Differential Equations

Differential equations

- A differential equations is: an equations that relate a function to its derivatives in such a way that the function can be determined
- In practice, differential equations typically describe quantities that changes in relation to each other
- Examples of such equations arise in several disciplines of Science and Technology (e.g. physics, chemistry, biology, economy, weather forecasting,...)
- During the last decades major progress in Science and Technology has evolved as a result of increased ability of solving differential equations
- These progresses are expected to continue in the future with increased strength

Differential equations

There are two main contributions that has made this possible

- the development of numerical algorithms and software
- the development of computers

Scientists have the last three centuries spend much effort on describing Nature with differential equations. Most of them could not be solved.
The last few decades Science recovers new opportunities.
We will in this course see how basic principles can be described with differential equations, and consider how these can be solved numerically.

Cultivation of rabbits

- A number of rabbits are placed on an isolated island with perfect environments for them
- How will the number of rabbits grow?

Note that this question can not be answered based on clever thinking only.

The simplest model

- Let $r = r(t)$ denote the number of rabbits
- Let $r_0 = r(0)$ denote the initial number of rabbits
- Assume that the change of rabbits per time is given by $f(t)$
- For a small period of time $\Delta t > 0$, we have

$$\frac{r(t+\Delta t) - r(t)}{\Delta t} = f(t) \tag{1}$$

- Assuming that $r(t)$ is continuous and differentiable and letting Δt go to zero, we obtain

$$r'(t) = f(t) \tag{2}$$

The simplest model

- From the fundamental theorem of Calculus, we get the solution
$$r(t) = r(0) + \int_0^t f(s)ds \tag{3}$$
- The integral can then be calculated as accurate as we want, with the methods presented in the previous lectures

Exponential growth

- We now assume that the growth in population is proportional to the number of rabbits, i.e

$$\frac{r(t+\Delta t)-r(t)}{\Delta t}=ar(t), \tag{4}$$

where a is a positive constant

- Letting Δt go to zero we get

$$r'(t)=ar(t) \tag{5}$$

- In practice a has to be measured

Analytical solution

- We want to solve the problem

$$r'(t) = ar(t) \tag{6}$$

  with initial condition

$$r(0) = r_0$$

- Since

$$\frac{dr}{dt} = ar$$

- we have

$$\frac{1}{r} dr = a\,dt$$

Analytical solution

- by integrating we get

$$\int \frac{1}{r}\,dr = \int a\,dt$$

- which gives

$$\ln(r) = at + c \tag{7}$$

  where c is a constant of integration

- The right value for c is received by putting $t = 0$

$$c = \ln(r_0)$$

Analytical solution

- From (7) we get

$$\ln(r(t)) - \ln(r_0) = at$$

- or

$$\ln\left(\frac{r(t)}{r_0}\right) = at$$

- and therefore

$$r(t) = r_0 e^{at} \tag{8}$$

- Conclusion: the number of rabbits increase exponentially in time

Uniqueness

Is the solution of (5) unique?

- Assume that there exists two solutions $r(t)$ and $q(t)$ of (5), i.e $r(t)$ and $q(t)$ solves the two systems

$$r'(t) = ar(t) \qquad\qquad q'(t) = aq(t)$$
$$r(0) = r_0 \qquad\qquad q(0) = r_0$$

- Define the difference by $e(t) = r(t) - q(t)$
- The difference fulfills the initial condition

$$e(0) = r(0) - q(0) = 0$$

(this value may be called $e_0 = 0$)

Uniqueness

- We see that $e'(t)$ has the property

$$\begin{aligned} e'(t) &= r'(t) - q'(t) \\ &= a(r(t) - q(t)) \\ &= ae(t) \end{aligned}$$

- Summarized we have

$$\begin{aligned} e'(t) &= ae(t) \\ e(0) &= 0 \end{aligned} \tag{9}$$

- From this $e(0) = 0 \Rightarrow e'(0) = 0$, and we can then conclude that $e(t) = 0$ for all time

Uniqueness

- Since

$$e(t) = r(t) - q(t)$$

we get

$$q(t) = r(t),$$

which means that the solution is unique
- i.e., only $r(t) = r_0 e^{at}$ solves (5)

Stability

How does wrong initial value influence on the solution?

- In practice initial values are often uncertain
- Just think about counting the number of rabbits on an island
- We will therefore study the difference in the solutions for the two systems

$$
\begin{aligned}
r'(t) &= ar(t) \qquad & q'(t) &= aq(t) \\
r(0) &= r_0 \qquad & q(0) &= q_0
\end{aligned}
$$

where $r_0 \neq q_0$

- Let the difference be denoted $d(t) = r(t) - q(t)$

Stability

- We see that $d(t)$ fulfills the following initial value problem

$$d'(t) = ad(t)$$
$$d(0) = r_0 - q_0$$

- By defining $d_0 = r_0 - q_0$, it follows that the solution can be written $d(t) = d_0 e^{at}$
- or $r(t) - q(t) = (r_0 - q_0)e^{at}$
- therefore

$$|r(t) - q(t)| = |r_0 - q_0|e^{at}$$

Stability

- We divide both sides with $r(t) = r_0 e^{at}$ and get

$$\frac{|r(t) - q(t)|}{r(t)} = \frac{|r_0 - q_0|}{r_0} \tag{10}$$

- We conclude that the relative error at time t is equal to the initial relative error
- We also conclude that $r(t) \rightarrow q(t)$ as $r_0 \rightarrow q_0$
- The problem is therefore referred to as stable

Logistic growth

The exponential growth is not realistic, since the number of rabbits will go to infinity as the time increase.

