2}[0-0] = 0 \end{aligned}$$

## 8.6

a) Using (8.71):

$$\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi x}{L}\right) dx = \frac{1}{2} \int_0^L \cos\left(\frac{(k-l)\pi x}{L}\right) dx - \frac{1}{2} \int_0^L \cos\left(\frac{(k+l)\pi x}{L}\right) dx$$

if  $k = l$ :

$$\begin{aligned} \frac{1}{2} \int_0^L \cos\left(\frac{(k-l)\pi x}{L}\right) dx - \frac{1}{2} \int_0^L \cos\left(\frac{(k+l)\pi x}{L}\right) dx &= \frac{1}{2} \int_0^L \cos 0 dx - \frac{1}{2} \left[ \frac{L}{2k\pi} \sin\left(\frac{2k\pi x}{L}\right) \right]_0^L \\ &= \frac{1}{2} \int_0^L 1 dx - \frac{1}{2}[0-0] = \frac{L}{2} \end{aligned}$$

if  $k \neq l$ :

$$\begin{aligned}
& \frac{1}{2} \int_0^L \cos \frac{(k-l)\pi x}{L} dx - \frac{1}{2} \int_0^L \cos \frac{(k+l)\pi x}{L} dx \\
&= \frac{1}{2} \left[ \frac{L}{(k-l)\pi} \sin \frac{(k-l)\pi x}{L} \right]_0^L - \frac{1}{2} \left[ \frac{L}{(k+l)\pi} \sin \frac{(k+l)\pi x}{L} \right]_0^L \\
&= \frac{1}{2}[0-0] - \frac{1}{2}[0-0] = 0
\end{aligned}$$

**b)** We want to find the constants  $c_1, c_2, \dots$  such that:

$$f(x) = \sum_{k=1}^{\infty} c_k \sin \left( \frac{k\pi x}{L} \right) \text{ for } x \in (0, L)$$

Multiply the left and right side of the equation by  $\sin(l\pi x/L)$  and integrate:

$$\int_0^L f(x) \sin \left( \frac{l\pi x}{L} \right) dx = \int_0^L \left( \sum_{k=1}^{\infty} c_k \sin \left( \frac{k\pi x}{L} \right) \right) \sin \left( \frac{l\pi x}{L} \right) dx$$

Interchange the order of integration and summation, and use the property from part **a**:

$$\begin{aligned}
\int_0^L f(x) \sin \left( \frac{l\pi x}{L} \right) dx &= \sum_{k=1}^{\infty} c_k \int_0^L \sin \left( \frac{k\pi x}{L} \right) \sin \left( \frac{l\pi x}{L} \right) dx \\
&= \frac{L}{2} c_l
\end{aligned}$$

**c)**

$$\begin{aligned}
u_t &= - \left( \frac{k\pi}{L} \right)^2 c e^{-\left(\frac{k\pi}{L}\right)^2 t} \sin \left( \frac{k\pi x}{L} \right) \\
u_{xx} &= c e^{-\left(\frac{k\pi}{L}\right)^2 t} \left( \frac{k\pi}{L} \right) \left( \frac{k\pi}{L} \right) \left( - \sin \left( \frac{k\pi x}{L} \right) \right) \\
u_t &= u_{xx}
\end{aligned}$$

**d)**

$$f(x) = \sum_{k=1}^{\infty} \frac{2k\pi (1 - e^L \cos(k\pi)) \sin \left( \frac{k\pi x}{L} \right)}{L^2 + k^2\pi^2}$$

$$u(x, t) = \sum_{k=1}^{\infty} \frac{2k\pi (1 - e^L \cos(k\pi))}{L^2 + k^2\pi^2} e^{-k^2\pi^2 t} \sin \left( \frac{k\pi x}{L} \right)$$

**e)+f)** See the program **ex86.m**.

## 8.7

a)-e) See the program **ex87.m**.

f)+g) See the program **ex87g.m**. This program can be altered to solve exercise f.

## 8.8

Use the property that

$$\begin{aligned}\frac{\partial}{\partial t}e^{-Kt}u(x,t) &= u_t e^{-Kt} - K u e^{-Kt} \\ &= u_{xx} e^{-Kt} - K u e^{-Kt}.\end{aligned}$$

This gives the result:

$$\begin{aligned}v_t + K v &= u_{xx} e^{-Kt} - K u e^{-Kt} + K u e^{-Kt} \\ &= u_{xx} e^{-Kt} = v_{xx}\end{aligned}$$

Boundary conditions:

$$\begin{aligned}v(0,t) &= e^{-Kt}u(0,t) = 0 \\ v(1,t) &= e^{-Kt}u(1,t) = 0\end{aligned}$$

