

NOTES ON PROJECTIVE REPRESENTATION THEORY

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ABSTRACT. The following exposition on projective representation theory for finite groups is taken from the introduction of the authors PhD-thesis.

I do not claim that any of the results presented in this text are original, and its exposition mimics, to some extent, the exposition given in [Ser77]. All the results I will present here can be found, or deduced from, results found in [CTS22; Che15; Ser77; Bro82]. Section 6 on Clifford theory especially follows [CTS22, Chapter 9].

1. BASIC RESULTS

In this chapter we will only consider finite groups and projective representations on finite dimensional complex vector spaces. If G is a finite group it is tempting to define a projective representation of G on a complex vector space V to simply be given by a group homomorphism $\pi: G \rightarrow PGL(V)$, where $PGL(V)$ is the quotient group of $GL(V)$ by the group of nonzero scalar multiples of id_V . We will in this chapter let the letter q always denote the quotient map $q: GL(V) \rightarrow PGL(V)$. This does not amount to a good representation theory, we will get back to this later. The following definition turns out to be correct.

Definition 1.1. Let G be a finite group and V be a complex vector space. A *projective representation of G on V* is given by a function $\pi: G \rightarrow GL(V)$ such that $q \circ \pi: G \rightarrow PGL(V)$ is a group homomorphism. We will use the notation π or (V, π) to denote this projective representation. The dimension of V is called the *degree of π* , and we will use the notation $d_\pi := \dim V$ to denote the degree of π .

We have the following equivalent definition.

Proposition 1.2. Let G be a finite group and V be a complex vector space. Then a function $\pi: G \rightarrow GL(V)$ is a projective representation if and only if there exists a function $\sigma: G \times G \rightarrow \mathbb{C}^\times$ such that

$$\pi(x)\pi(y) = \sigma(x, y)\pi(xy)$$

for all $x, y \in G$.

Proof. If such a function σ exists, then for any $x, y \in G$ we have that

$$q(\pi(x))q(\pi(y)) = q(\pi(x)\pi(y)) = q(\sigma(x, y)\pi(xy)) = q(\pi(xy))$$

since $\sigma(x, y)\text{id}_V \in \ker q$. Thus, $q \circ \pi$ is a group homomorphism.

For the converse suppose that $q \circ \pi$ is a group homomorphism. Then for any $x, y \in G$ we have that the element

$$\pi(x)\pi(y)\pi(xy)^{-1}$$

is in $\ker q$. Hence, there exists a unique element $\sigma(x, y) \in \mathbb{C}^\times$ such that

$$\pi(x)\pi(y)\pi(xy)^{-1} = \sigma(x, y)\text{id}_V.$$

As we range over all elements in G this produces the desired function $\sigma: G \times G \rightarrow \mathbb{C}^\times$. $\square$

If we want to emphasize the dependence on the function σ we will say that π is a σ -projective representation of G on V . By associativity of the product in $GL(V)$ we have that the function σ satisfies the 2-cocycle equation, i.e. for any $x, y, z \in G$ we have that

$$(1) \quad \sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z).$$

This means that the σ -projective representation π determines a class in group cohomology, namely the class $[\sigma] \in H^2(G; \mathbb{C}^\times)$. For the sake of completeness we give a definition of what group cohomology is.

Definition 1.3. Let G be a group and M be a G -module, i.e. an abelian group with a G -action. For each $n \geq 0$ let $C^n(G; M)$ be the set of all functions from G^n to M , by convention we set G^0 equal to the trivial group. The elements of $C^n(G; M)$ are called *inhomogeneous n -cochains of G with values in M* . Let $\delta^n: C^n(G; M) \rightarrow C^{n+1}(G; M)$ be defined by

$$(\delta^n f)(g_0, \dots, g_n) = g_0 f(g_1, \dots, g_n) - f(g_0 g_1, g_2, \dots, g_n) + \dots + (-1)^n f(g_0, \dots, g_{n-1}).$$

This defines a cochain complex $(C^*(G; M), \delta)$. We define the *n -cocycles*, the *n -coboundaries*, and the *n -th cohomology group* as follows:

$$\begin{aligned} Z^n(G; M) &:= \ker \delta^n \\ B^n(G; M) &:= \operatorname{im} \delta^n \\ H^n(G; M) &:= \frac{Z^n(G; M)}{B^n(G; M)}. \end{aligned}$$

We say that two elements in $Z^n(G; M)$ are *cohomologous* if they represent the same class in $H^n(G; M)$. Finally, we will often omit the superscript of the maps δ^n and only write δ .

There are of course some details that we are skipping here, for example the fact that $(C^*(G; M), \delta)$ is indeed a complex, ensuring that the definition of $H^n(G; M)$ is well-defined. Since this chapter is mainly going to focus on projective representation theory we leave it to the interested reader to fill in all the necessary prerequisites on group cohomology. The book [Bro82] is an excellent resource for this.

With this definition in mind we see that the function σ associated to a projective representation is an element in $C^2(G; \mathbb{C}^\times)$, where $\mathbb{C}^\times$ is regarded as a G -module with the trivial action. One checks that $\sigma \in Z^2(G; \mathbb{C}^\times)$ precisely because (1) holds. Note that a 2-coboundary of G with values in $\mathbb{C}^\times$ is of the form

$$(\delta\mu)(x, y) = \frac{\mu(x)\mu(y)}{\mu(xy)},$$

where $\mu: G \rightarrow \mathbb{C}^\times$ is a function.

We furthermore say that a 2-cocycle σ is *normalized* if $\sigma(1, 1) = 1$. Note that any projective representation π of G on V with $\pi(1) = \operatorname{id}_V$, will give rise to a normalized 2-cocycle. It is furthermore true that any 2-cocycle is cohomologous to a normalized 2-cocycle: Let $\sigma \in Z^2(G; \mathbb{C}^\times)$ and define $\mu_\sigma: G \rightarrow \mathbb{C}^\times$ by

$$\mu_\sigma(x) = \frac{1}{\sigma(1, x)}.$$

Then $(\delta\mu_\sigma)\sigma$ is normalized. Note that there is nothing special about $\mathbb{C}^\times$ that allows us to define μ_σ , it is indeed true that a 2-cocycle with values in an arbitrary G -module is cohomologous to a normalized 2-cocycle.

A map of σ -projective representations is defined as follows.

Definition 1.4. Let G be a finite group, and π_1 and π_2 be σ -projective representations of G on V_1 and V_2 respectively. An *intertwiner from π_1 to π_2* is a linear map $F: V_1 \rightarrow V_2$ such

that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{F} & V_2 \\ \pi_1(x) \downarrow & & \downarrow \pi_2(x) \\ V_1 & \xrightarrow{F} & V_2 \end{array}$$

commutes for all $x \in G$. We will use the notation $\text{Hom}_G(\pi_1, \pi_2)$ to denote the set of all intertwiners from π_1 to π_2 , and it is clear that this set has the structure of a complex vector space. We say that two σ -projective representations are *isomorphic* if there exists an intertwiner from π_1 to π_2 that is also a linear isomorphism. We will sometimes write $\pi_1 \simeq \pi_2$ to denote that π_1 and π_2 are isomorphic.