- We assume that there is a *carrying capacity* R of the island
- This number tells how many rabbits the island can feed, host etc.
- The *logistic model* reads

$$r'(t) = ar(t)\left(1 - \frac{r(t)}{R}\right) \tag{11}$$

where $a > 0$ is the growth rate and $R > 0$ is the carrying capacity

Logistic growth

- If $r_0 << R$, we see that

$$\frac{r(t)}{R} \approx 0$$

  for small t

- and thus

$$r'(t) \approx ar(t)$$

- This means that logistic and the exponential model give similar results as long as $r(t) << R$

Logistic growth

- Note that $r(t) < R \;\Rightarrow\; (1 - r(t)/R) \geq 0$
- therefore

$$r'(t) = ar(t)\left(1 - \frac{r(t)}{R}\right) > 0 \tag{12}$$

- Thus the rabbit population will grow as long as the number of rabbits is less than the carrying capacity
- The growth will decrease as the number of rabbits approach the carrying capacity
- If $r(t) = R$ is reached at some time $t = t^*$ we get

$$r'(t^*) = 0,$$

so the growth will stop

Exceeding the carrying capacity

- Assume that we place a lot of rabbits at the island initially - more than the carrying capacity

$$r_0 > R$$

- Since $(1 - \frac{r_0}{R}) < 0$ we have that

$$r'(0) = ar_0 \left(1 - \frac{r_0}{R}\right) < 0,$$

  i.e. the number of rabbits decrease
- The population will decrease as long as $r(t) > R$

Analytical solution

Solve

$$r'(t) = ar(t)\left(1 - \frac{r(t)}{R}\right)$$
$$r(0) = r_0.$$

We write

$$\frac{dr}{dt} = ar\left(1 - \frac{r}{R}\right),$$

or

$$\frac{dr}{r\left(1 - \frac{r}{R}\right)} = a\,dt.$$

Analytical solution

By integration we get

$$\ln \frac{r}{R-r} = at + c$$

where c is a integration constant. This constant is determined by the initial condition

$$\ln \frac{r_0}{R-r_0} = c$$

and thus

$$\ln \left[\frac{\frac{r}{R-r}}{\frac{r_0}{R-r_0}} \right] = at,$$

Analytical solution

or

$$\frac{r}{R-r} = \frac{r_0}{R-r_0} e^{at}.$$

Solving this with respect to r gives

$$r(t) = \frac{r_0}{r_0 + e^{-at}(R-r_0)} R \tag{13}$$

(see Figure 1.)