Initial condition:

$$v(x,0) = e^{-K \cdot 0}u(x,0) = u(x,0) = f(x)$$

## 8.9

a) (8.85):

$$\begin{aligned}u_i^{l+1} &= \alpha u_{i-1}^l + (1 - 2\alpha)u_i^l + \alpha u_{i+1}^l \\ &= G(u_{i-1}^l, u_i^l, u_{i+1}^l)\end{aligned}$$

b)

$$\begin{aligned}\frac{\partial G}{\partial U_-} &= \alpha = \frac{\Delta t}{\Delta x^2} \geq 0 \\ \frac{\partial G}{\partial U} &= 1 - 2\alpha \geq 0 \quad \left(\text{since } \alpha \leq \frac{1}{2}\right) \\ \frac{\partial G}{\partial U_+} &= \alpha = \frac{\Delta t}{\Delta x^2} \geq 0\end{aligned}$$

c) The result from exercise b means that  $G(u_{i-1}^l, u_i^l, u_{i+1}^l)$  is never higher than  $G(\bar{u}_{\max}^l, \bar{u}_{\max}^l, \bar{u}_{\max}^l)$ .

$$u_i^{l+1} = G(u_{i-1}^l, u_i^l, u_{i+1}^l) \leq G(\bar{u}_{\max}^l, \bar{u}_{\max}^l, \bar{u}_{\max}^l) = \bar{u}_{\max}^l$$

As this inequality is true for all  $u_i^{l+1}$ , it is true for  $\bar{u}_{\max}^{l+1}$ .

- d)** Using induction on the above result gives  $\bar{u}_{\max}^l \leq \bar{u}_{\max}^1$ . For  $u_i^1$  the inner points ( $i = 2, \dots, n-1$ ) must have a maximum smaller or equal to  $\bar{u}_{\max}^0$  and the boundaries the value is known to be 0. This gives the result

$$\bar{u}_{\max}^l \leq \max \left( \max_x f(x), 0 \right).$$

- e)** The result from exercise **b** also indicate that  $G(u_{i-1}^l, u_i^l, u_{i+1}^l)$  is never lower than  $G(\bar{u}_{\min}^l, \bar{u}_{\min}^l, \bar{u}_{\min}^l)$ .

$$u_i^{l+1} = G(u_{i-1}^l, u_i^l, u_{i+1}^l) \geq G(\bar{u}_{\min}^l, \bar{u}_{\min}^l, \bar{u}_{\min}^l) = \bar{u}_{\min}^l$$

This inequality is true for all  $u_i^{l+1}$ , including  $\bar{u}_{\min}^{l+1}$ . Using induction on this result leads to  $\bar{u}_{\min}^l \geq \bar{u}_{\min}^1$ . When Including the boundary conditions this leads to the result

$$\bar{u}_{\min}^l \geq \min \left( \min_x f(x), 0 \right).$$

- f)** Equation (8.100) is a stronger result than equation (8.98) because we can derive (8.98) from (8.100), but not the other way around. Equation (8.100) can be separated into three cases:

$$\begin{array}{ll} \text{if } f(x) > 0 \quad \forall x & \text{then } 0 \leq u_i^l \leq \max f(x) \\ \text{if } f(x) < 0 \quad \forall x & \text{then } \min f(x) \leq u_i^l \leq 0 \\ \text{if } \max f(x) \geq 0 \geq \min f(x) & \text{then } \min f(x) \leq u_i^l \leq \max f(x) \end{array}$$

All three results in turn indicate that  $\max |u_i| \leq \max |f(x)|$  i.e. (8.100) implies (8.98).

## 8.10

$$\begin{aligned} \frac{\partial}{\partial t} E_1(t) &= \frac{\partial}{\partial t} \int_0^1 u^2(x, t) dx \\ &= \int_0^1 \frac{\partial}{\partial t} u^2(x, t) dx \\ &= \int_0^1 2uu_t dx = \int_0^1 2uu_{xx} dx \\ &= 2 \left( u(1, t)u_x(1, t) - u(0, t)u_x(0, t) - \int_0^1 u_x u_x dx \right) \\ &= -2 \int_0^1 u_x^2 dx \leq 0 \end{aligned}$$

$E'(t) \leq 0$  i.e.  $E_1(t_2) \leq E_1(t_1)$  for all  $t_2 \geq t_1 \geq 0$ .



## 8.11

For  $x \in (0, 1)$  and  $t < 0$ :

$$\frac{\partial}{\partial t} g(t) = \frac{\partial}{\partial t} f(x) = 0$$

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x} g(t) = 0$$

Both  $f(x)$  and  $g(t)$  are constants. Since  $f(x) = g(t)$ , they must be the same constant.