The collection of σ -projective representations and intertwiners form a category which we denote by $\mathbf{Rep}^\sigma(G)$. We have the following result for this category.

Proposition 1.5. *Let G be a finite group. If $\sigma_1, \sigma_2 \in Z^2(G; \mathbb{C}^\times)$ are cohomologous, then $\mathbf{Rep}^{\sigma_1}(G)$ and $\mathbf{Rep}^{\sigma_2}(G)$ are isomorphic categories.*

Proof. That σ_1 and σ_2 are cohomologous means that there exists a function $\mu: G \rightarrow \mathbb{C}^\times$ such that $\sigma_1 = (\delta\mu)\sigma_2$. Define a functor $\mathcal{F}_\mu: \mathbf{Rep}^{\sigma_2}(G) \rightarrow \mathbf{Rep}^{\sigma_1}(G)$ as follows: if π is a σ_2 -projective representation of G , let $\mathcal{F}_\mu(\pi) = \mu\pi$, and if π' is another σ_2 -projective representation of G , and $F \in \text{Hom}_G(\pi, \pi')$, let $\mathcal{F}_\mu(F) = F$. This is indeed well-defined: let $x, y \in G$, then we have that

$$\begin{aligned} \mathcal{F}_\mu(\pi)(x)\mathcal{F}_\mu(\pi)(y) &= \mu(x)\mu(y)\pi(x)\pi(y) \\ &= \mu(x)\mu(y)\sigma_2(x, y)\pi(xy) \\ &= (\delta\mu)(x, y)\sigma_2(x, y)\mu(xy)\pi(xy) \\ &= \sigma_1(x, y)\mathcal{F}_\mu(\pi)(xy). \end{aligned}$$

Furthermore, it is clear that the two diagrams

$$\begin{array}{ccc} V & \xrightarrow{F} & V' \\ \pi(x) \downarrow & & \downarrow \pi'(x) \\ V & \xrightarrow{F} & V' \end{array} \quad \begin{array}{ccc} V & \xrightarrow{F} & V \\ \mu(x)\pi(x) \downarrow & & \downarrow \mu(x)\pi'(x) \\ V & \xrightarrow{F} & V \end{array}$$

commute if and only if one of them commutes, showing that $\mathcal{F}_\mu(F) \in \text{Hom}_G(\mathcal{F}_\mu(\pi), \mathcal{F}_\mu(\pi'))$. Functoriality now follows since $\mathcal{F}_\mu$ is the identity on morphisms. It is clear that the inverse functor is given by $\mathcal{F}_{1/\mu}$ defined analogously. $\square$

This result is very important because it tells us that we can understand the projective representation theory of a group by only considering the associated 2-cocycle up to cohomology. We will use this to great extents; since one is often able to perturb any given 2-cocycle to a cohomologous 2-cocycle with desirable properties.

One of the most important examples of a σ -projective representation is the following.

Example 1.6. Let G be a finite group and $\sigma \in Z^2(G; \mathbb{C}^\times)$. The *left regular σ -projective representation of G* is the representation $\lambda_\sigma: G \rightarrow GL(\ell^2(G))$ defined by

$$\lambda_\sigma(x)\delta_g = \sigma(x, g)\delta_{xg},$$

where $\{\delta_g\}_{g \in G}$ is the standard basis for $\ell^2(G)$. To be precise, $\delta_g: G \rightarrow \mathbb{C}$ is defined by

$$\delta_g(x) = \begin{cases} 1, & \text{if } x = g, \\ 0, & \text{if } x \neq g. \end{cases}$$

We list some more properties of the category $\mathbf{Rep}^\sigma(G)$: For any two σ -projective representations $\pi_1, \pi_2 \in \mathbf{Rep}^\sigma(G)$ we can form their direct sum $\pi_1 \oplus \pi_2: G \rightarrow GL(V_1 \oplus V_2)$ defined by

$$(\pi_1 \oplus \pi_2)(x)(v_1, v_2) = (\pi_1(x)v_1, \pi_2(x)v_2).$$

This direct sum will be a σ -projective representation as well.

If $\pi_1 \in \mathbf{Rep}^{\sigma_1}(G)$ and $\pi_2 \in \mathbf{Rep}^{\sigma_2}(G)$, we can form their tensor product $\pi_1 \otimes \pi_2: G \rightarrow GL(V_1 \otimes V_2)$ defined by

$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) = (\pi_1(x)v_1) \otimes (\pi_2(x)v_2).$$

Note that the tensor product will be a $(\sigma_1\sigma_2)$ -projective representation.

If $\pi \in \mathbf{Rep}^\sigma(G)$, we can form the dual representation $\pi^*: G \rightarrow GL(V^*)$ defined by

$$(\pi^*(x)\varphi)(v) = \varphi(\pi(x)^{-1}v).$$

With this we have that π^* is a $(1/\sigma)$ -projective representation.

For now, these are all the categorical facts we are interested in. We will get back to these constructions when we introduce characters. Now we present the most fundamental result about projective representations.

Lemma 1.7 (The averaging Lemma). *Let G be a finite group, π_1 and π_2 be σ -projective representations, and $T: V_1 \rightarrow V_2$ be a linear map. Then the average*

$$\tilde{T} := \frac{1}{|G|} \sum_{x \in G} \pi_2(x)^{-1} T \pi_1(x)$$

is an intertwiner from π_1 to π_2 .

Proof. We want to show that for any $y \in G$, the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\tilde{T}} & V_2 \\ \pi_1(y) \downarrow & & \downarrow \pi_2(y) \\ V_1 & \xrightarrow{\tilde{T}} & V_2 \end{array}$$

commutes. That is the case by the following computation:

$$\begin{aligned} \tilde{T}\pi_1(y) &= \frac{1}{|G|} \sum_{x \in G} \pi_2(x)^{-1} T \pi_1(x) \pi_1(y) \\ &= \frac{1}{|G|} \sum_{x \in G} \sigma(x, y) \pi_2(x)^{-1} T \pi_1(xy) \\ &= \frac{1}{|G|} \sum_{x \in G} \sigma(x, y) \pi_2(xy y^{-1})^{-1} T \pi_1(xy) \\ &= \frac{1}{|G|} \sum_{x \in G} \sigma(x, y) (1/\sigma(xy, y^{-1}) \pi_2(xy) \pi_2(y^{-1}))^{-1} T \pi_1(xy) \\ &= \frac{1}{|G|} \sum_{x \in G} \sigma(x, y) \sigma(xy, y^{-1}) \pi_2(y^{-1})^{-1} \pi_2(xy)^{-1} T \pi_1(xy) \\ &= \frac{1}{|G|} \sum_{x \in G} \frac{\sigma(x, y) \sigma(xy, y^{-1})}{\sigma(y, y^{-1}) \sigma(1, 1)} \pi_2(y) \pi_2(xy)^{-1} T \pi_1(xy) \\ &= \pi_2(y) \tilde{T}. \end{aligned}$$

Where the second to last equality follows because

$$\pi_2(y)^{-1} = \frac{1}{\sigma(y^{-1}, y) \sigma(1, 1)} \pi_2(y^{-1}).$$

The last equality follows because $\sigma(x, 1) = \sigma(1, 1)$ for all $x \in G$ by inserting $y = z = 1$ in (1), and

$$\frac{\sigma(x, y)\sigma(xy, y^{-1})}{\sigma(y, y^{-1})\sigma(x, 1)} = \frac{1}{(\delta\sigma)(x, y, y^{-1})} = 1,$$

since $\sigma \in Z^2(G, \mathbb{C}^\times)$. $\square$

Definition 1.8. Let G be a finite group, and π be a σ -projective representation of G on V . A subspace W of V is said to be G -invariant if for every $w \in W$, and for every $x \in G$, we have that $\pi(x)w \in W$. We say that the representation π is *irreducible* if the only G -invariant subspaces of V are V itself and 0 . We will use the notation $\pi|_W$ for the induced σ -projective representation of G on W whenever W is a G -invariant subspace.