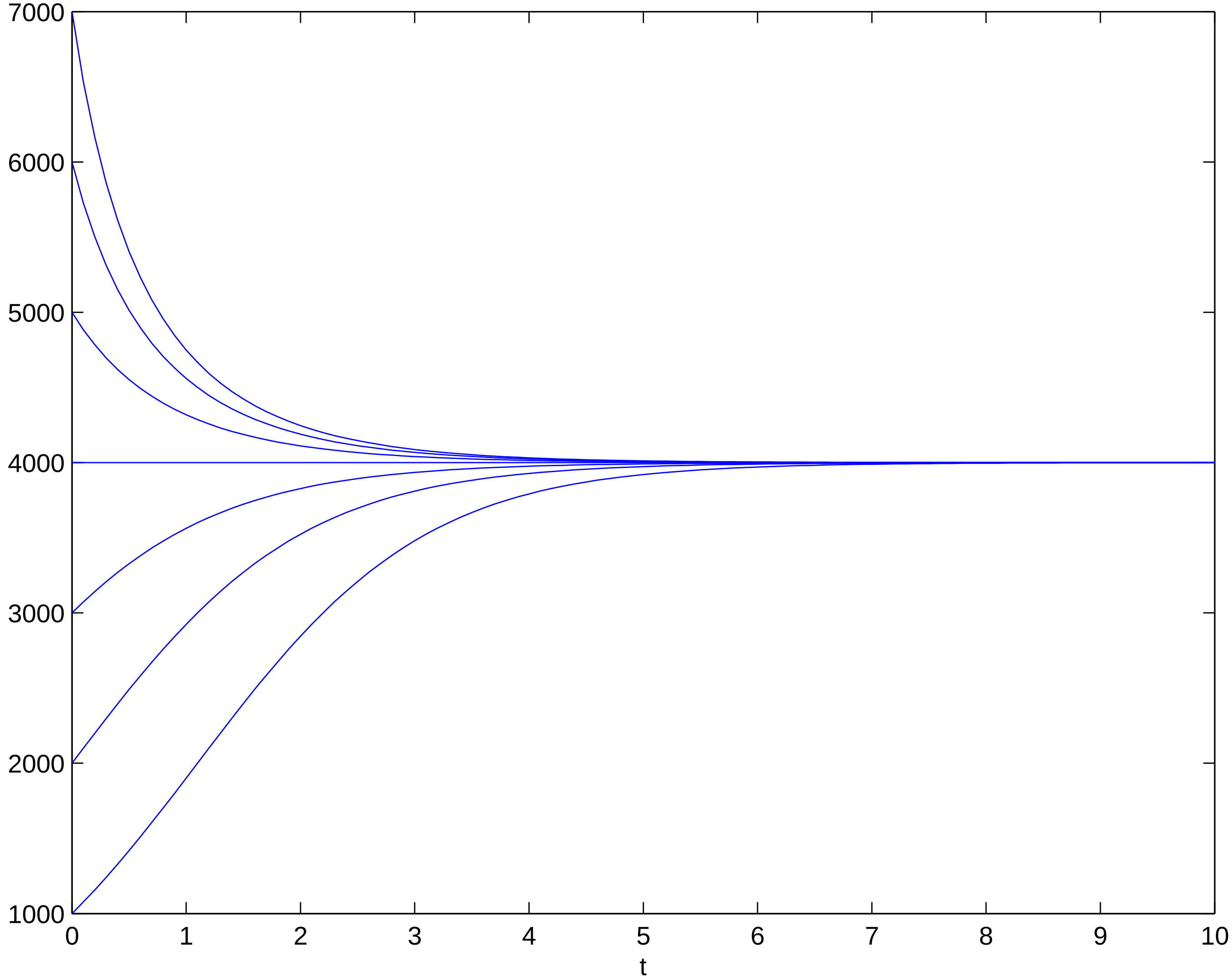

Figure 1: Different solutions of (13) using different values of r_0.

Numerical solution

- For simple examples of differential equations we can find analytical solutions
- This is not the case for most of the realistic models of nature
- Analytical solutions are still important for testing numerical methods
- Analytical insight is very important for designing good numerical methods
- An example of this is the insight we got above from the arguments about increase and decrease in rabbit population

The simplest model

We now pretend that we do not know the exact solution of

$$r'(t) = f(t)$$

with $r(0) = r_0$, and solve the problem in $t \in (0,1)$.

- Pick a positive integer N, and define the time-step

$$\Delta t = \frac{1}{N}$$

- Define time-levels $t_n = n\Delta t$
- Let r_n denote the approximation of $r(t_n)$

$$r_n \approx r(t_n)$$

The simplest model

- Remember the series expansion

$$r(t+\Delta t) = r(t) + \Delta t r'(t) + O(\Delta t^2)$$

- or

$$r'(t) = \frac{r(t+\Delta t) - r(t)}{\Delta t} + O(\Delta t)$$

- By setting $t = t_n$, we therefore see that

$$r'(t_n) \approx \frac{r(t_{n+1}) - r(t_n)}{\Delta t}$$

- By using the approximate solutions $r_n \approx r(t_n)$ and $r_{n+1} \approx r(t_{n+1})$, the numerical scheme is defined

$$\frac{r_{n+1} - r_n}{\Delta t} = f(t_n)$$

The simplest model

- This can be written

$$r_{n+1} = r_n + \Delta t\, f(t_n)$$

- For the first time step we get

$$r_1 = r_0 + \Delta t\, f(t_0)$$

- and for the next

$$r_2 = r_1 + \Delta t\, f(t_1) = r_0 + \Delta t\, \left(f(t_0) + f(t_1) \right)$$

The simplest model

- And for time-step N we see that

$$r_N = r_0 + \Delta t \sum_{n=0}^{N-1} f(t_n) \tag{14}$$

- The sum can be recognized as the Riemann sum approximation of the integral in (3) (this approximation is not covered in this course)

Example 6

We will test this on a problem with $f(t) = t^2$ and $r(0) = 0$

- The exact solution is

$$r(1) = r(0) + \int_0^1 t^2 dt = 1/3$$

- When $N = 10$, we get

$$\begin{aligned} r(1) &\approx r_{10} \\ &= 0 + \Delta t \left(\Delta t^2 + (2\Delta t)^2 + \cdots + (9\Delta t)^2 \right) \\ &= \Delta t^3 (1 + 2^2 + \cdots + 9^2) \\ &= \frac{1}{6} \frac{9 \cdot 10 \cdot 19}{10^3} \\ &= 0.285 \end{aligned}$$

Example 6

- and when $N = 100$, we get

$$\begin{aligned} r(1) &\approx r_{100} \\ &= \Delta t^3 (1 + 2^2 + \cdots + 99^2) \\ &= \frac{1}{6} \frac{99 \cdot 100 \cdot 199}{100^3} \\ &= 0.3285 \end{aligned}$$

- We see that the approximation improves with increased number of approximation points

Example 7

The same example, but using the Trapezoid method

- Recall that for a given N the general formula is

$$r(t) \approx r_0 + \Delta t \left(\frac{1}{2} f(0) + \sum_{n=1}^{N-1} f(t_n) + \frac{1}{2} f(t) \right) \tag{15}$$