## 8.12

Use the property that

$$\int_0^1 f(x) dx = F(1) - F(0) = 0,$$

where  $F'(x) = f(x)$ . Because  $f(x) \geq 0$  in  $x \in (0, 1)$ ,  $F(x)$  is an *increasing* function. In order to have  $F(1) = F(0)$ ,  $F'(x)$  (i.e.  $f(x)$ ) must be zero for all  $x \in (0, 1)$ .

## 8.13

We know that  $f(x) \geq 0$ , and that  $S$  is a subset of  $[0, 1]$ . This means that the sum of  $\int f(x) dx$  on each interval that is part of  $S$  must be equal to or smaller than  $\int f(x) dx$  on the entire interval  $[0, 1]$ .

$$\sum_{i=1}^{\infty} \int_{a_i}^{b_i} f(x) dx \leq \int_0^1 f(x) dx \leq \epsilon$$

On each interval  $[a_i, b_i]$  we have that

$$\int_{a_i}^{b_i} f(x) dx \geq (b_i - a_i) \min_{a_i \leq x \leq b_i} f(x) \geq (b_i - a_i) \sqrt{\epsilon}.$$

Putting this together gives the result

$$|S| = \sum_{i=1}^{\infty} (b_i - a_i) \leq \sum_{i=1}^{\infty} \frac{1}{\sqrt{\epsilon}} \int_{a_i}^{b_i} f(x) dx \leq \frac{1}{\sqrt{\epsilon}} \int_0^1 f(x) dx \leq \sqrt{\epsilon},$$

which we wanted to prove.

## Projects

### 8.5.1; Neumann boundary conditions

a) Explicit scheme for solving (8.101)-(8.103) (using  $\alpha = \Delta t / \Delta x^2$ ):

$$\begin{aligned} u_i^{l+1} &= u_i^l + \alpha(u_{i-1}^l - 2u_i^l + u_{i+1}^l) \text{ for } i = 2, \dots, n-1 \\ u_1^{l+1} &= 2\alpha(u_2^l - u_1^l) + u_1^l \\ u_n^{l+1} &= 2\alpha(u_{n-1}^l - u_n^l) + u_n^l \\ u_i^0 &= f(x_i) \end{aligned}$$

b) See the program **p851.m**.

c) (8.104) and (8.105) are approximations of the integrals from (8.7), using the composite trapezoid rule. For the inequality from (8.7) to be valid, the result of (8.104) must be less than or equal to the result of (8.105). The table shows some results for different  $\Delta t$  and  $\Delta x$ , for  $T = 1$ :

$\Delta x$	$\Delta t$	(8.104)	(8.105)
0.1	0.001	0.0776	0.0972
0.05	0.001	0.0776	0.0973
0.02	0.0002	0.0793	0.0973
0.02	0.000201	$1.37 \cdot 10^{36}$	0.0973

The results indicate that the inequality (8.7) is valid for the problem defined in (8.101)-(8.103). The last result is an example of what can happen if  $\alpha > \frac{1}{2}$ .

d) Multiply by  $u$  on both sides of equation (8.101), replace  $uu_t$  with  $\frac{1}{2} \frac{\partial}{\partial t} u^2$ , and integrate with respect to  $x$ :

$$\begin{aligned} \int_0^1 \frac{1}{2} \frac{\partial}{\partial t} u^2 dx &= \int_0^1 u_{xx}(x, t) u(x, t) dx \\ &= u_x(1, t) u(1, t) - u_x(0, t) u(0, t) - \int_0^1 u_x(x, t) u_x(x, t) dx \end{aligned}$$

With the boundary conditions,  $u_x(0, t) = u_x(1, t) = 0$ , this becomes

$$\frac{1}{2} \frac{\partial}{\partial t} \int_0^1 u^2 dx = - \int_0^1 u_x(x, t) u_x(x, t) dx \quad \text{for } t > 0.$$

This means that the time derivative of  $\int_0^1 u^2 dx$  is negative, i.e.

$$\int_0^1 u^2(x, t) dx \leq \int_0^1 u^2(x, 0) dx \quad \text{for } t > 0.$$

e) The two sums represent approximations of the integral from 0 to 1 of  $u$  and  $f$ . By trying out different values of  $\Delta x$  and  $\Delta t$  we find that the two integrals are approximately equal (as long as we remember to use  $\Delta t \leq \Delta x^2/2$ ).

f) The derivative of  $\int_0^1 u dx$  with respect to time is

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^1 u(x, t) dx &= \int_0^1 u_t(x, t) dx \\ &= \int_0^1 u_{xx}(x, t) dx \\ &= u_x(1, t) - u_x(0, t) = 0 - 0 = 0. \end{aligned}$$