Lemma 1.9. Let G be a finite group, π_1 and π_2 be σ -projective representations, and $T \in \text{Hom}_G(\pi_1, \pi_2)$. Then $\ker T$ is a G -invariant subspace of V_1 , and $\text{im } T$ is a G -invariant subspace of V_2 .

Proof. Let $v \in \ker T$ and $x \in G$. We want to show that $\pi_1(x)v \in \ker T$. We simply compute the following:

$$T(\pi_1(x)v) = (T\pi_1(x))v = (\pi_2(x)T)v = \pi_2(x)(Tv) = 0.$$

Hence, we can conclude that $\ker T$ is a G -invariant subspace of V_1 .

To see that $\text{im } T$ is a G -invariant subspace of V_2 we use the fact that any vector in $\text{im } T$ is of the form Tw for some $w \in V_1$. We then compute that

$$\pi_2(x)Tw = T(\pi_1(x)w),$$

which shows that $\text{im } T$ is a G -invariant subspace of V_2 . $\square$

With this we can prove some of the most important results about projective representations, namely that analogs of Maschke's Theorem and Schur's Lemma holds.

Proposition 1.10. Let G be a finite group and π be a σ -projective representation of G on V . If W is a proper G -invariant subspace of V , then there exists a G -invariant complement U of W such that $V = W \oplus U$.

Proof. Let W be such a proper G -invariant subspace and $p: V \rightarrow W$ be any projection onto W . We claim that the average $\tilde{p} \in \text{Hom}_G(\pi, \pi|_W)$ is still a projection onto W . If this is the case we define $U := \ker \tilde{p}$, and we have that

$$V = \text{im } \tilde{p} \oplus \ker \tilde{p} = W \oplus U.$$

By Lemma 1.9 we have that U is also a G -invariant subspace of V .

To see that $\tilde{p}$ is a projection onto W we need to check that $\text{im } \tilde{p} = W$ and for any $w \in W$ that $\tilde{p}w = w$. For any $x \in G$ and any $v \in V$ we have that $p\pi(x)v \in W$. Furthermore, since W is a G -invariant subspace of V we have that $\pi(x)^{-1}p\pi(x)v \in W$. From this we conclude that

$$\tilde{p}v = \frac{1}{|G|} \sum_{x \in G} \pi(x)^{-1}p\pi(x)v$$

is an element of W for any $v \in V$. Hence, $\text{im } \tilde{p} \subset W$. If $w \in W$ we have that $\pi(x)w \in W$, again since W is a G -invariant subspace of V . Therefore, $p\pi(x)w = \pi(x)w$, hence

$$\tilde{p}w = \frac{1}{|G|} \sum_{x \in G} \pi(x)^{-1}p\pi(x)w = \frac{1}{|G|} \sum_{x \in G} w = w.$$

From this we conclude that $\tilde{p}$ is indeed a projection onto W , completing the proof. $\square$

Corollary 1.11 (Maschke's Theorem). Let G be a finite group and π be a σ -projective representation of G on V . Then π is isomorphic to a direct sum of irreducible σ -projective representations.

Proof. We will prove this by induction on the dimension of V . If $\dim V = 1$ we have that π is necessarily irreducible, proving the claim. Suppose therefore that $\dim V = n > 1$ and that any σ -projective representation of degree less than n is isomorphic to a direct sum of irreducible σ -projective representations of G . If π is irreducible there is nothing to prove. If π is not irreducible then there exists a proper G -invariant subspace W of V . By Proposition 1.10 we have that there exists a G -invariant complement U of W such that $V = W \oplus U$. Since $\dim W < n$ and $\dim U < n$ we have by the induction hypothesis that both $\pi|_W$ and $\pi|_U$ decompose into a direct sum of irreducible σ -projective representations of G , completing the proof. $\square$

Proposition 1.12 (Schur's Lemma). *Let G be a finite group and π_1, π_2 be two irreducible σ -projective representations of G . Then we have that*

$$\dim \operatorname{Hom}_G(\pi_1, \pi_2) = \begin{cases} 1, & \text{if } \pi_1 \simeq \pi_2, \\ 0, & \text{if } \pi_1 \not\simeq \pi_2. \end{cases}$$

Proof. Let $T \in \operatorname{Hom}_G(\pi_1, \pi_2)$. By Lemma 1.9 we have that $\ker T$ is a G -invariant subspace of V_1 and that $\operatorname{im} T$ is a G -invariant subspace of V_2 . Since π_1 and π_2 are irreducible we furthermore have that

$$\ker T \in \{0, V_1\} \text{ and } \operatorname{im} T \in \{0, V_2\}.$$

If $\ker T = V_1$ we necessarily have that $T = 0$, in which case we also conclude that $\operatorname{im} T = 0$. If $\ker T = 0$ we have that T is injective, and since $V_1 \neq 0$ this means that $\operatorname{im} T = V_2$, which shows that T is also surjective, hence an isomorphism from π_1 to π_2 . Therefore, any intertwiner between two irreducible σ -projective representations is either an isomorphism, or the zero intertwiner. From this we can conclude that if $\pi_1 \not\simeq \pi_2$, the only intertwiner from π_1 to π_2 is the zero intertwiner, showing that $\dim \operatorname{Hom}_G(\pi_1, \pi_2) = 0$ in this case.

If $\pi_1 \simeq \pi_2$, we have that every nonzero intertwiner from π_1 to π_2 is an isomorphism, in particular this means that

$$\dim \operatorname{Hom}_G(\pi_1, \pi_2) = \dim \operatorname{Hom}_G(\pi_1, \pi_1).$$

Let therefore $T \in \operatorname{Hom}_G(\pi_1, \pi_1)$ be a nonzero intertwiner. Since T is a linear isomorphism from V_1 to V_1 we have that there exists a nonzero eigenvalue λ of T with associated nonzero eigenvector $v \in V_1$. Then $T - \lambda \operatorname{id}_V$ is an intertwiner from π_1 to π_1 , and by construction we have that $v \in \ker(T - \lambda \operatorname{id}_V)$. Since π_1 is irreducible, we have that $\ker(T - \lambda \operatorname{id}_V) = V_1$, which shows that $T = \lambda \operatorname{id}_V$. Since T was an arbitrary intertwiner from π_1 to π_1 we have that every intertwiner from π_1 to π_1 is a scalar multiple of id_V . From this we conclude that

$$\dim \operatorname{Hom}_G(\pi_1, \pi_1) = 1,$$

which completes the proof. $\square$

2. COCYCLE PERTURBATIONS

We will need the following fact about group cohomology of finite groups. See [Bro82, Corollary 10.2, chapter 3] for a proof.

