Systems of Ordinary Differential Equations

Systems of ordinary differential equations

Last two lectures we have studied models of the form

$$y'(t) = F(y), \qquad y(0) = y_0 \tag{1}$$

this is an scalar ordinary differential equation (ODE).
In the next two lectures we shall study systems of ODEs.
Especially we will consider numerical methods for systems of two ODEs on the form

$$\begin{aligned} y'(t) &= F(y,z), \qquad & y(0) &= y_0, \\ z'(t) &= G(y,z), \qquad & z(0) &= z_0. \end{aligned} \tag{2}$$

Here y_0 and z_0 are given initial states and F and G are smooth functions.

Rabbits and foxes

- Earlier we have studied the evolution of a rabbit population, and studied the Logistic model

$$y' = \alpha y(1 - y/\beta), \qquad y(0) = y_0 \tag{3}$$

  where now y is the number of rabbits, $\alpha > 0$ denotes the growth rate and β is the carrying capacity.

- Note that this model is the same as the Exponential growth model if $\beta = \infty$
- In the next two lectures we consider the case where foxes are introduced to the model
- This model is called a predator-prey system, and is similar to models describing populations of fish (prey) and sharks (predators)

Fish and Sharks

The first mathematician to study predator-pray models was Vito Volterra. He studied shark-fish populations, but his results are valid for rabbit-fox populations as well.

- Let $F = F(t)$ denote the number of fishes and $S = S(t)$ the number of sharks for a given time t
- If there is no sharks we assume that the number of fishes follows the logistic model

$$F' = \alpha F(1 - F/\beta) \tag{4}$$

- Expressed with relative growth it reads

$$\frac{F'}{F} = \alpha(1 - F/\beta) \tag{5}$$

Fish and Sharks

- Introducing sharks to the model, we assume the relative growth rate of fish is reduced linearly with respect to S

$$\frac{F'}{F} = \alpha(1 - F/\beta - \gamma S), \tag{6}$$

  where $\gamma > 0$
- or

$$F' = \alpha(1 - F/\beta - \gamma S)F \tag{7}$$

Fish and Sharks

- If there is no fish, we expect the number of sharks to decrease, and assume the relative change of sharks to be expressed as

$$\frac{S'}{S} = -\delta, \tag{8}$$

  where $\delta > 0$ is the decay rate

- We also assume that the relative change of sharks increase linearly with the number of fish

$$\frac{S'}{S} = -\delta + \varepsilon F \tag{9}$$

Fish and Sharks

We now have a 2×2 system which predicts the development of fish- and shark- population

$$F' = \alpha(1 - F/\beta - \gamma S)F, \qquad F(0) = F_0, \qquad (10)$$

$$S' = (\varepsilon F - \delta)S, \qquad S(0) = S_0. \qquad (11)$$

- In practice the parameters α, β, γ and ε, and initial values F_0 and S_0 must be determined with some estimation methods

Numerical method; Unlimited resources

- First we study the system (10)-(11) with $\beta = \infty$, i.e. unlimited resources of food and space for the fish
- For the other parameters we choose $\alpha = 2, \quad \gamma = 1/2, \quad \varepsilon = 1$ and $\delta = 1$, which gives the system

$$\begin{aligned} F' &= (2-S)F, \qquad F(0) = F_0, \qquad (12) \\ S' &= (F-1)S, \qquad S(0) = S_0. \qquad (13) \end{aligned}$$

- We introduce $\Delta t > 0$ and define $t_n = n\Delta t$, and let F_n and S_n denote approximations of $F(t_n)$ and $S(t_n)$ respectively

Numerical method

- The derivatives, F' and S', are approximated with

$$\frac{F(t_{n+1}) - F(t_n)}{\Delta t} \approx F'(t_n) \quad \text{and} \quad \frac{S(t_{n+1}) - S(t_n)}{\Delta t} \approx S'(t_n),$$

  which correspond to the explicit scheme
- The numerical scheme can then be written

$$\frac{F_{n+1} - F_n}{\Delta t} = (2 - S_n)F_n \tag{14}$$

$$\frac{S_{n+1} - S_n}{\Delta t} = (F_n - 1)S_n \tag{15}$$

Numerical method

- This can then be rewritten on an explicit form

$$F_{n+1} = F_n + \Delta t (2 - S_n) F_n \tag{16}$$

$$S_{n+1} = S_n + \Delta t (F_n - 1) S_n \tag{17}$$

- When F_0 and S_0 are given, this formula gives us F_1 and S_1 by setting $n = 0$, and then we can compute F_2 and S_2 by putting $n = 1$ in the formula, and so on
- In Figure 1 we have tested the explicit scheme (16)-(17) with $F_0 = 1.9$, $S_0 = 0.1$ and $\Delta t = 1/1000$