This means that  $\int_0^1 u dx$  is constant in time, i.e.

$$\int_0^1 u(x, t) dx = \int_0^1 u(x, 0) dx = \int_0^1 f(x) dx.$$

- g)** A simple numerical test of (8.106) is to have the program **p851.m** calculate the difference between the maximum and minimum values of  $u$ . If the conjecture is correct the difference will decrease as time increases. This experiment gives the following results:

$t$	$\max(u) - \min(u)$
0	2
0.01	1.34
0.1	$3.68 \cdot 10^{-2}$
0.5	$4.24 \cdot 10^{-9}$

- h)** When  $x(x, t) = X(x)T(t)$  equation (8.101) becomes

$$X(x)T'(t) = X''(x)T(t).$$

By dividing both sides by  $X(x)T(t)$  we get

$$\frac{T'(t)}{X(x)} = \frac{X''(x)}{X(x)}.$$

Now the left hand side depends only on  $t$  and the right hand side depends only on  $x$ . This means both sides must be equal to a constant, which we can call  $\lambda$ , i.e.

$$\frac{T'(t)}{X(x)} = \lambda, \quad \frac{X''(x)}{X(x)} = \lambda.$$

- i)** If  $T(t) = ce^{\lambda t}$  we get:

$$T'(t) = \frac{\partial}{\partial t} ce^{\lambda t} = \lambda ce^{\lambda t} = \lambda T(t)$$

- j)** If  $X(x) = \cos(k\pi x)$  we get

$$X''(x) = \frac{\partial^2}{\partial x^2} \cos(k\pi x) = -k^2 \pi^2 \cos(k\pi x) = -k^2 \pi^2 X(x),$$

which means that  $\lambda = -k^2 \pi^2$ . We must also check the boundary conditions:

$$X'(0) = -k\pi \sin(0) = 0,$$

$$X'(1) = -k\pi \sin(k\pi) = 0, \text{ for } k = 0, 1, 2, \dots$$

- k)** We now know that  $u(x, t) = X(x)T(t) = e^{-k^2 \pi^2 t} \cos(k\pi x)$  is a solution of (8.101) and (8.102). By the super-positioning principle any linear combination of these solutions is also a solution. We can write the solutions as a series: (Using  $\cos(-y) = \cos(y)$  and  $\cos(0) = e^0 = 1$ .)

$$u(x, t) = c_0 + \sum_{k=1}^{\infty} c_k e^{-k^2 \pi^2 t} \cos(k\pi x)$$

**l)** If  $k = l = 0$ :

$$\begin{aligned}\int_0^1 \cos(k\pi x) \cos(l\pi x) dx &= \int_0^1 \cos(0)\cos(0) dx \\ &= \int_0^1 1^2 dx = 1\end{aligned}$$

If  $k = l > 0$ :

$$\begin{aligned}\int_0^1 \cos(k\pi x) \cos(l\pi x) dx &= \frac{1}{2} \int_0^1 \cos((k+l)\pi x) dx + \frac{1}{2} \int_0^1 \cos((k-l)\pi x) dx \\ &= \frac{1}{2} \int_0^1 \cos(2k\pi x) dx + \frac{1}{2} \int_0^1 \cos(0) dx \\ &= \frac{1}{2} \left[ \frac{\sin(2k\pi x)}{2k\pi} \right]_0^1 + \frac{1}{2} \int_0^1 1 dx \\ &= \frac{1}{2}\end{aligned}$$

If  $k \neq l$ :

$$\begin{aligned}\int_0^1 \cos(k\pi x) \cos(l\pi x) dx &= \frac{1}{2} \int_0^1 \cos((k+l)\pi x) dx + \frac{1}{2} \int_0^1 \cos((k-l)\pi x) dx \\ &= \frac{1}{2} \left[ \frac{\sin((k+l)\pi x)}{(k+l)\pi} \right]_0^1 + \frac{1}{2} \left[ \frac{\sin((k-l)\pi x)}{(k-l)\pi} \right]_0^1 \\ &= 0\end{aligned}$$

**m)** If (8.109) holds then

$$\begin{aligned}c_0 &= f(x) - \sum_{k=1}^{\infty} c_k \cos(k\pi x) \\ \int_0^1 c_0 dx &= \int_0^1 f(x) dx - \int_0^1 \sum_{k=1}^{\infty} c_k \cos(k\pi x) dx \\ c_0 &= \int_0^1 f(x) dx - \sum_{k=1}^{\infty} \int_0^1 c_k \cos(k\pi x) dx \\ &= \int_0^1 f(x) dx - \sum_{k=1}^{\infty} \left[ c_k \frac{\sin(k\pi x)}{k\pi} \right]_0^1 \\ &= \int_0^1 f(x) dx.\end{aligned}$$