- For $N = 10$, we get

$$\begin{aligned} r(1) &\approx \Delta t \left(\frac{1}{2} f(0) + \sum_{n=1}^{9} f(t_n) + \frac{1}{2} f(t) \right) \\ &= \Delta t^3 \left(\sum_{n=1}^{9} n^2 + \frac{1}{2} 10^2 \right) = \frac{1}{10^3} \left(\frac{9 \cdot 10 \cdot 19}{6} + 50 \right) \\ &= 0.335 \end{aligned}$$

Example 7

- And for $N = 100$, we get

$$
\begin{aligned}
r(1) &\approx \Delta t \left(\frac{1}{2} f(0) + \sum_{n=1}^{99} f(t_n) + \frac{1}{2} f(t) \right) \\
&= \Delta t^3 \left(\sum_{n=1}^{99} n^2 + \frac{1}{2} 100^2 \right) = \frac{1}{100^3} \left(\frac{99 \cdot 100 \cdot 199}{6} + 5000 \right) \\
&= 0.33335
\end{aligned}
$$

- Notice that these approximations are much closer to the true value $1/3$, than the approximations in Example 6

Exponential growth

We now want study numerical solution of the problem $r'(t) = ar(t)$, $t \in (0, T)$, where a is a given constant, and initial condition $r(0) = r_0$.

- Similar to above we choose an integer $N > 0$, define the time-steps $\Delta t = T/N$ and the time-levels $t_n = n\Delta t$, r_n is the approximation of $r(t_n)$ and the derivative is approximated by

$$r'(t_n) \approx \frac{r(t_{n+1}) - r(t_n)}{\Delta t}$$

- The numerical scheme is defined by

$$\frac{r_{n+1} - r_n}{\Delta t} = ar_n \tag{16}$$

Exponential growth

- Which can be written

$$r_{n+1} = (1 + a\Delta t) r_n \tag{17}$$

- This formula gives initially

$$\begin{aligned} r_1 &= (1 + a\Delta t) r_0 \\ r_2 &= (1 + a\Delta t) r_1 = (1 + a\Delta t)^2 r_0 \end{aligned}$$

- and for general n we can see that

$$r_n = (1 + a\Delta t)^n r_0 \tag{18}$$

Example 8

We test an example where $a = 1$, $r_0 = 1$ and $T = 1$.

- The exact solution is $r(t) = e^t$ and therefore

$$r(1) = e \approx 2.718$$

- Using $N = 10$ in the numerical scheme gives

$$r(1) \approx r_{10} = (1 + \frac{1}{10})^{10} \approx 2.594$$

- Choosing $N = 100$, gives

$$r_{100} = (1 + \frac{1}{100})^{100} \approx 2.705$$

Example 8 - Convergence

- The general formula is

$$r(1) \approx r_N = (1 + \frac{1}{N})^N$$

- From Calculus we know that

$$\lim_{N \longrightarrow \infty} (1 + \frac{1}{N})^N = e = r(1)$$

- Thus the numerical scheme will converge to the right solution in this example

Numerical stability

Consider the initial value problem

$$\begin{aligned} y'(t) &= -100y(t), \quad t \in (0,1) \\ y(0) &= 1, \end{aligned} \tag{19}$$

with analytic solution

$$y(t) = e^{-100t}$$

- For a given N and corresponding Δt we have

$$y_{n+1} = (1 - 100\Delta t)y_n \tag{20}$$

- which gives

$$y_n = \left(1 - \frac{100}{N}\right)^n$$

Numerical stability

- Note that the analytical solution is always positive, but decreases rapidly and monotonically towards zero
- For $N = 10$ we get the formula

$$y_n = \left(1 - \frac{100}{10}\right)^n = (-9)^n$$

- which gives $y_0 = 1,\ y_1 = -9,\ y_2 = 18,\ y_3 = -729$
- This is referred to as *numerical instability*

Numerical stability

- For y_n to stay positive we get from (20) that

$$1 - 100\Delta t > 0$$

- or

$$\Delta t < \frac{1}{100} \tag{21}$$

- which means

$$N \geq 101$$

- This is referred to as *stability condition*
- A numerical scheme that is stable for all Δt is called *unconditionally stable*
- A scheme that needs a stability condition is called *conditionally stable*

An implicit scheme

We still study the exponential model, $r'(t) = ar(t)$.

- Above the observation

$$r'(t_n) = \frac{r(t_{n+1}) - r(t_n)}{\Delta t} + O(\Delta t)$$

- led to the scheme

$$\frac{r_{n+1} - r_n}{\Delta t} = ar_n$$

- Similarly we could have observed that

$$r'(t_{n+1}) = \frac{r(t_{n+1}) - r(t_n)}{\Delta t} + O(\Delta t)$$

An implicit scheme

- This leads to

$$\frac{r_{n+1} - r_n}{\Delta t} = ar(t_{n+1})$$

- which can be written

$$r_{n+1} = \frac{1}{1 - \Delta t a} r_n$$

- This leads to

$$r_n = \left(\frac{1}{1 - \Delta t a} \right)^n r_0$$

An implicit scheme

- Reconsider the initial value problem

$$\begin{aligned} y'(t) &= -100y(t), \\ y(0) &= 1 \end{aligned}$$

- The implicit scheme gives

$$\begin{aligned} y_n &= \left(\frac{1}{1+100\Delta t}\right)^n \\ &= \left(\frac{N}{N+100}\right)^n \end{aligned}$$

- We see that y_n is positive for all choices of N
- The scheme is therefore unconditionally stable

An implicit scheme

N	y_N
10^1	$3.85 \cdot 10^{-11}$
10^2	$7.89 \cdot 10^{-31}$
10^3	$4.05 \cdot 10^{-42}$
10^7	$3.72 \cdot 10^{-44}$

The exact solution is $e^{-100} \approx 3.72 \cdot 10^{-44}$.