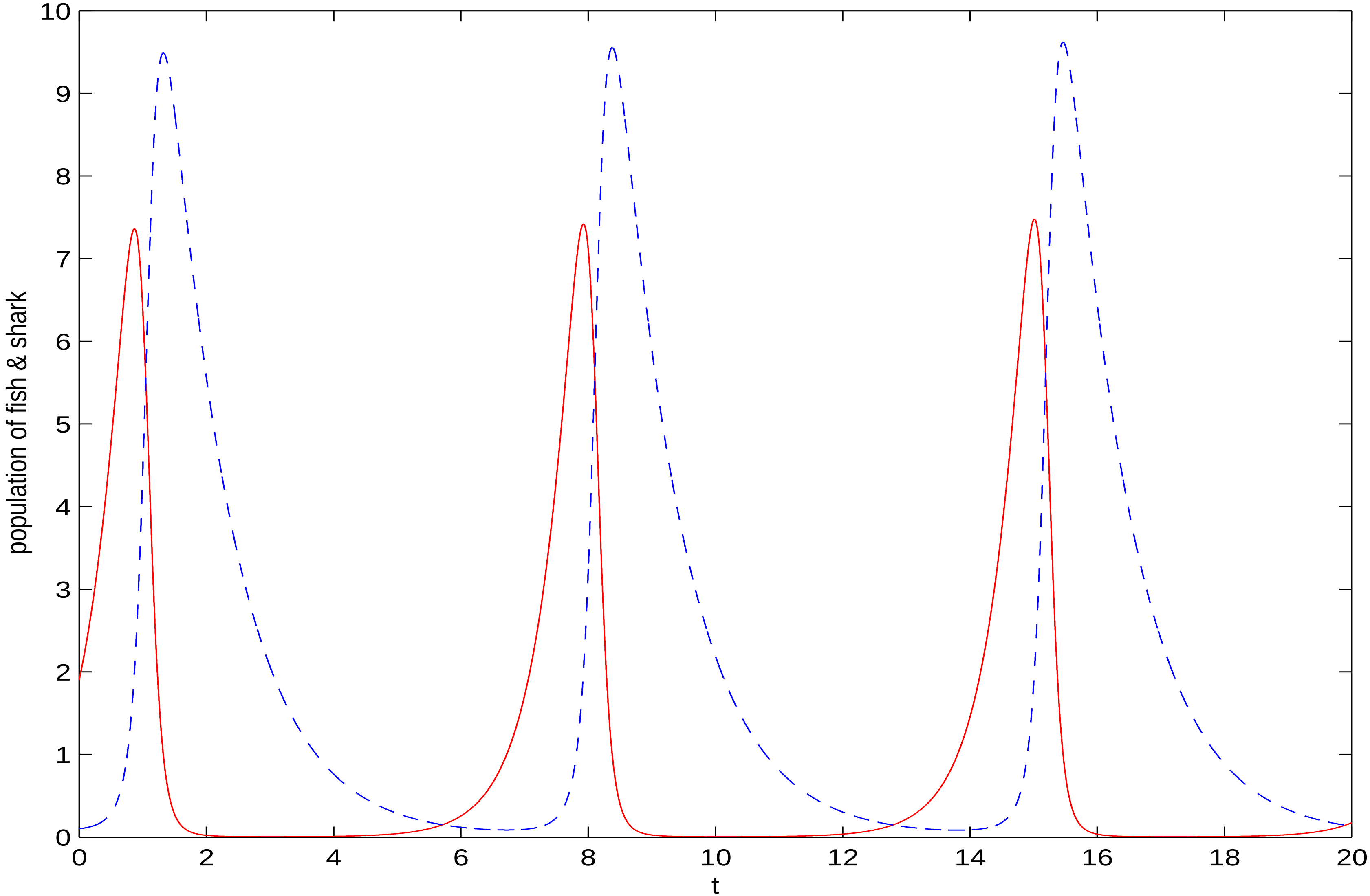

Figure 1: The solid curve is the solution for F, and the dashed curve is the solution for S.

Numerical methods; limited resources

- We do the same as above, but use $\beta = 2$, which corresponds to quite limited resources
- The system now reads

$$F' = (2 - F - S)F, \quad F(0) = F_0, \tag{18}$$
$$S' = (F - 1)S, \quad S(0) = S_0 \tag{19}$$

- Similar to above we can define an explicit numerical scheme

$$F_{n+1} = F_n + \Delta t(2 - F_n - S_n)F_n, \tag{20}$$
$$S_{n+1} = S_n + \Delta t(F_n - 1)S_n \tag{21}$$

- The results for $F_0 = 1.9$, $S_0 = 0.1$ and $\Delta t = 1/1000$ are shown in Figure 2