Furthermore

$$f(x) \cos(l\pi x) = c_0 \cos(l\pi x) + \sum_{k=1}^{\infty} \cos(k\pi x) \cos(l\pi x)$$

$$\begin{aligned}
\int_0^1 f(x) \cos(l\pi x) dx &= \int_0^1 c_0 \cos(l\pi x) dx + \int_0^1 \sum_{k=1}^{\infty} \cos(k\pi x) \cos(l\pi x) dx \\
&= c_0 \int_0^1 \cos(l\pi x) dx + \int_0^1 \sum_{k=1}^{\infty} \cos(k\pi x) \cos(l\pi x) dx \\
&= c_0 \left[ \frac{\sin(2k\pi x)}{2k\pi} \right]_0^1 + c_k \frac{1}{2} \\
2 \int_0^1 f(x) \cos(l\pi x) dx &= c_k
\end{aligned}$$

n)

$$u(x, t) = 3.14 + e^{-4\pi^2 t} \cos(2\pi x)$$

o)

$$\begin{aligned}
\lim_{t \leftarrow \infty} u(x, t) &= \lim_{t \leftarrow \infty} \left( c_0 + \sum_{k=1}^{\infty} c_k \cos(k\pi x) \right) \\
&= \lim_{t \leftarrow \infty} c_0 + \sum_{k=1}^{\infty} \lim_{t \leftarrow \infty} c_k \cos(k\pi x) \\
&= c_0 + 0 \\
&= \int_0^1 f(x) dx
\end{aligned}$$

## 8.5.2; variable coefficients

a) Using the approximations (8.113)-(8.116), we generate a scheme for solving (8.110):

$$\begin{aligned}
\frac{u_i^{l+1} - u_i^l}{\Delta t} &= \frac{(ku_x)(x_{i+1/2}, t_l) - (ku_x)(x_{i-1/2}, t_l)}{\Delta x} \\
&= \frac{k_{i+1/2}(u_{i+1}^l - u_i^l) - k_{i-1/2}(u_i^l - u_{i-1}^l)}{\Delta x^2} \\
u_i^{l+1} &= u_i^l + \frac{\Delta t}{\Delta x^2} (k_{i+1/2}(u_{i+1}^l - u_i^l) - k_{i-1/2}(u_i^l - u_{i-1}^l)) \\
&= \alpha k_{i-1/2} u_{i-1}^l + (1 - \alpha(k_{i+1/2} + k_{i-1/2})) u_i^l + \alpha k_{i+1/2} u_{i+1}^l
\end{aligned}$$

Where  $\alpha = \frac{\Delta t}{\Delta x^2}$ . See program **p852.m**.

b) We know that  $k \leq M$  and that  $\alpha \leq \frac{1}{2M}$ . This means that  $(1 - \alpha(k_{i-1/2} + k_{i+1/2})) \geq (1 - 2\alpha M) \geq 0$  and it follows from the triangle inequality that

$$\begin{aligned}
|u_i^{l+1}| &= |\alpha k_{i-1/2} u_{i-1}^l + (1 - \alpha(k_{i-1/2} + k_{i+1/2})) u_i^l + \alpha k_{i+1/2} u_{i+1}^l| \\
&\leq |\alpha k_{i-1/2} u_{i-1}^l| + |(1 - \alpha(k_{i-1/2} + k_{i+1/2})) u_i^l| + |\alpha k_{i+1/2} u_{i+1}^l| \\
&= \alpha k_{i-1/2} |u_{i-1}^l| + (1 - \alpha(k_{i-1/2} + k_{i+1/2})) |u_i^l| + \alpha k_{i+1/2} |u_{i+1}^l| \\
&\leq \alpha k_{i-1/2} \bar{u}^l + (1 - \alpha(k_{i-1/2} + k_{i+1/2})) \bar{u}^l + \alpha k_{i+1/2} \bar{u}^l \\
&= \bar{u}^l
\end{aligned}$$

for  $i = 2, \dots, n - 1$ . Combined with the boundary conditions this means we can write

$$\bar{u}^{l+1} \leq \bar{u}^l,$$

and, using induction, we conclude that

$$\max_i |u_i^l| \leq \max_i |u_i^0| = \max_i |f(x_i)|.$$

c) (8.119) and (8.120) are approximations of the integrals from (8.7), using the composite trapezoid rule. For the inequality from (8.7) to be valid, the result of (8.119) must be less than or equal to the result of (8.120). The results from program **p852.m** indicate that the inequality is valid for this problem.