Proposition 2.1. *Let G be a finite group and M be a G -module. Then multiplication with $|G|$ as a map from $H^n(G; M) \rightarrow H^n(G; M)$ equals the trivial group homomorphism for every $n \geq 1$. In particular, if $|G|$ is invertible in M , then $H^n(G; M) = 0$ for $n \geq 1$.*

If π is a σ -projective representation of G on V , we would like to say that there exists an inner product on V for which $\pi(x)$ is unitary for every $x \in G$. However, this is only true if $\sigma(x, y) \in \mathbb{T}$ for every $x, y \in G$. Luckily, with Proposition 1.5 in mind, every 2-cocycle of a finite group with values in $\mathbb{C}^\times$ is cohomologous to a 2-cocycle with values in $\mathbb{T}$.

Proposition 2.2. *Let G be a finite group. Then the inclusion $j: \mathbb{T} \rightarrow \mathbb{C}^\times$ induces an isomorphism $H^2(G; \mathbb{T}) \simeq H^2(G; \mathbb{C}^\times)$.*

Proof. Note that the inclusion $j: \mathbb{T} \rightarrow \mathbb{C}^\times$ fits into a short exact sequence

$$1 \longrightarrow \mathbb{T} \xrightarrow{j} \mathbb{C}^\times \longrightarrow \mathbb{R} \longrightarrow 1$$

where the map $\mathbb{C}^\times \rightarrow \mathbb{R}$ is given by $re^{i\theta} \mapsto \ln(r)$. Since G is finite we have by Proposition 2.1 that $H^n(G; \mathbb{R}) = 0$ for all $n \geq 1$, hence the long exact sequence in cohomology shows that $H^n(G; \mathbb{T}) \simeq H^n(G; \mathbb{C}^\times)$ for all $n \geq 2$. $\square$

Proposition 2.3. *Let G be a finite group and π be a σ -projective representation of G on V with σ taking values in $\mathbb{T}$. Then there exists an inner product on V such that $\pi(x)$ is unitary with respect to this inner product for all $x \in G$.*

Proof. Let $(-, -): V \times V \rightarrow \mathbb{C}$ be any inner product on V . This exists because V is a finite dimensional complex vector space. Define $\langle -, - \rangle: V \times V \rightarrow \mathbb{C}$ by

$$\langle u, v \rangle = \frac{1}{|G|} \sum_{x \in G} (\pi(x)u, \pi(x)v).$$

It is clear that this defines an inner product on V , and for any $y \in G$ we have that

$$\begin{aligned} \langle \pi(y)^{-1}u, v \rangle &= \frac{1}{|G|} \sum_{x \in G} (\pi(x)\pi(y)^{-1}u, \pi(x)v) \\ &= \frac{1}{|G|} \sum_{x \in G} \frac{\sigma(x, y)}{\sigma(y^{-1}, y)\sigma(1, 1)} (\pi(xy^{-1})u, \pi(x)v) \\ &= \frac{1}{|G|} \sum_{x \in G} \frac{\sigma(x, y)}{\sigma(y^{-1}, y)\sigma(1, 1)} (\pi(xy^{-1})u, \pi(xy^{-1}y)v) \\ &= \frac{1}{|G|} \sum_{x \in G} \frac{\sigma(x, y)}{\sigma(y^{-1}, y)\sigma(1, 1)} \frac{1}{\overline{\sigma(xy^{-1}, y)}} (\pi(xy^{-1})u, \pi(xy^{-1})\pi(y)v) \\ &= \frac{1}{|G|} \sum_{x \in G} \frac{\sigma(x, y)\sigma(xy^{-1}, y)}{\sigma(y^{-1}, y)\sigma(1, 1)} (\pi(xy^{-1})u, \pi(xy^{-1})\pi(y)v) \\ &= \langle u, \pi(y)v \rangle. \end{aligned}$$

Note that we really needed that $\sigma(x, y) \in \mathbb{T}$ for every $x, y \in G$ to conclude that

$$\overline{\sigma(x, y)} = \frac{1}{\sigma(x, y)}.$$

We also have that

$$\frac{\sigma(x, y)\sigma(xy^{-1}, y)}{\sigma(y^{-1}, y)\sigma(1, 1)} = \frac{\sigma(x, y)\sigma(xy^{-1}, y)}{\sigma(y^{-1}, y)\sigma(x, 1)} = \frac{1}{(\delta\sigma)(x, y^{-1}, y)} = 1,$$

which justifies the last equality. Hence, $\pi(y)^* = \pi(y)^{-1}$ for every $y \in G$, which is what we wanted to show. $\square$

In the rest of this section we present further results in the direction of Proposition 2.2. First we show that the group $H^2(G; \mathbb{C}^\times)$ is finite. In the following, $G_{ab} = G/[G, G]$ denotes the *abelianization* of G .

Proposition 2.4. *Let G be a finite group, then there is a short exact sequence*

$$0 \longrightarrow G_{ab} \longrightarrow H^2(G; C_{|G|}) \longrightarrow H^2(G; \mathbb{C}^\times) \longrightarrow 0.$$

In particular, $H^2(G; \mathbb{C}^\times)$ is finite.

Proof. By Proposition 2.2 we have that $H^2(G; \mathbb{C}^\times) \simeq H^2(G; \mathbb{T})$. By Proposition 2.1 we have that multiplication by $|G|$ as a map $H^n(G; \mathbb{T}) \rightarrow H^n(G; \mathbb{T})$ is the zero map for any $n \geq 1$. Hence, the short exact sequence

$$1 \longrightarrow C_{|G|} \longrightarrow \mathbb{T} \xrightarrow{(-)^{|G|}} \mathbb{T} \longrightarrow 1$$

induces a long exact sequence in group cohomology which contains the short exact sequence

$$0 \longrightarrow H^1(G; \mathbb{T}) \longrightarrow H^2(G; C_{|G|}) \longrightarrow H^2(G; \mathbb{T}) \longrightarrow 0.$$

Since we are considering $\mathbb{T}$ as a G -module with the trivial action we have the following isomorphisms,

$$H^1(G; \mathbb{T}) \simeq \text{Hom}(G, \mathbb{T}) \simeq G_{ab}.$$

Along with the isomorphism $H^2(G; \mathbb{C}^\times) \simeq H^2(G; \mathbb{T})$ we get that there is a short exact sequence

$$0 \longrightarrow G_{ab} \longrightarrow H^2(G; C_{|G|}) \longrightarrow H^2(G; \mathbb{C}^\times) \longrightarrow 0.$$

Note that $H^2(G; C_{|G|})$ is necessarily finite, since it is a quotient of the group $Z^2(G; C_{|G|})$, which has order less than the cardinality of the set of all functions $G \times G \rightarrow C_{|G|}$, which is $|G|^{|G|^2}$. Since $H^2(G; C_{|G|})$ surjects onto $H^2(G; \mathbb{C}^\times)$, this in particular implies that $H^2(G; \mathbb{C}^\times)$ is a finite group. $\square$

Since $H^2(G; \mathbb{C}^\times)$ is finite it makes sense to talk about the order of its group elements. For notational convenience we make the following definition.