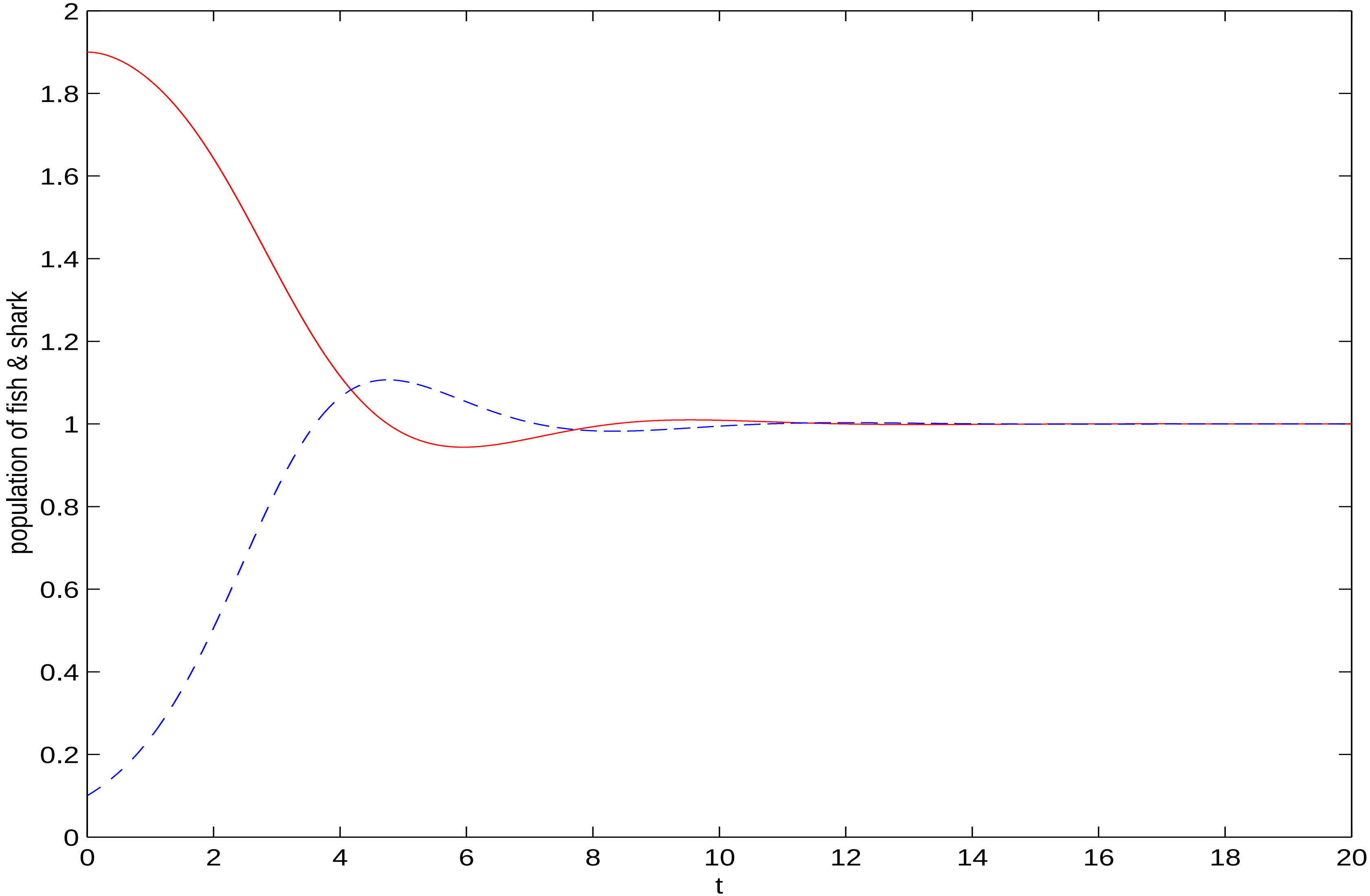

Figure 2: The solution for F is the solid curve, whereas the solution for S is the dashed curve.

Numerical methods

- We see from Figure 1 that the solutions for both $F(t)$ and $S(t)$ seem to be periodic
- From Figure 2 it seems that the solutions converge to an equilibrium solution represented by $S = F = 1$
- Therefore it is interesting to notice that, different parameter values can give different quantitative behavior of the solution

Phase plane analysis

We shall now study a simplified version of the fish-shark model

$$F'(t) = 1 - S(t), \quad F(0) = F_0,$$
$$S'(t) = F(t) - 1, \quad S(0) = S_0. \tag{22}$$

- Using the notation as above an explicit numerical scheme for this problem reads

$$\begin{aligned} F_{n+1} &= F_n + \Delta t(1 - S_n), \\ S_{n+1} &= S_n + \Delta t(F_n - 1), \end{aligned} \tag{23}$$

  where F_0 and S_0 are given initial states
- Figure 3 show a solution of this scheme when $F_0 = 0.9$, $S_0 = 0.1$ and $\Delta t = 1/1000$

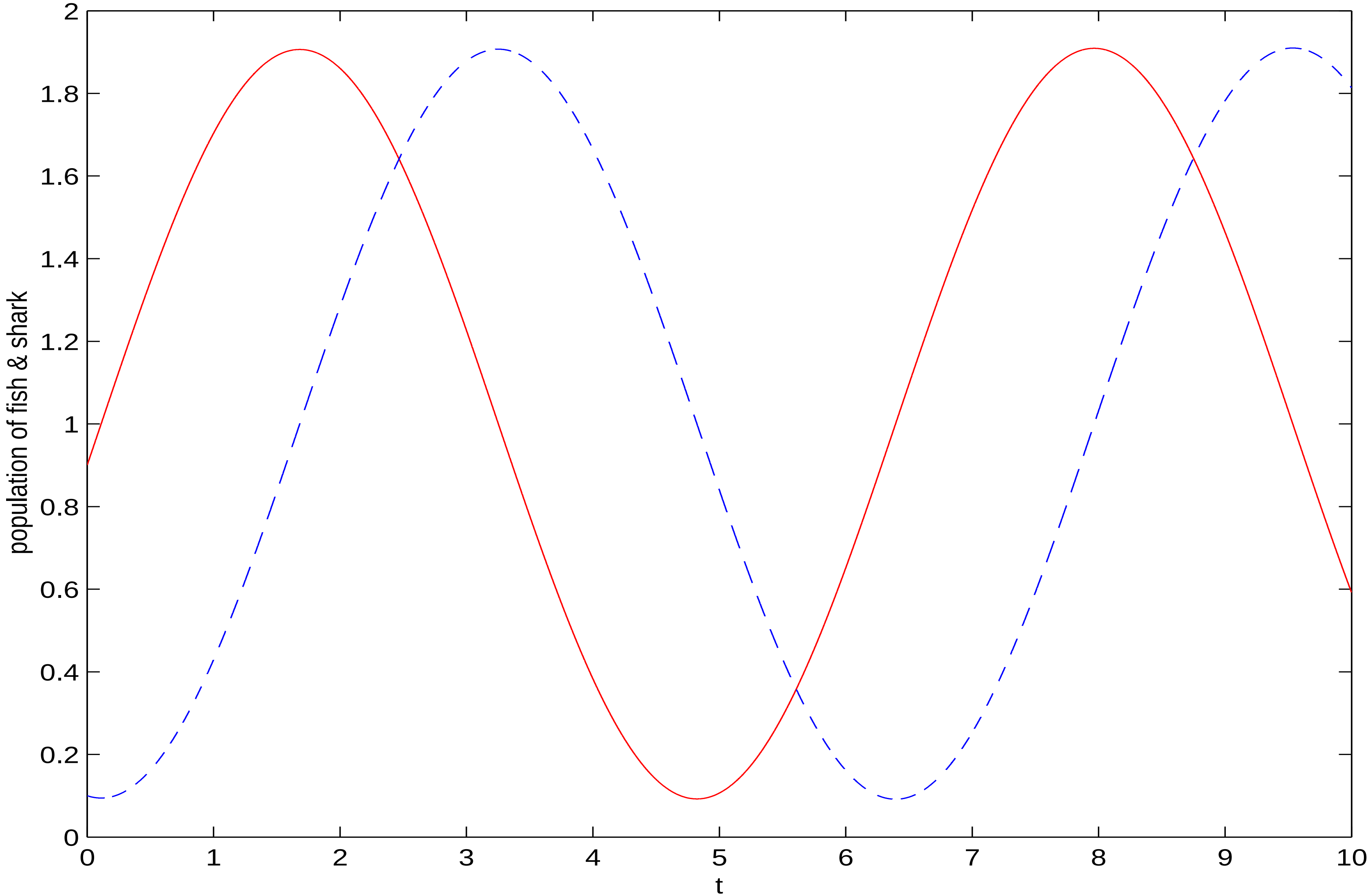

Figure 3: The solution for F is the solid curve, whereas the solution for S is the dashed curve.