d) We define  $E(t) = \int_0^1 u^2 dx$  and

$$\begin{aligned} u_t u &= (ku_x)_x u \\ \frac{1}{2} \frac{\partial}{\partial t} \int_0^1 u^2(x, t) dx &= \int_0^1 (ku_x)_x u dx \\ &= k(1)u_x(1, t)u(1, t) - k(0)u_x(0, t)u(0, t) - \int_0^1 ku_x^2(x, t) dx \\ &= - \int_0^1 ku_x^2(x, t) dx \end{aligned}$$

Since both  $k(x)$  and  $u_x^2(x, t)$  are non-negative on  $x \in [0, 1]$  this means that  $E'(t) \leq 0$ , i.e.

$$\int_0^1 u^2(x, t) dx \leq \int_0^1 u^2(x, 0) dx = \int_0^1 f^2(x) dx$$

e) We can test the bound by modifying the program **p852.m** so that it also calculates a trapezoidal approximation of the two integrals in (8.121). Running the program a few times with different values for  $\Delta t$  and  $\Delta x$  (mindful of the stability condition) indicates that the bound (8.121) holds true.

f)

$$\begin{aligned} \int_0^1 u_t^2(x, t) dx &= \int_0^1 (ku_x)_x u_t dx \\ &= k(1)u_x(1, t)u_t(1, t) - k(0)u_x(0, t)u_t(0, t) - \int_0^1 ku_x(x, t)u_{xt}(x, t) dx \\ &= - \int_0^1 ku_x(x, t)u_{xt}(x, t) dx \\ &= - \frac{1}{2} \frac{\partial}{\partial t} \int_0^1 ku_x^2(x, t) dx \end{aligned}$$

$$\frac{\partial}{\partial t} \int_0^1 k u_x^2(x, t) dx = -2 \int_0^1 u_t^2(x, t) dx$$

the time derivative of the integral is always negative, leading to (8.121).

- g) We have assumed that there exist numbers  $m$  and  $M$  such that  $0 < m \leq k(x) \leq M$ , for  $x \in [0, 1]$ . We can use this to find the following bounds for the integral expressions in (8.121).

$$\begin{aligned} \int_0^1 k(x) u_x^2(x, t) dx &\geq \int_0^1 m u_x^2(x, t) dx = m \int_0^1 u_x^2(x, t) dx \\ \int_0^1 k(x) u_x^2(x, 0) dx &\leq \int_0^1 M u_x^2(x, 0) dx = M \int_0^1 u_x^2(x, 0) dx \end{aligned}$$

Combining this with expression (8.121) leads to

$$\int_0^1 u_x^2(x, t) dx \leq \frac{M}{m} \int_0^1 u_x^2(x, 0) dx = \frac{M}{m} \int_0^1 f_x^2(x) dx$$

## Programs (MATLAB)

### ex82.m

```
function ex82(N)
% Plots the Nth partial sum of the Fourier series
% of f(x)=1, for x in (0,1).
% example: ex82(7)

x=0:0.02:1;
f=x.*0;

for k=1:N
c=2/(k*pi)*(1-cos(k*pi));
f=f+c*sin(x.*pi*k);
end

plot(x, f);
xlabel('x');
ylabel('f');
title(['8b N=', num2str(N)]);
```

### ex82d.m

```
function ex82d(N,t)
% Plots the Nth partial sum of the solution to exercise 8.2c
% at time t
% example: ex82d(7,0.25)
```

```

x=0:0.02:1;
u=x.*0;

for k=1:N
c=2/(k*pi)*(1-cos(k*pi));
u=u+c*exp(-k*k*pi*pi*t)*sin(x.*pi*k);
end

plot(x,u);
xlabel('x'); ylabel('u');
title(['8d N=',num2str(N),', t=',num2str(t)]);

```

### ex82e.m

```

function ex82e(N,t)
% Plots the Nth partial sum of the solution to exercise 8.2e
% at time t
% example: ex82e(7,0.25)

x=0:0.02:1;
u=x.*0;

for k=1:N
c=-2*cos(k*pi)/(k*pi);
u=u+c*exp(-k*k*pi*pi*t)*sin(x.*pi*k);
end

plot(x,u,x,x,'r--');
xlabel('x'); ylabel('u');
title(['8e N=',num2str(N),', t=',num2str(t)]);

```

### ex83.m

```

function ex83(N,t)
% Solves the problem from exercise 8-3 at time t.
% both by an explicit scheme and by using the
% N first terms of the formal solution.
% example: ex83(50,2)

x=0:0.02:1;
Dx= 0.02;
Dt= Dx*Dx/2;

% explicit scheme:
us=x.^2 - x.^3;
for t=0:Dt:t
for i=2:length(us)-1
u(i)=us(i) +Dt/(Dx*Dx)*(us(i-1) -2*us(i)+us(i+1));
end