Definition 2.5. Let G be a finite group and $\sigma \in Z^2(G; \mathbb{C}^\times)$. We define $|\sigma| \in \mathbb{N}$ to be the order of $[\sigma]$ in $H^2(G; \mathbb{C}^\times)$ and call it the *degree* of σ .

Proposition 2.6. Let G be a finite group, $\sigma \in Z^2(G; \mathbb{C}^\times)$, and $d \in \mathbb{N}$. Then σ is cohomologous to a 2-cocycle taking values in C_d if and only if $|\sigma|$ divides d .

Proof. The short exact sequence

$$1 \longrightarrow C_d \xrightarrow{i} \mathbb{T} \xrightarrow{(-)^d} \mathbb{T} \longrightarrow 1$$

induces a long exact sequence in group cohomology containing the following segment:

$$H^2(G; C_d) \xrightarrow{i_*} H^2(G; \mathbb{T}) \xrightarrow{d} H^2(G; \mathbb{T}).$$

By Proposition 2.2 we may swap $H^2(G; \mathbb{T})$ with the isomorphic group $H^2(G; \mathbb{C}^\times)$ and obtain the exact sequence

$$H^2(G; C_d) \xrightarrow{(j \circ i)_*} H^2(G; \mathbb{C}^\times) \xrightarrow{d} H^2(G; \mathbb{C}^\times).$$

Note that the map $j \circ i: C_d \rightarrow \mathbb{C}^\times$ really is the inclusion of C_d as the d -roots of unity. We have that $[\sigma] \in \ker(d)$ if and only if $|\sigma|$ divides d . Hence, there exists a class $[\sigma'] \in H^2(G; C_d)$ such that

$$[\sigma] = (j \circ i)_*[\sigma'] = [j \circ i \circ \sigma'],$$

if and only if $|\sigma|$ divides d . In which case, the 2-cocycle $j \circ i \circ \sigma'$ takes values in C_d as a subgroup of $\mathbb{C}^\times$ and it is, by the above argument, cohomologous to σ . This completes the proof. $\square$

This result has some very nice consequences.

Theorem 2.7. Let G be a finite group and π be a σ -projective representation of G on V . Then $|\sigma|$ divides d_π .

Proof. By possibly replacing π with another projective representation on V , we may, by Proposition 2.2 and Proposition 2.3, assume that σ takes values in $\mathbb{T}$ and that V is endowed with an inner product such that $\pi(x)$ is unitary for all $x \in G$. Let

$$\mu(x) = \frac{1}{\det(\pi(x))^{1/d_\pi}}.$$

Then $\pi' : G \rightarrow GL(V)$ defined by

$$\pi'(x) = \mu(x)\pi(x)$$

is a $(\delta\mu)\sigma$ -projective representation of G on V such that $\pi'(x)$ is special unitary for each $x \in G$, i.e. that $\pi'(x)$ is unitary, and $\det(\pi'(x)) = 1$ for every $x \in G$. The 2-cocycle $(\delta\mu)\sigma$ takes values in C_{d_π} : Let $x, y \in G$, then

$$\begin{aligned} 1 &= \det(\pi'(x)) \det(\pi'(y)) \\ &= \det(\pi'(x)\pi'(y)) \\ &= \det(((\delta\mu)\sigma)(x, y)\pi'(xy)) \\ &= ((\delta\mu)\sigma)(x, y)^{d_\pi} \det(\pi'(xy)) \\ &= ((\delta\mu)\sigma)(x, y)^{d_\pi}. \end{aligned}$$

Hence, σ is cohomologous to a 2-cocycle taking values in C_{d_π} , which by Proposition 2.6 is equivalent to $|\sigma|$ dividing d_π . $\square$

3. CHARACTER THEORY

The character theory for projective representations is best formalized for *unitary* projective representations. With Proposition 1.5, Proposition 2.2, and Proposition 2.3 in mind, we feel justified in restricting ourselves to this setting of unitary projective representations. Thus, every 2-cocycle appearing in this section will be assumed to take values in $\mathbb{T}$.

Definition 3.1. Let G be a finite group and π be a σ -projective representation of G on V . The *character of G associated to π* is the function $\chi_\pi : G \rightarrow \mathbb{C}$ defined by

$$\chi_\pi(x) = \text{Tr}(\pi(x)).$$

We will sometimes say that a function $\chi : G \rightarrow \mathbb{C}$ is a *(σ -projective) character of G* if there exists a σ -projective representation π of G such that $\chi = \chi_\pi$. We will furthermore say that the character χ is *irreducible* if the σ -projective representation π can be chosen to be irreducible.

Proposition 3.2. *Let G be a finite group and π be a σ -projective representation of G on V . Then,*

- (1) $\chi_\pi(1) = \sigma(1, 1)d_\pi$,
- (2) $\chi_\pi(x^{-1}) = \sigma(x^{-1}, x)\sigma(1, 1)\overline{\chi_\pi(x)}$,
- (3) $\chi_\pi(yxy^{-1}) = \overline{\sigma(y, x)}\sigma(yxy^{-1}, y)\chi_\pi(x)$.

Proof. We have that $\pi(1) = \sigma(1, 1)\text{id}_V$. Then,

$$\chi_\pi(1) = \text{Tr}(\pi(1)) = \sigma(1, 1)\text{Tr}(\text{id}_V) = \sigma(1, 1)d_\pi,$$

which proves the first point. Since π is assumed to be unitary we have that

$$\pi(x^{-1}) = \sigma(x^{-1}, x)\sigma(1, 1)\pi(x)^{-1} = \sigma(x^{-1}, x)\sigma(1, 1)\pi(x)^*.$$

Taking trace of the extreme ends of these equalities yields

$$\chi_\pi(x^{-1}) = \sigma(x^{-1}, x)\sigma(1, 1)\overline{\chi_\pi(x)}.$$

Finally, we simply compute that

$$\begin{aligned}
\chi_\pi(yxy^{-1}) &= \text{Tr}(\pi(yxy^{-1})) \\
&= \text{Tr}(\pi(yxy^{-1})\pi(y)\pi(y)^{-1}) \\
&= \sigma(yxy^{-1}, y) \text{Tr}(\pi(yx)\pi(y)^{-1}) \\
&= \sigma(yxy^{-1}, y) \text{Tr}(\pi(yx)\pi(y)^{-1}) \\
&= \overline{\sigma(y, x)}\sigma(yxy^{-1}, y) \text{Tr}(\pi(y)\pi(x)\pi(y)^{-1}) \\
&= \overline{\sigma(y, x)}\sigma(yxy^{-1}, y) \text{Tr}(\pi(x)) \\
&= \overline{\sigma(y, x)}\sigma(yxy^{-1}, y)\chi_\pi(x).
\end{aligned}$$

□

With the third point in the above proposition in mind we make the following definition.