Phase plane analysis

- The solution of (22) seems to be periodic like the solution of (12)-(13)
- In order to study how F and S interact we will plot the solution in the $F-S$ coordinate system, i.e. we plot the points (F_n, S_n) for all n-values
- In Figure 4 we plot the solution of (23) in the $F-S$ coordinate system, with the same specifications as above ($F_0 = 0.9$, $S_0 = 0.1$, $\Delta t = 1/1000$)
- In Figure 5 we do the same, but $\Delta t = 1/100$

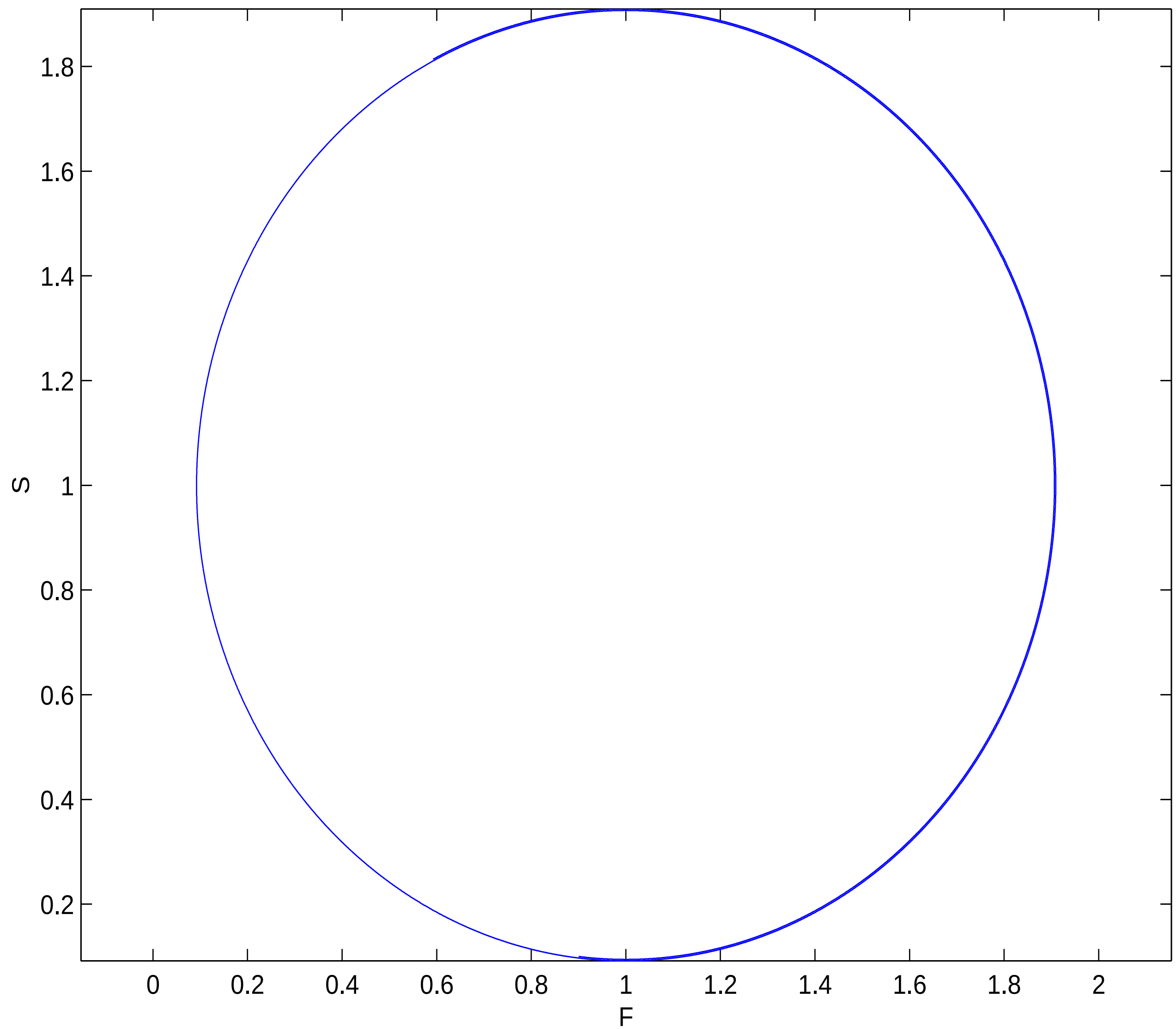

Figure 4: Explicit scheme (23) using $\Delta t = 1/1000$, $F_0 = 0.9$ and $S_0 = 0.1$, plotted in the F-S coordinate system