```



```

    u(1)=0; u(length(us))=0;
    us=u;
end

% formal solution:
f=x.*0;
for k=1:N
    c= -4*(1+2*cos(k*pi))/(k*k*k*pi*pi*pi);
    f=f+c*exp(-k*k*pi*pi*t)*sin(x.*pi*k);
end

plot(x,u,x,f,'r--')
xlabel('x'); ylabel('f');
title(['8-3 N=',num2str(N),', t=',num2str(t)]);
legend('Explicit scheme','Formal solution',1)

```

### ex84.m

```

function u = ex84(N,x)
% Finds the sum of the N first terms of the
% Fourier sine series for e^x
% example: ex84(5,0.5)

u=0;
for k=1:N
c=2*k*pi*(1-exp(1)*cos(k*pi))/(1+k*k*pi*pi);
u=u+c*sin(x*pi*k);
end

```

### ex84c.m

```

function ex84c(N)
% Plots the partial sum of the Fourier series
% for e^x. Also plots e^x.
% Calls the function ex84.m.
% example: ex84c(50)

x=0:0.02:1;
u=x.*0;

for k=1:50
u(k)=ex84(N,x(k));
end

plot(x,exp(x),'r--',x,u);
legend('e^x','Fourier series',2);
title(['N=',int2str(N)]);

```

### ex86.m

```
function ex86(L,N,T,Dx,Dt)
% Solves the problem in exercise 8.6.
% in two different ways.
% example: ex86(2,6,0.01,0.05,0.001)

x=0:0.05:L;
f=exp(x);u=x.*0;

% Fourier:
for k=1:N
c=2*k*pi*(1-exp(L)*cos(k*pi))/(L*L+k*k*pi*pi);
u=u+c*exp(-k*k*pi*pi*T/(L*L))*sin(x.*pi*k/L);
end

% explicit scheme
ex=f;
for t=0:Dt:T
    for i=2:length(ex)-1
        e(i)=ex(i)+Dt/(Dx*Dx)*(ex(i-1)-2*ex(i)+ex(i+1));
    end
    e(1)=0;e(length(ex))=0;
    ex=e;
end

plot(x,u,x,ex,'r--');
legend('Fourier series','explicit scheme',2);
xlabel('x');
title(['N=',num2str(N),' t=',num2str(T) ] );
```

### ex87.m

```
function ex87(N)
% for exercise 8.7.a-e
% Computes the Nth partial sum of the Fourier series
% for  $e^{x^2}$ , using the trapezoidal and Simpson's rule
% example: ex87(100)

x=0:0.01:1;
f=x.*0; s=x.*0;

for k=1:N
ck=TrapRuleFourier(100,k);
f=f+ck*sin(x.*k*pi);
cks=SimpsonFourier(100,k);
s=s+cks*sin(x.*k*pi);
end

g=exp(-x.*x);
```

```

plot(x,f,x,s,'c--',x,g,'r--');
xlabel('x');
legend('Trapez','Simpson','exact solution',1)
title(['8-7 N=',num2str(N)]);

function ck=TrapRuleFourier(n,k)
% n - intervals
% k - coefficient c_k

dx=1/n; x=0; ck=0;
for i=1:n-1
x=x+dx;
ck= ck + ex87a(x)*sin(k*pi*x)/n;
end
ck=ck*2;

function ck=SimpsonFourier(n,k)
% n - intervals
% k - coefficient c_k

dx=1/n; x=0; ck=0;
for i=1:n-1
x=x+dx;
ck= ck + 2*ex87a(x)*sin(k*pi*x) +4*ex87a(x-dx/2)*sin(k*pi*(x-dx/2));
end
ck=ck*2/(n*6);

function f = ex87a(x)

f=exp(-x*x);

```

### **ex87g.m**

```

function ex87g(N)
% Computes an approximation of
% the formal solution to exercise 8.7
% for  $f(x)=\sin(\exp(\cos(x)))$ , and  $t=3$ , using Simpson's rule.
% example: ex87g(100)

t=3;
x=0:0.01:1;
s=x.*0;

for k=1:N
cks=SimpsonFourier(100,k);
s=s+cks*exp(-k*k*pi*pi*t)*sin(x.*k*pi);
end

f= sin(exp(cos(x)));
plot(x,s,x,f,'r--'); xlabel('x');