Definition 3.3. Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. A function $f: G \rightarrow \mathbb{C}$ is said to be a σ -class function if

$$f(yxy^{-1}) = \overline{\sigma(y, x)}\sigma(yxy^{-1}, y)f(x)$$

holds for all $x, y \in G$. We will use the notation $\text{Class}^\sigma(G)$ to denote the set of all σ -class functions of G . It is clear that $\text{Class}^\sigma(G)$ has the structure of a complex vector space, and it attains the structure of a Hilbert space with the inner product $\langle -, - \rangle$ defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{x \in G} f_1(x) \overline{f_2(x)}.$$

From Proposition 3.2 we have that $\chi_\pi \in \text{Class}^\sigma(G)$ for any σ -projective representation π of G .

Lemma 3.4. Let G be a finite group, π be an irreducible σ -projective representation of G on V , and $T: V \rightarrow V$ be any linear map. Then the average of T satisfies

$$\tilde{T} = \frac{\text{Tr}(T)}{d_\pi} \text{id}_V.$$

Proof. By Schur's Lemma we have that there exists $\lambda \in \mathbb{C}$ such that

$$\frac{1}{|G|} \sum_{x \in G} \pi(x)^{-1} T \pi(x) = \tilde{T} = \lambda \text{id}_V.$$

Computing the trace of this we see that

$$\text{Tr}(T) = \lambda d_\pi.$$

Rearranging this equation completes the proof. □

Lemma 3.5. Let G be a finite group, and π_1 and π_2 be two isomorphic unitary σ -projective representations of G . Then $\chi_{\pi_1} = \chi_{\pi_2}$.

Proof. Let $T: \pi_1 \rightarrow \pi_2$ be a unitary isomorphism. Then we have that

$$\pi_1(x) = T^* \pi_2(x) T$$

for all $x \in G$. Computing the trace of this yields the desired equality. □

We will later see that the converse to this result is also true.

Theorem 3.6 (Orthogonality relations). Let G be a finite group, and π_1 and π_2 be two irreducible unitary σ -projective representations of G . Then

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = \begin{cases} 1, & \text{if } \pi_1 \simeq \pi_2, \\ 0, & \text{if } \pi_1 \not\simeq \pi_2. \end{cases}$$

Proof. Let $\{e_i\}_{i=1}^{d_{\pi_1}}$ and $\{f_j\}_{j=1}^{d_{\pi_2}}$ be two orthonormal bases for V_1 and V_2 respectively. Let $E_{k,l}: V_1 \rightarrow V_2$ be the linear map defined by

$$E_{l,k}e_i = \delta_{k,i}f_l.$$

Then, if we write the average of this map as a matrix (with respect to the chosen bases above), we get that the number in the (j, i) -position is

$$(\tilde{E}_{l,k})_{j,i} = \frac{1}{|G|} \sum_{x \in G} \sum_{m=1}^{d_{\pi_1}} \sum_{n=1}^{d_{\pi_2}} \pi_2(x)_{j,n}^* (E_{l,k})_{n,m} \pi_1(x)_{m,i} = \frac{1}{|G|} \sum_{x \in G} \pi_2(x)_{j,l}^* \pi_1(x)_{k,i}$$

If $\pi_1 \not\simeq \pi_2$, then we have by Schur's Lemma that this is necessarily zero for all j, i, k , and l . In particular,

$$\begin{aligned} \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle &= \frac{1}{|G|} \sum_{x \in G} \chi_{\pi_1}(x) \overline{\chi_{\pi_2}(x)} \\ &= \frac{1}{|G|} \sum_{x \in G} \sum_{i=1}^{d_{\pi_1}} \sum_{j=1}^{d_{\pi_2}} \pi_1(x)_{i,i} \overline{\pi_2(x)_{j,j}} \\ &= \frac{1}{|G|} \sum_{x \in G} \sum_{i=1}^{d_{\pi_1}} \sum_{j=1}^{d_{\pi_2}} \pi_2(x)_{j,j}^* \pi_1(x)_{i,i} \\ &= 0. \end{aligned}$$

If $\pi_1 \simeq \pi_2$, we have by Lemma 3.5 that $\chi_{\pi_1} = \chi_{\pi_2}$, so we may assume that $\pi_1 = \pi_2$. In this case, we have by Lemma 3.4 that

$$\tilde{E}_{l,k} = \frac{\delta_{l,k}}{d_{\pi_1}} \text{id}_{V_1}.$$

Hence,

$$\frac{1}{|G|} \sum_{x \in G} \pi_2(x)_{j,l}^* \pi_1(x)_{k,i} = (\tilde{E}_{l,k})_{j,i} = \frac{1}{d_{\pi_1}} \delta_{l,k} (\text{id}_{V_1})_{j,i} = \frac{1}{d_{\pi_1}} \delta_{l,k} \delta_{j,i}.$$

Thus,

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = \frac{1}{|G|} \sum_{x \in G} \sum_{i=1}^{d_{\pi_1}} \frac{1}{d_{\pi_1}} = 1. \quad \square$$

Corollary 3.7. *Let G be a finite group, and π_1 and π_2 be two irreducible σ -projective representations. Then $\pi_1 \simeq \pi_2$ if and only if $\chi_{\pi_1} = \chi_{\pi_2}$. Furthermore, there are finitely many irreducible σ -projective representations of G up to isomorphism.*

Proof. The only if implication is Lemma 3.5. Suppose therefore that $\chi_{\pi_1} = \chi_{\pi_2}$, in particular this means that

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 1.$$

From Theorem 3.6 we conclude that $\pi_1 \simeq \pi_2$.

With this in hand we conclude that the number of irreducible σ -projective representations of G up to isomorphism equals the size of the set.

$$\{\chi_{\pi} : \pi \text{ is an irreducible } \sigma\text{-projective representation of } G\}.$$

Since this is an orthonormal set in $\text{Class}^{\sigma}(G)$ by Theorem 3.6, and the dimension of $\text{Class}^{\sigma}(G)$ is less than or equal to $|G|$, we get that there necessarily are only finitely many irreducible σ -projective representations of G up to isomorphism. $\square$

In the next section we determine precisely how many irreducible σ -projective representations of G (up to isomorphism) there are. Through the remainder of this section we prove some immediate consequences of the orthogonality relations and the fact that there are only finitely many irreducible σ -characters. First we present the following result which says how characters interact with direct sums, tensor products, and adjoints.

Proposition 3.8. *Let G be a finite group, π_1 be a σ_1 -projective representation of G , and π_2 be a σ_2 -projective representation of G . Then*

- (1) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$, if $\sigma_1 = \sigma_2$,
- (2) $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$,
- (3) $\chi_{\pi_1^*} = \overline{\chi_{\pi_1}}$.

Proof. This is perhaps easiest to prove by analyzing the eigenvalues of the respective representations. For $x \in G$ let $\{\alpha_1(x), \dots, \alpha_{d_{\pi_1}}(x)\}$ be the eigenvalues of $\pi_1(x)$, and $\{\beta_1(x), \dots, \beta_{d_{\pi_2}}(x)\}$ be the eigenvalues of $\pi_2(x)$. Then $(\pi_1 \oplus \pi_2)(x)$ has eigenvalues

$$\{\alpha_1(x), \dots, \alpha_{d_{\pi_1}}(x), \beta_1(x), \dots, \beta_{d_{\pi_2}}(x)\},$$

which proves that $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$, where this of course only makes sense if $\sigma_1 = \sigma_2$.