Explicit and implicit schemes

We consider problems on the form

$$v'(t) = \mathit{something}(t) \tag{22}$$

The term $v'(t)$ is replaced

$$\frac{v_{n+1} - v_n}{\Delta t}$$

The right hand side can be evaluated in $t = t_n$ or $t = t_{n+1}$.

- Explicit scheme: $v_{n+1} = v_n + \Delta t \mathit{something}(t_n)$
- Implicit scheme: $v_{n+1} = v_n + \Delta t \mathit{something}(t_{n+1})$

Implicit schemes are often unconditionable stable, but might be harder to use. Explicit schemes are often only conditionable stable, but are very simple to implement.

Example 9

The following initial value problem

$$\begin{aligned} y'(t) &= y^2(t) \\ y(0) &= 1, \end{aligned} \tag{23}$$

has the analytical solution

$$y(t) = \frac{1}{1-t}.$$

- The explicit scheme reads

$$\frac{y_{n+1} - y_n}{\Delta t} = y_n^2$$

- which gives an explicit expression for y_{n+1}

Example 9

- The implicit scheme is

$$\frac{z_{n+1} - z_n}{\Delta t} = z_{n+1}^2$$

- or

$$z_{n+1} - \Delta t z_{n+1}^2 = z_n$$

- When z_n is known, z_{n+1} can be found by solving the following nonlinear equation

$$x - \Delta t x^2 = z_n \tag{25}$$

- which has solution

$$z_{n+1} = x = \frac{1}{2\Delta t}\left(1 - \sqrt{1 - 4\Delta t y_n}\right) \tag{26}$$