```

```

title(['8-7g) N=', num2str(N)]);
legend('Solution at t=3', 'Initial condition', 2)

function ck=SimpsonFourier(n,k)
% n - intervals
% k - coefficient c_k

dx=1/n; x=0; ck=0;
for i=1:n-1
x=x+dx;
ck= ck + 2*ex87g(x)*sin(k*pi*x) +4*ex87g(x-dx/2)*sin(k*pi*(x-dx/2));
end
ck=ck*2/(n*6);

function f = ex87g(x)

    f=sin(exp(cos(x)));

```

### **p851.m**

```

function p851(Dx,Dt,t)
% Solves the problem from Project 1 at time t.
% The initial condition
% is placed in subfunctions.
% example: p851(0.1,0.0025,1)

x=0:Dx:1;
% Dt= Dx*Dx/4;
alpha=Dt/(Dx*Dx);

us=init2_f(Dx);

for t=Dt:Dt:t
for i=2:length(us)-1
u(i)=us(i)+alpha*(us(i-1)-2*us(i)+us(i+1));
end

u(1)=us(1)+2*alpha*(us(2)-us(1));
u(length(us))=us(length(us))+2*alpha*(us(length(us)-1)-us(length(us)));
% max(us)-max(u)
us=u;
end

ft=init2_f(Dx);
plot(x,us,x,ft,'r--');

c1= 0.5*us(1)*us(1) +...
    sum(us(2:length(us)-1).*us(2:length(us)-1)) +...

```

```

    0.5*us(length(us))*us(length(us));
Dx*c1;

c2= 0.5*ft(1)*ft(1) +...
    sum(ft(2:length(ft)-1).*ft(2:length(ft)-1)) +...
    0.5*ft(length(ft))*ft(length(ft));
Dx*c2;

% find the integrals, using the trapezoid method.
c3= Dx*(0.5*us(1)+sum(us(2:length(us)-1)) +0.5*us(length(us)))
c4= Dx*(0.5*ft(1)+sum(ft(2:length(ft)-1)) +0.5*ft(length(ft)))
% find

max(us)-min(us)

% initial condition
function f=init_f(Dx)
x=0:Dx:1;
f=exp(x).*x.*(1-x);

% initial condition 2
function f=init2_f(Dx)
x=0:Dx:1;
f=3.14+cos(2*pi.*x);

```

## **p852.m**

```

function p852(Dx,Dt)
% Solves the problem from Project 5.8.2. at time t=1.
% The initial condition and the thermal conductivity
% are placed in subfunctions.
% example: p852(0.1,0.0025)

x=0:Dx:1;
t=1;
% Dt= Dx*Dx/4;
alpha=Dt/(Dx*Dx);

us=init_f(Dx);
for t=Dt:Dt:t
for i=2:length(us)-1
    u(i)=alpha*tc_k((i-0.5)*Dx)*us(i-1) +...
        (1-alpha*(tc_k((i-0.5)*Dx)+tc_k((i+0.5)*Dx)))*us(i) +...
        alpha*tc_k((i+0.5)*Dx)*us(i+1);
end
u(1)=0; u(length(us))=0;
% max(us)-max(u)

```

```

us=u;
end

% Equation (8.119)
c1= sum(us(2:length(us)-1).*us(2:length(us)-1))
Dx*c1

% Equation (8.120)
ft=init_f(Dx);
c2= 0.5*ft(1)*ft(1) +...
    sum(ft(2:length(ft)-1).*ft(2:length(ft)-1)) +...
    0.5*ft(length(ft))*ft(length(ft));
Dx*c2

% Equation (8.121)

c3=0.5*tc_k(Dx/2)*us(2)*us(2)/Dx;
for i=2:length(us)-2
    c3 = c3 +tc_k((i-0.5)*Dx)*(us(i+1)*us(i+1) -...
        2*us(i+1)*us(i)+ us(i)*us(i) )/Dx;
end
c3 = c3+0.5*tc_k(1-Dx/2)*us(length(us)-1)*us(length(us)-1);
c3

c4=0.5*tc_k(Dx/2)*(ft(1)*ft(1) -2*ft(1)*ft(2) +ft(2)*ft(2))/Dx;
for i=2:length(ft)-2
c4=c4 + tc_k((i-0.5)*Dx)*(ft(i+1)*ft(i+1) -...
    2*ft(i+1)*ft(i) +ft(i)*ft(i))/Dx;
end

c4=c4 + 0.5*tc_k(1-Dx/2)*(ft(length(ft))*ft(length(ft)) -...
    2*ft(length(ft))*ft(length(ft)-1) +ft(length(ft)-1)*ft(length(ft)-1)

c4

% initial condition
function f=init_f(Dx)
x=0:Dx:1;
f=x.*(1-x);

% thermal conductivity
function k=tc_k(x)
    k=1+x;

```