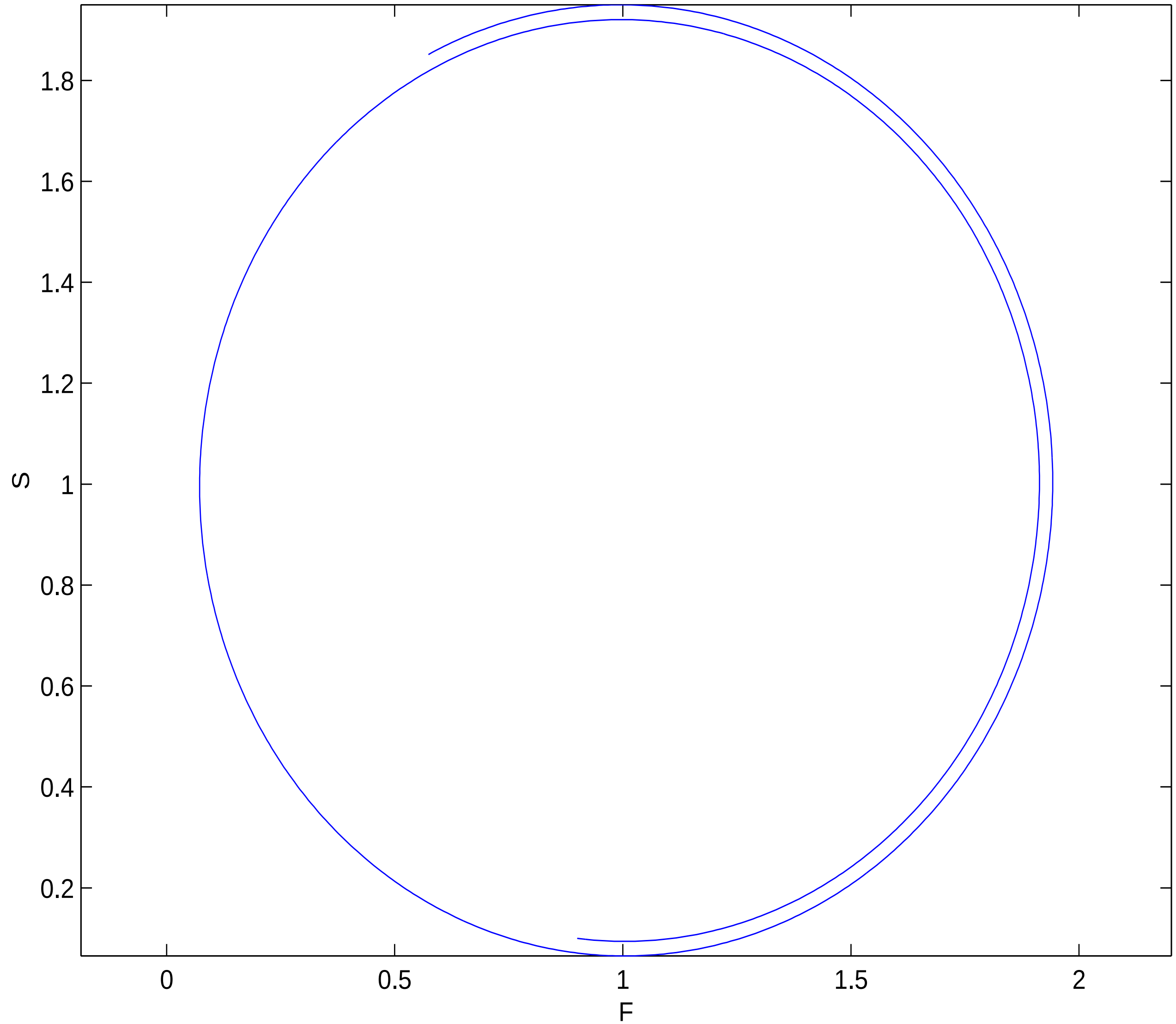

Figure 5: Explicit scheme (23) using $\Delta t = 1/100$, $F_0 = 0.9$ and $S_0 = 0.1$, plotted in the F-S coordinate system

Phase plane analysis

In Figure 4 it seems that the solution is almost a perfect circle with radius 1 and center in $(1,1)$, and in Figure 5 the solution is a circle of lower quality. Based on these observations we expect that

- The analytical solutions $(F(t),S(t))$ form circles in the F-S coordinate system
- A good numerical method generates values (F_n,S_n) that are placed almost exactly on a circle and the numerical solution get closer to a circle when Δt is smaller

In the following we shall study this hypothesis in more detail.

Analysis of the analytical solution

We shall try to do some analysis of the analytical solution. In order to study the behavior of $F(t)$ and $S(t)$ we will define the function

$$r(t) = (F(t) - 1)^2 + (S(t) - 1)^2, \tag{24}$$

which is the distance function from the point $(1, 1)$. In Figure 6 we have plotted an approximation to this function given by

$$r_n = (F_n - 1)^2 + (S_n - 1)^2 \tag{25}$$

for the case of $F_0 = 0.9$, $S_0 = 0.1$ and $\Delta t = 1/1000$.