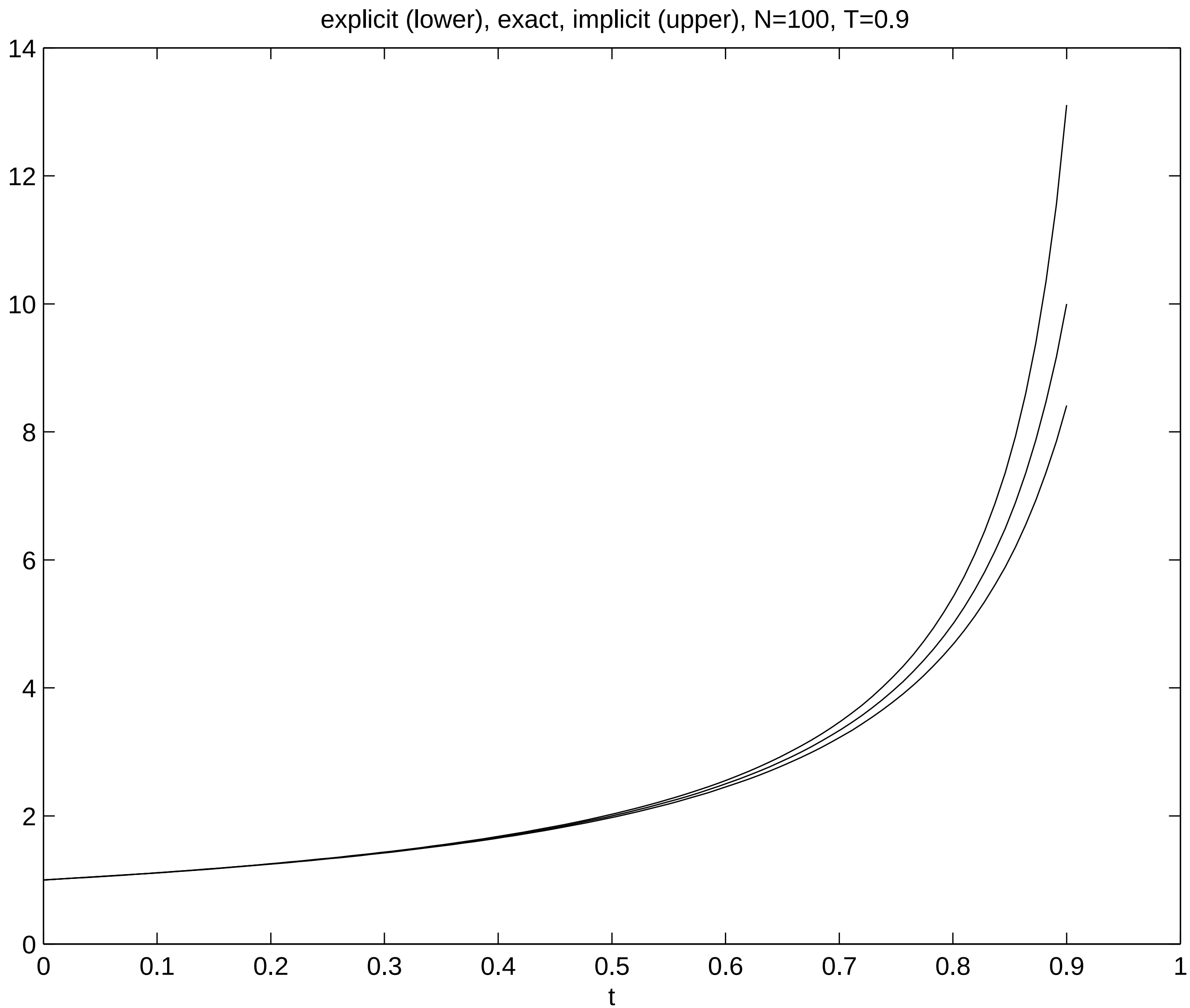

Figure 2: Example 9 with $N = 100$.

Example 10

We here study the same example, but with negative initial condition

$$\begin{aligned} y'(t) &= y^2(t) \\ y(0) &= -10, \end{aligned} \tag{27}$$

with solution

$$y(t) = \frac{-10}{1+10t}.$$

The exact, explicit and implicit solutions with $N = 25$ are plotted in Figure 3.
Table 1 shows the three solutions with different values for N and Δt.

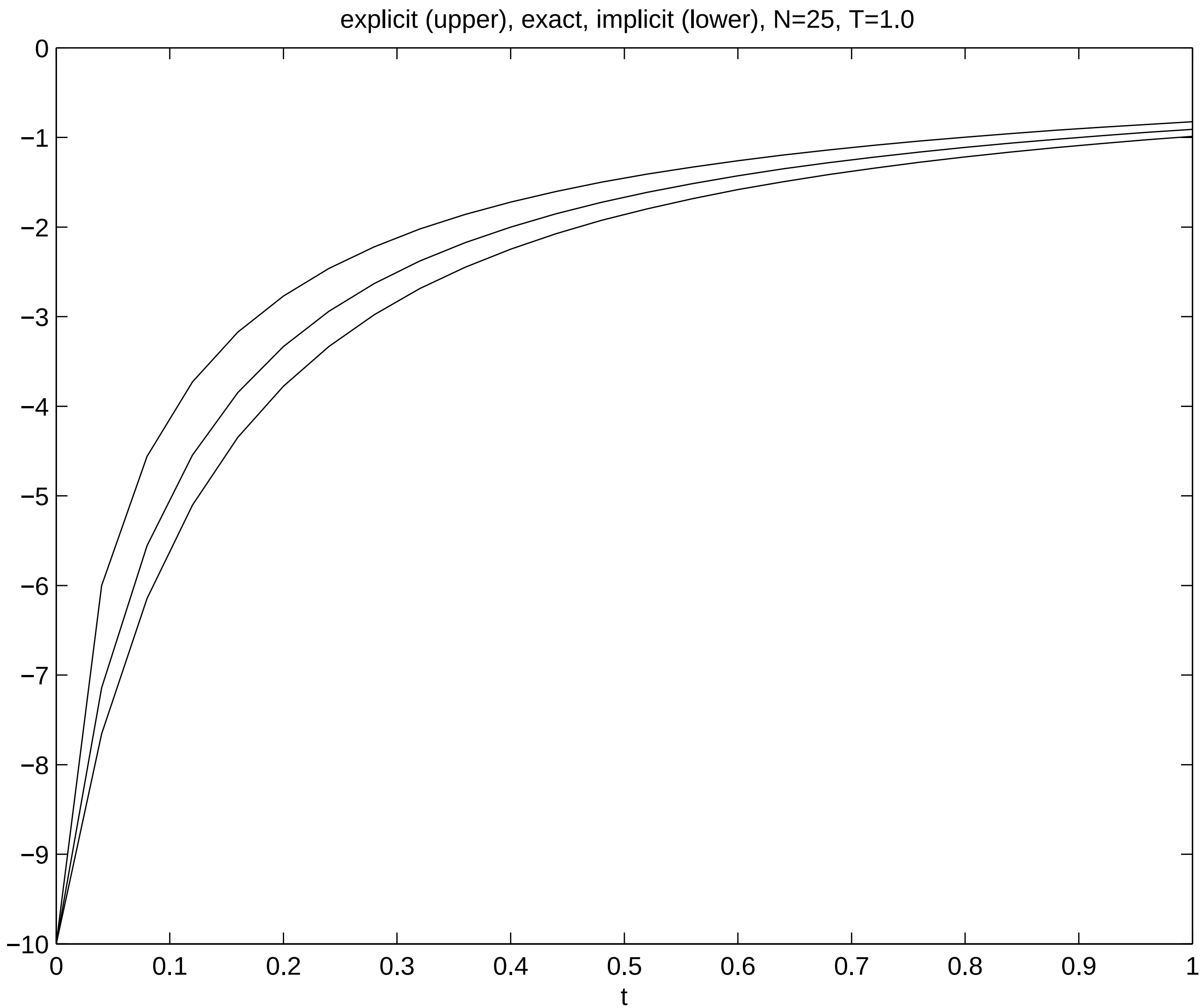

Figure 3: Example 10 with $N = 25$.