The eigenvalues of $(\pi_1 \otimes \pi_2)(x)$ are

$$\{\alpha_i(x)\beta_j(x) : 1 \leq i \leq d_{\pi_1}, 1 \leq j \leq d_{\pi_2}\}.$$

From this it follows that $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$.

For the last statement, let $\{v_i\}$ be an orthonormal basis of V_1 such that v_i is an eigenvector of $\pi_1(x)$ with eigenvalue $\alpha_i(x)$. Let $\{v_i^*\}$ be the dual basis. Then

$$\begin{aligned} (\pi_1^*(x)v_i^*)(v_i) &= v_i^*(\pi_1(x)^{-1}v_i) \\ &= \langle v_i, \pi_1(x)^*v_i \rangle \\ &= \langle \pi_1(x)v_i, v_i \rangle \\ &= \overline{\alpha_i(x)} \langle v_i, v_i \rangle \\ &= \overline{\alpha_i(x)}. \end{aligned}$$

Which shows that $\{\overline{\alpha_1(x)}, \dots, \overline{\alpha_{d_{\pi_1}}(x)}\}$ are the eigenvalues of $\pi_1^*(x)$. This allows us to conclude that $\chi_{\pi_1^*} = \overline{\chi_{\pi_1}}$, completing the proof. $\square$

Corollary 3.9. *Let G be a finite group. If π_1 and π_2 are two σ -projective representations of G , then*

$$\dim \text{Hom}_G(\pi_1, \pi_2) = \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle.$$

Proof. In the case where π_1 and π_2 are irreducible, this follows by the orthogonality relations and Schur's Lemma. If π_1 and π_2 are not irreducible, then Maschke's Theorem in combination with Schur's Lemma and Proposition 3.8 allows us to reduce to the irreducible case. $\square$

Corollary 3.10. *Let G be a finite group and $\{\chi_1, \dots, \chi_n\}$ be the irreducible σ -projective characters of G with associated irreducible σ -projective representations $\{\pi_1, \dots, \pi_n\}$. If π is a σ -projective representation of G , then the number of subrepresentations of π isomorphic to π_i is given by*

$$\langle \chi_\pi, \chi_i \rangle.$$

Furthermore, we have that π is irreducible if and only if

$$\langle \chi_\pi, \chi_\pi \rangle = 1.$$

Proof. By Maschke's Theorem, Lemma 3.5, and Proposition 3.8 we have that

$$\chi_\pi = \sum_{j=1}^n m_j \chi_j,$$

where m_i is the number of subrepresentations of π isomorphic to π_i . Using the orthogonality relations we then have that

$$\langle \chi_\pi, \chi_i \rangle = \sum_{j=1}^n m_j \langle \chi_j, \chi_i \rangle = m_i.$$

Furthermore, using this computation we see that

$$\langle \chi_\pi, \chi_\pi \rangle = \sum_{i=1}^n m_i^2.$$

This is equal to 1 if and only if $m_i = 1$ for precisely one of the indices i , and $m_j = 0$ for all $j \neq i$. In which case we see that π is necessarily isomorphic to π_i . $\square$

Corollary 3.11. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. Then for any irreducible σ -projective representation π of G we have that*

$$\langle \chi_{\lambda_\sigma}, \chi_\pi \rangle = d_\pi.$$

In particular,

$$|G| = \sum d_\pi^2,$$

where the sum ranges over the isomorphism classes of irreducible σ -projective representations of G .

Proof. From the definition of the left regular σ -projective representation of G , we have that

$$\chi_{\lambda_\sigma}(x) = \text{Tr}(\lambda_\sigma(x)) = \begin{cases} \sigma(1, 1)d_\sigma, & \text{if } x = 1, \\ 0, & \text{if } x \neq 1. \end{cases}$$

Furthermore, $d_{\lambda_\sigma} = |G|$. Hence,

$$\langle \chi_{\lambda_\sigma}, \chi_\pi \rangle = \sigma(1, 1) \overline{\chi_\pi(1)} = \sigma(1, 1) \overline{\sigma(1, 1)} d_\pi = d_\pi.$$

Using Maschke's Theorem along with Corollary 3.10 we get that

$$|G| = \langle \chi_{\lambda_\sigma}, \chi_{\lambda_\sigma} \rangle = \sum d_\pi^2. \quad \square$$

Corollary 3.12. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{C}^\times)$. Then $|\sigma|^2$ divides $|G|$.*

Proof. By Theorem 2.7 and Corollary 3.11 we have that

$$|G| = \sum d_\pi^2 = \sum |\sigma|^2 k_\pi^2 = |\sigma|^2 \sum k_\pi^2$$

for some $k_\pi \geq 1$. $\square$

4. THE NUMBER OF IRREDUCIBLE PROJECTIVE REPRESENTATIONS

In the linear setting it is well-known that the number of irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes in G . A similar result holds true for σ -projective representations, except we will see that not every conjugacy class of G corresponds to an irreducible σ -projective representation of G .

Definition 4.1. Let G be a finite group, $\sigma \in Z^2(G; \mathbb{C}^\times)$, and $x \in G$. We say that x is σ -regular if

$$\sigma(x, y) = \sigma(y, x)$$

for all $y \in C_G(x) = \{y \in G : xy = yx\}$.

Lemma 4.2. *Let G be a finite group, $\sigma \in Z^2(G; \mathbb{C}^\times)$, and $x \in G$. Then x is σ -regular if and only if x is σ' -regular for every σ' cohomologous to σ .*

Proof. If σ and σ' are cohomologous, then there exists a function $\mu: G \rightarrow \mathbb{C}^\times$ such that $\sigma' = (\delta\mu)\sigma$. For any $x \in G$ and $y \in C_G(x)$ we have that

$$(\delta\mu)(x, y) = \frac{\mu(x)\mu(y)}{\mu(xy)} = \frac{\mu(x)\mu(y)}{\mu(yx)} = (\delta\mu)(y, x).$$

Hence, x is σ -regular if and only if x is σ' -regular. $\square$

Lemma 4.3. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{C}^\times)$. Then $x \in G$ is σ -regular if and only if y is σ -regular for all $y \in C_x = \{zxz^{-1} : z \in G\}$.*

Proof. Note that $x \in C_x$, so if every element of C_x is σ -regular, then in particular x is σ -regular. To prove the converse, suppose that x is σ -regular, and $z \in G$. By the Lemma 4.2 we may assume that σ is normalized. Note that $C_G(zxz^{-1}) = zC_G(x)z^{-1}$, we therefore want to show that

$$\sigma(zxz^{-1}, zyz^{-1}) = \sigma(zyz^{-1}, zxz^{-1})$$

for all $y \in C_G(x)$. We have that $xy = yx$ and $\sigma(x, y) = \sigma(y, x)$. Multiplying the following identities:

$$\begin{aligned} (\delta\sigma)(zx, z^{-1}, zyz^{-1}) &= 1 \\ (\delta\sigma)(z, x, yz^{-1}) &= 1 \\ (\delta\sigma)(zy, z^{-1}, zxz^{-1}) &= 1 \\ (\delta\sigma)(z, y, xz^{-1}) &= 1 \\ (\delta\sigma)(y, x, z^{-1}) &= 1 \end{aligned}$$

shows that

$$\frac{\sigma(zxz^{-1}, zyz^{-1})}{\sigma(zyz^{-1}, zxz^{-1})} = 1,$$

completing the proof. $\square$

With this result we say that a conjugacy class in G is σ -regular if any element in the conjugacy class is σ -regular.