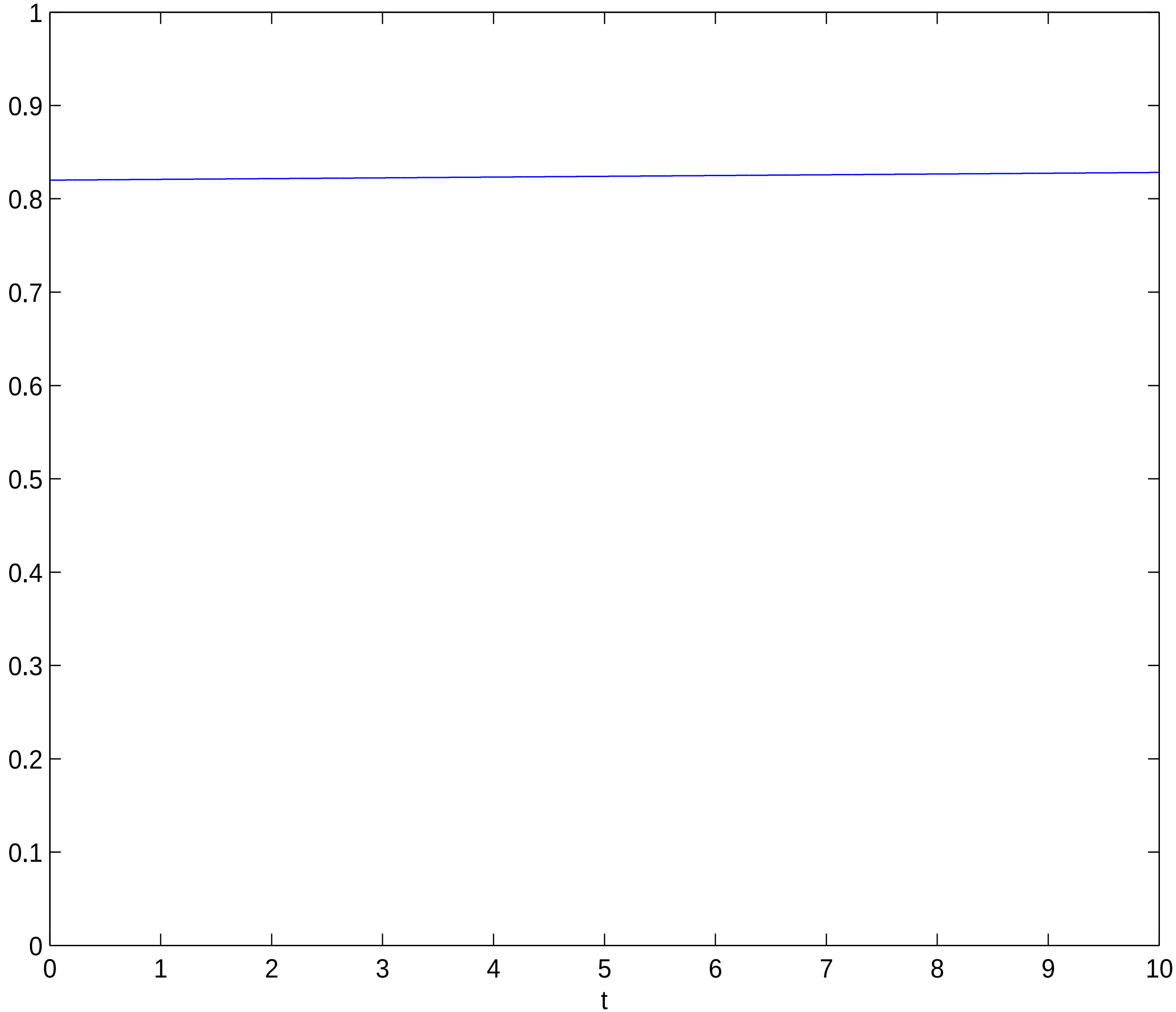

Figure 6: r_n from (25), which is produced by the explicit scheme (23) using $\Delta t = 1/1000$, $F_0 = 0.9$ and $S_0 = 0.1$.

Analysis of the analytical solution

- We see from Figure 6 that r_n is almost a constant
- We therefore assume that $r(t)$ is constant in time
- If this is true, we should be able to see that $r'(t) = 0$ for all t
- By differentiating (24) on both sides, we see that

$$r'(t) = 2(F-1)F' + 2(S-1)S' \tag{26}$$

- Recall the original system

$$F' = 1 - S \quad \text{and} \quad S' = F - 1 \tag{27}$$

Analysis of the analytical solution

- We can now calculate

$$r'(t) = 2(F-1)(1-S) + 2(S-1)(F-1) = 0, \qquad (28)$$

- This means that $r(t)$ is constant in the analytical case
- In general we can conclude that the analytical solutions of (22) are circles in the $F-S$ plane, with radius $((F_0-1)^2+(S_0-1)^2)^{1/2}$ and centered at (1,1)

Alternative analysis

We present an alternative strategy for proving that the graph of $(F(t),S(t)),\ t>0$ defines a circle in the $F-S$ plane. From the original system we see that

$$F'(t) = 1 - S(t) \quad \text{and} \quad S'(t) = F(t) - 1.$$

By multiplying the equations together we get

$$(F(t)-1)F'(t) = (1-S(t))S'(t). \tag{29}$$

Then integration in time from 0 to t gives

$$\int_0^t (F(\tau)-1)F'(\tau)d\tau = \int_0^t (1-S(\tau))S'(\tau)d\tau. \tag{30}$$

Alternative analysis

This leads to

$$\frac{1}{2}\left[(F(\tau)-1)^2\right]_0^t = -\frac{1}{2}\left[(S(\tau)-1)^2\right]_0^t, \tag{31}$$

which gives

$$(F(t)-1)^2 + (S(t)-1)^2 = (F_0-1)^2 + (S_0-1)^2 \tag{32}$$

for all $t \geq 0$.
This proves that

$$r(t) = (F(t)-1)^2 + (S(t)-1)^2$$

is constant.

Numerical solution

We shall continue our study of this behavior, but now we return to the numerical scheme

$$\begin{aligned} F_{n+1} &= F_n + \Delta t\,(1 - S_n), \\ S_{n+1} &= S_n + \Delta t\,(F_n - 1), \end{aligned} \tag{33}$$

where F_0 and S_0 are given.
We have observed from Figure 6 that for this solution

$$r_n = (F_n - 1)^2 + (S_n - 1)^2$$

is almost constant, i.e. $r_n \approx r_0$ for all $n \geq 0$.

Numerical solution

- We have shown that $r(t)$ is constant analytically
- This fact can be used to evaluate the quality of the numerical solution
- We can e.g. use $r_0 - r_n$ as as measure of the error in our computation
- In Table 1 we list $\frac{r_N - r_0}{r_0}$ and $\frac{r_N - r_0}{r_0 \Delta t}$ and compare for different Δt and N values, where we have used the explicit scheme from $t = 0$ to $t = 10$

Numerical solution

Δt	N	$\frac{r_N - r_0}{r_0}$	$\frac{r_N - r_0}{r_0 \Delta t}$
10^{-1}	10^{2}	1.7048	17.0481
10^{-2}	10^{3}	$1.0517 \cdot 10^{-1}$	10.5165
10^{-3}	10^{4}	$1.0050 \cdot 10^{-2}$	10.0502
10^{-4}	10^{5}	$1.0005 \cdot 10^{-3}$	10.0050

Table 1: The table shows Δt, the number of time steps N, the "error" $\frac{r_N - r_0}{r_0}$ and $\frac{r_N - r_0}{r_0 \Delta t}$. Note that the numbers in the last column seem to tend towards a constant.