N	Δt	$y(1)$	Explicit at $t=1$	Implicit at $t=1$
1000	$\frac{1}{1000}$	$-\frac{10}{11} \approx -0.9091$	-0.9071	-0.9111
100	$\frac{1}{100}$	$-\frac{10}{11} \approx -0.9091$	-0.8891	-0.9288
25	$\frac{1}{25}$	$-\frac{10}{11} \approx -0.9091$	-0.9871	-0.8256
12	$\frac{1}{12}$	$-\frac{10}{11} \approx -0.9091$	-0.6239	-1.0703
11	$\frac{1}{11}$	$-\frac{10}{11} \approx -0.9091$	-0.4835	-1.0848
10	$\frac{1}{10}$	$-\frac{10}{11} \approx -0.9091$	0.0	-1.1022
9	$\frac{1}{9}$	$-\frac{10}{11} \approx -0.9091$	5.7500	-1.1235
8	$\frac{1}{8}$	$-\frac{10}{11} \approx -0.9091$	$6.4 * 10^3$	-1.1501
7	$\frac{1}{7}$	$-\frac{10}{11} \approx -0.9091$	$1.8014 * 10^7$	-1.1843
5	$\frac{1}{5}$	$-\frac{10}{11} \approx -0.9091$	$1.6317 * 10^7$	-1.2936
2	$\frac{1}{2}$	$-\frac{10}{11} \approx -0.9091$	840	-1.8575

Table 1: Comparison of the three solutions.

Stability - explicit scheme

In order to keep y_{n+1} negative in the explicit scheme

$$y_{n+1} = y_n + \Delta t y_n^2.$$

we must have

$$y_n + \Delta t y_n^2 < 0,$$

or because $y_n < 0$

$$1 + \Delta t y_n > 0.$$

For $n = 0$, we have $y_0 = -10$ and therefore we require

$$1 - 10\Delta t > 0$$

which implies that $\Delta t < \frac{1}{10}$, or

$$N > 10.$$

Implicit scheme

For the implicit scheme z_{n+1} is given by the solution of

$$x - \Delta t x^2 = z_n \tag{28}$$

We want to show that the solution of (28) remains negative when z_n is negative, i.e.

$$z_n < 0 \Rightarrow x < 0.$$

Therefore we study the function

$$f(x) = x - \Delta t x^2 - z_n,$$

with derivative

$$f'(x) = 1 - 2\Delta t x.$$

Implicit scheme

- Thus $f'(x) > 0$ for all $x < 0$, which means that f is monotonically increasing for all $x < 0$.
- It can be seen that

$$z_n < 0 \Rightarrow z_{n+1} < 0$$

which means that the scheme is stable (see Figure 4)

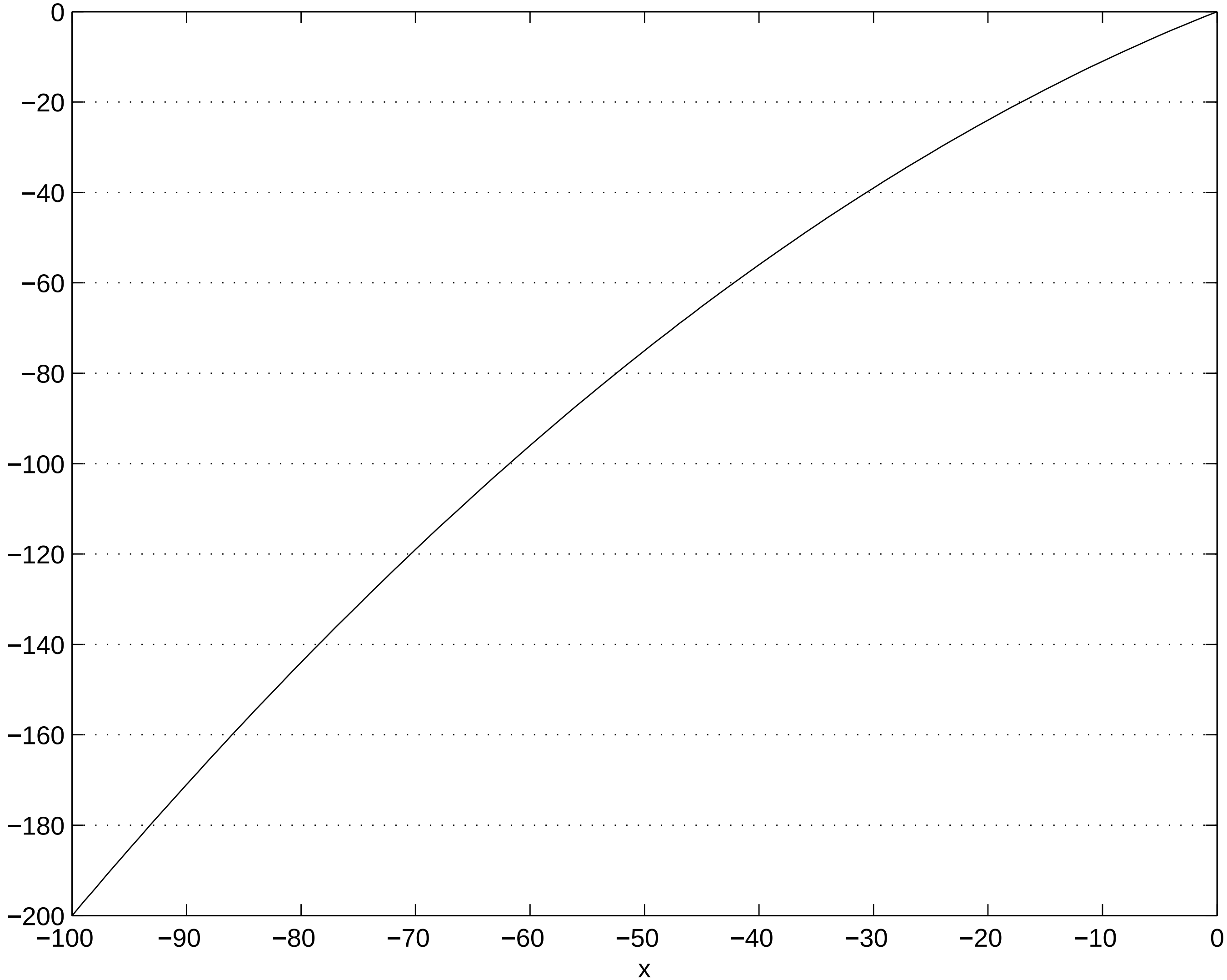

Figure 4: $f(x) = x - \Delta t x^2$ with $\Delta t = 1/100$.

Logistic equation

We study the explicit scheme for the logistic equation

$$
\begin{aligned}
r'(t) &= ar(t)\left(1-\frac{r(t)}{R}\right) && (29)\\
r(0) &= r_0, && (30)
\end{aligned}
$$

where $a > 0$ is the growth rate and R is the carrying capacity. The discussion above gives the properties

- If $R >> r_0$, then for small t, we have $r'(t) \approx ar(t)$ and thus exponential growth
- If $0 < r_0 < R$, then the solution satisfies $r_0 \leq r(t) \leq R$ and $r'(t) \geq 0$ for all time
- If $r_0 > R$, then the solution satisfies $R \leq r(t) \leq r_0$ and $r'(t) \leq 0$ for all time

Explicit scheme

An explicit scheme for this model reads

$$\frac{r_{n+1} - r_n}{\Delta t} = a r_n \left(1 - \frac{r_n}{R}\right),$$

or

$$r_{n+1} = r_n + a r_n \Delta t \left(1 - \frac{r_n}{R}\right). \tag{31}$$

We assume the same stability conditions for this scheme as for the exponential growth because of the exponential growth, i.e.

$$\Delta t < 1/a. \tag{32}$$

Properties

- If $R >> r_0$, we have $r_{n+1} \approx r_n + a\Delta t r_n$, for small values of n, which corresponds to the explicit scheme for the exponential growth model
- Assume that $0 < r_n < R$, then

  $$(1 - \frac{r_n}{R}) \geq 0,$$

  and therefore

  $$r_{n+1} \geq r_n \tag{33}$$

- If $0 < r_0 < R$, then

  $$r_0 \leq r_n \leq R$$

  see the discussion below

Properties

- Similarly, if $r_0 > R$, then $r_{n+1} \leq r_n$ and

$$r_0 \geq r_n \geq R \tag{34}$$

- If $r_0 = R$, then

$$r_1 = r_0 + ar_0 \Delta t (1 - \frac{r_0}{R}) = R,$$

and

$$r_n = R \tag{35}$$

for all $n > 0$

Properties

To understand that $r_0 \le r_n \le R$, when $0 \le r_0 \le R$, we study the function

$$g(x) = x + ax\Delta t(1 - \frac{x}{R}),\ x \in [0, R]$$

The derivative is

$$g'(x) = 1 + a\Delta t - \frac{2a\Delta t}{R}x.$$

Therefore for x in $[0, R]$, we have

$$\begin{aligned} g'(x) &\ge 1 + a\Delta t - \frac{2a\Delta t}{R}R = 1 - a\Delta t \\ &> 0, \end{aligned}$$

where the fact that $\Delta t < 1/a$, is used.

Properties

Note that $r_{n+1} = g(r_n)$ and $g(R) = R$, and because $g'(x) > 0$ we get

$$r_{n+1} = g(r_n) \leq g(R) = R,$$

and

$$r_{n+1} = g(r_n) \geq g(0) = 0.$$

Which means

$$0 \leq r_n \leq R \quad \Rightarrow \quad 0 \leq r_{n+1} \leq R,$$

By induction $0 \leq r_n \leq R$ holds for all $n \geq 0$, thus

$$r_0 \leq r_n \leq R, \tag{36}$$

for all $n \geq 0$.

Implicit scheme

The implicit scheme for the logistic model reads

$$\frac{r_{n+1} - r_n}{\Delta t} = a r_{n+1}\left(1 - \frac{r_{n+1}}{R}\right),$$

or

$$r_{n+1} - \Delta t\, a r_{n+1}\left(1 - \frac{r_{n+1}}{R}\right) = r_n.$$

- For r_n given, this is a nonlinear equation in r_{n+1}
- This is easy to solve since it is only a second order polynomial equation

The scheme is unconditionally stable and it fulfills the same properties as the explicit scheme did (see Exercise 6).