Numerical solution

- From Table 1, it seems that

$$\frac{r_N - r_0}{r_0 \Delta t} \approx 10$$

- or

$$r_N \approx (1 + 10\Delta t) r_0$$

- If this assumption is true, the numerical solution will approach a perfect circle as Δt goes to zero
- We shall study the assumption in more detail

Analysis of the numerical scheme

- Note that

$$r_{n+1} = (F_{n+1} - 1)^2 + (S_{n+1} - 1)^2 \tag{34}$$

- By using the numerical scheme (33), we get

$$\begin{aligned} r_{n+1} &= (F_n - 1 + \Delta t(1 - S_n))^2 + (S_n - 1 + \Delta t(F_n - 1))^2 \\ &= (F_n - 1)^2 + 2\Delta t(F_n - 1)(1 - S_n) + \Delta t^2(1 - S_n)^2 \\ &\quad + (S_n - 1)^2 + 2\Delta t(F_n - 1)(S_n - 1) + \Delta t^2(1 - F_n)^2 \\ &= r_n + \Delta t^2 r_n \\ &= (1 + \Delta t^2) r_n \end{aligned}$$

- From this it follows that

$$r_m = (1 + \Delta t^2)^m r_0 \tag{35}$$

Analysis of the numerical scheme

- Using e.g. $\Delta t = 10/N$, we get

$$r_N = \left(1 + \frac{10^2}{N^2}\right)^N r_0 \tag{36}$$

- Using Taylor-series expansion we have

$$(1+x)^N = 1 + Nx + O(x^2) \tag{37}$$

for a given x

- We therefore see that

$$\left(1 + \frac{10^2}{N^2}\right)^N \approx 1 + N\frac{10^2}{N^2}$$

$$= 1 + \frac{10^2}{N}$$

Analysis of the numerical scheme

- From (36), we get

$$
\begin{aligned}
r_N - r_0 &= \left(\left(1 + 10^2/N^2\right)^N - 1\right) r_0 \\
&\approx \left(1 + 10^2/N - 1\right) r_0 \\
&= \frac{10^2}{N} r_0 \\
&= 10\Delta t \, r_0
\end{aligned}
$$

- or

$$
\frac{r_N - r_0}{r_0} \approx 10\Delta t
$$

- Thus this analysis gives the same conclusion as the numerical study above

Crank-Nicolson scheme

The Crank-Nicolson scheme for the system

$$
\begin{aligned}
F'(t) &= 1 - S(t), \quad F(0) = F_0, \\
S'(t) &= F(t) - 1, \quad S(0) = S_0.
\end{aligned}
\tag{38}
$$

reads

$$
\begin{aligned}
\frac{F_{n+1} - F_n}{\Delta t} &= \frac{1}{2}\left[(1 - S_n) + (1 - S_{n+1})\right], \\
\frac{S_{n+1} - S_n}{\Delta t} &= \frac{1}{2}\left[(F_n - 1) + (F_{n+1} - 1)\right].
\end{aligned}
\tag{39}
$$

Crank-Nicolson scheme

The Crank-Nicolson scheme can be rewritten as

$$\begin{aligned} F_{n+1} + \frac{\Delta t}{2} S_{n+1} &= F_n + \Delta t - \frac{\Delta t}{2} S_n, \\ -\frac{\Delta t}{2} F_{n+1} + S_{n+1} &= S_n - \Delta t + \frac{\Delta t}{2} F_n. \end{aligned} \tag{40}$$

- We see that when F_n and S_n are given, we have to solve a 2×2 system of linear equations, to find F_{n+1} and S_{n+1}

Crank-Nicolson scheme

Define

$$\mathbf{A} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix}, \tag{41}$$

and

$$\mathbf{b}_n = \begin{pmatrix} F_n + \Delta t - \frac{\Delta t}{2} S_n \\ S_n - \Delta t + \frac{\Delta t}{2} F_n \end{pmatrix}. \tag{42}$$

Crank-Nicolson scheme

Solving (40) for one time-step can now be done by:

- Solve

$$\mathbf{A}\mathbf{x}_{n+1} = \mathbf{b}_n, \tag{43}$$

  where x_{n+1} is the unknown vector with two components
- The new solution for F and S is then

$$\begin{pmatrix} F_{n+1} \\ S_{n+1} \end{pmatrix} = \mathbf{x}_{n+1} \tag{44}$$

Crank-Nicolson scheme

In general, a 2×2 matrix

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{45}$$

is non-singular if $ad \neq cb$. And when $ad \neq cb$ the inverse is given by

$$\mathbf{B}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{46}$$

Crank-Nicolson scheme

- In order for the problem to be well defined we need the matrix $\mathbf{A}$ to be non-singular
- But we have that

$$\text{det}(\mathbf{A}) \;=\; 1+\Delta t^2/4, \tag{47}$$

which ensures $\text{det}(\mathbf{A}) > 0$ for all values of Δt, and $\mathbf{A}$ is always non-singular

Crank-Nicolson scheme

- For the matrix (41), the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{1+\Delta t^2/4}\begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \tag{48}$$

- This fact together with (43) and (44) gives

$$\begin{pmatrix} F_{n+1} \\ S_{n+1} \end{pmatrix} = \frac{1}{1+\Delta t^2/4}\begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix}\begin{pmatrix} F_n + \Delta t - \frac{\Delta t}{2} S_n \\ S_n - \Delta t + \frac{\Delta t}{2} F_n \end{pmatrix}$$

Crank-Nicolson scheme

- We get

$$
\begin{aligned}
F_{n+1} &= \tfrac{1}{1+\Delta t^2/4}\left[\left(1-\Delta t^2/4\right)F_n + \Delta t\left(\tfrac{\Delta t}{2}+1\right) - \Delta t S_n\right] \\
S_{n+1} &= \tfrac{1}{1+\Delta t^2/4}\left[\left(1-\Delta t^2/4\right)S_n + \Delta t\left(\tfrac{\Delta t}{2}-1\right) + \Delta t F_n\right]
\end{aligned} \qquad (49)
$$

- Figure 7 plots the solution of this scheme for $S_0 = 0.1$, $F_0 = 0.9$ and $\Delta t = 1/1000$, t is from $t = 0$ to $t = 10$ and the solution is plotted in the F-S coordinate system

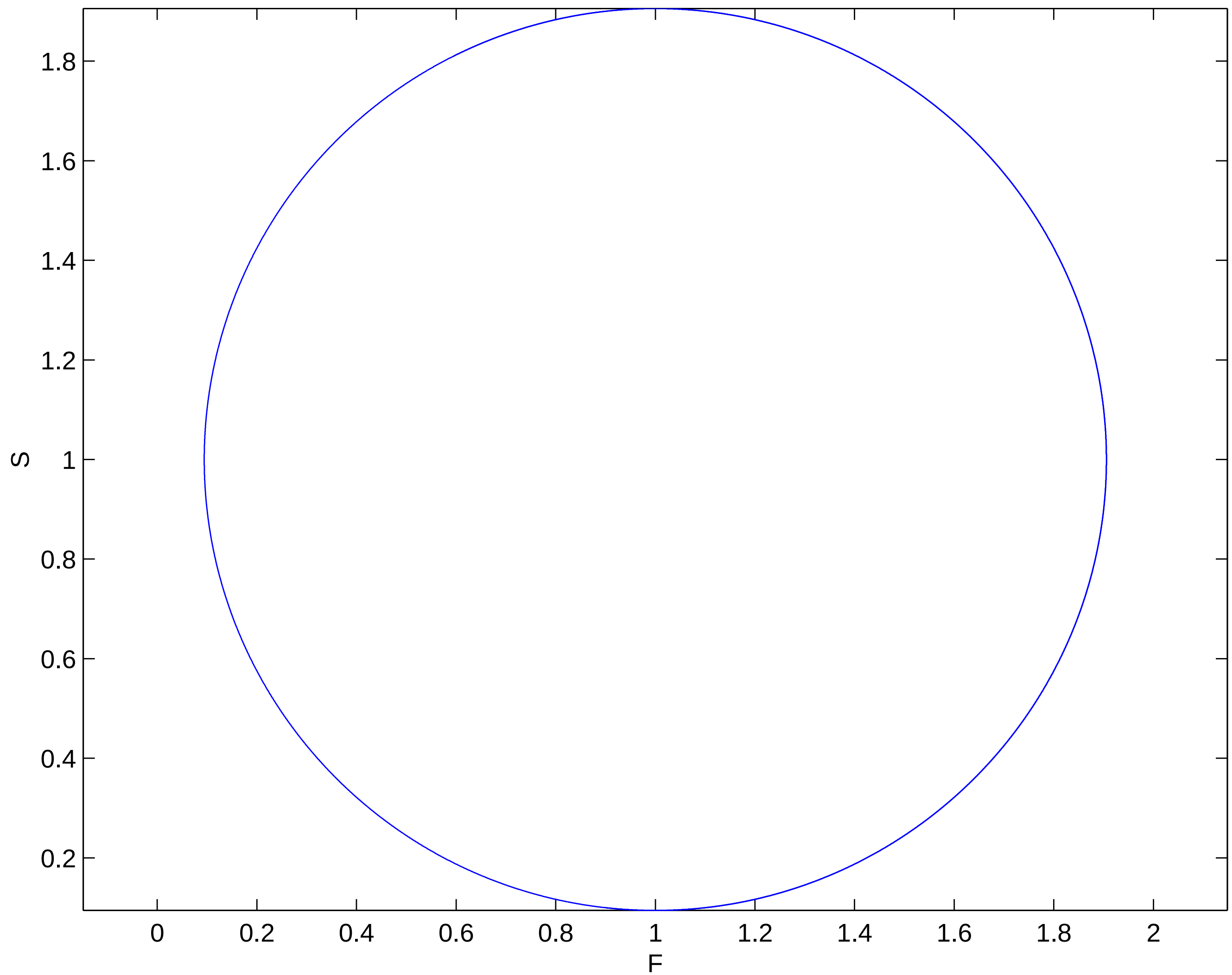

Figure 7: The numerical solution for the Crank-Nicholson scheme

Crank-Nicolson scheme

- In Figure 7 we observe that the solution again seems to form a perfect circle
- To study this closer we define, as above

$$r_n = (F_n - 1)^2 + (S_n - 1)^2 \tag{50}$$

- and study the relative change

$$\frac{r_N - r_0}{r_0} \tag{51}$$

in Table 7

Crank-Nicolson scheme

Δt	N	$\frac{r_N - r_0}{r_0}$
10^{-1}	10^2	$-2.6682 \cdot 10^{-16}$
10^{-2}	10^3	$-1.59986 \cdot 10^{-17}$
10^{-3}	10^4	$3.97982 \cdot 10^{-17}$
10^{-4}	10^5	$7.06021 \cdot 10^{-15}$

Table 2: The table shows Δt, the number of time steps N, and the "error" $\frac{r_N - r_0}{r_0}$.

Crank-Nicolson scheme

- We observe that the relative error $\frac{r_N - r_0}{r_0}$ is much smaller for the Crank-Nicolson scheme (50) than for the explicit scheme (23)
- We therefore conclude that the Crank-Nicolson scheme produces better solutions than the explicit scheme