Proposition 4.4. *Let G be a finite group, and $\sigma \in Z^2(G; \mathbb{T})$. Suppose that $g \in G$ is σ -regular and define $c_g: G \rightarrow \mathbb{C}$ by*

$$c_g(x) = \begin{cases} \overline{\sigma(z, g)}\sigma(zgz^{-1}, z), & \text{if } x = zgz^{-1} \in C_g, \\ 0, & \text{if } x \in G \setminus C_g. \end{cases}$$

Then $c_g \in \text{Class}^\sigma(G)$. Furthermore, for any $f \in \text{Class}^\sigma(G)$ we have that

$$f|_{C_g} = f(g)c_g.$$

Proof. Notice first that c_g is well-defined: if $zgz^{-1} = ygy^{-1}$ then we have that $gy^{-1}z = y^{-1}zg$. Along with the fact that g is σ -regular and the identities

$$\begin{aligned} (\delta\sigma)(y^{-1}, zgz^{-1}, z) &= 1 \\ (\delta\sigma)(y^{-1}, ygy^{-1}, y) &= 1 \\ (\delta\sigma)(g, y^{-1}, z) &= 1 \\ (\delta\sigma)(y^{-1}, z, g) &= 1 \\ (\delta\sigma)(gy^{-1}, y, g) &= 1 \\ (\delta\sigma)(g, y^{-1}, yg) &= 1 \end{aligned}$$

We get that

$$\overline{\sigma(z, g)}\sigma(zgz^{-1}, z) = \overline{\sigma(y, g)}\sigma(ygy^{-1}, y),$$

hence c_g is well-defined.

To see that $c_g \in \text{Class}^\sigma(G)$ we want to show that

$$c_g(yxy^{-1}) = \overline{\sigma(y, x)}\sigma(yxy^{-1}, y)c_g(x)$$

for every $x, y \in G$. Note that x and yxy^{-1} are necessarily in the same conjugacy class, hence

$$c_g(yxy^{-1}) = 0 = \overline{\sigma(y, x)}\sigma(yxy^{-1}, y)c_g(x)$$

if $x \in G \setminus C_g$. Now suppose that $x \in C_g$, and let $z \in G$ be such that $x = zgz^{-1}$. Then

$$\begin{aligned} c_g(yxy^{-1}) &= \overline{\sigma(yz, g)}\sigma(yzg(yz)^{-1}, yz), \\ c_g(x) &= \overline{\sigma(z, g)}\sigma(zgz^{-1}, z). \end{aligned}$$

Hence, we want to show that

$$\overline{\sigma(yz, g)}\sigma(yzg(yz)^{-1}, yz) = \overline{\sigma(y, zgz^{-1})}\sigma(yzg(yz)^{-1}, y)\overline{\sigma(z, g)}\sigma(zgz^{-1}, z).$$

This follows by the following identities:

$$\begin{aligned} (\delta\sigma)(yzg(yz)^{-1}, y, z) &= 1 \\ (\delta\sigma)(y, zgz^{-1}, z) &= 1 \\ (\delta\sigma)(y, z, g) &= 1, \end{aligned}$$

hence we conclude that $c_g \in \text{Class}^\sigma(G)$.

Finally, if $f \in \text{Class}^\sigma(G)$ and $z \in G$, then

$$f(zgz^{-1}) = \overline{\sigma(z, g)}\sigma(zgz^{-1}, z)f(g) = f(g)c_g(zgz^{-1}). \quad \square$$

Corollary 4.5. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. Then the dimension of $\text{Class}^\sigma(G)$ equals the number of σ -regular conjugacy classes in G .*

Proof. Let $f \in \text{Class}^\sigma(G)$, and let $\{C_i\}_{i=1}^n$ denote all the conjugacy classes in G . Choose a representative $g_i \in C_i$ of the i -th conjugacy class. If g_i is σ -regular, we have by Proposition 4.4 that

$$f|_{C_i} = f(g_i)c_{g_i}.$$

If g_j is not σ -regular, then there exists an element $z \in C_G(g_j)$ such that

$$\sigma(z, g_j) \neq \sigma(g_j, z).$$

Since $f \in \text{Class}^\sigma(G)$ we get that

$$f(g_j) = f(zg_jz^{-1}) = \overline{\sigma(z, g_j)}\sigma(zg_jz^{-1}, z)f(g_j) = \overline{\sigma(z, g_j)}\sigma(g_j, z)f(g_j),$$

which can only be true if $f(g_j) = 0$. From this it follows that $f|_{C_j} = 0$, which completes the proof. $\square$

Proposition 4.6. *Let G be a finite group, $\sigma \in Z^2(G; \mathbb{T})$, $f \in \text{Class}^\sigma(G)$, and π be a σ -projective representation of G on V . Define $T_f: V \rightarrow V$ by*

$$T_f = \sum_{x \in G} f(x)\pi(x),$$

then T_f is an intertwiner. Furthermore, if π is irreducible, then

$$T_f = \frac{|G|}{d_\pi} \langle \chi_\pi, f \rangle \text{id}_V.$$

Proof. Let $z \in G$, then

$$\begin{aligned}
\pi(z)T_f &= \sum_{x \in G} \overline{f(x)}\pi(z)\pi(x) \\
&= \sum_{x \in G} \overline{f(x)}\sigma(z, x)\pi(zx) \\
&= \sum_{x \in G} \overline{f(x)}\sigma(z, x)\pi(zxz^{-1}z) \\
&= \sum_{x \in G} \overline{\sigma(z, x)\sigma(zxz^{-1}, z)}\overline{f(zxz^{-1})}\sigma(z, x)\overline{\sigma(zxz^{-1}, z)}\pi(zxz^{-1})\pi(z) \\
&= \sum_{x \in G} \overline{f(zxz^{-1})}\pi(zxz^{-1})\pi(z) \\
&= T_f\pi(z).
\end{aligned}$$

Hence, T_f is an intertwiner. If π is irreducible we have by Schur's Lemma that $T_f = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}$. Computing the trace of both of these sides yields the desired equality. $\square$

Theorem 4.7. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. Then*

$$\{\chi_\pi : \pi \text{ is an irreducible } \sigma\text{-projective representation of } G\}$$

is an orthonormal basis for $\text{Class}^\sigma(G)$.

Proof. Assume, to get a contradiction, that this is not the case. Then there exists a nonzero function $f \in \text{Class}^\sigma(G)$ such that $\langle \chi_\pi, f \rangle = 0$ for all irreducible σ -projective representations π of G . By Proposition 4.6, the intertwiner T_f is the zero-intertwiner for all irreducible σ -projective representations. By Maschke's Theorem, and Schur's Lemma, T_f is the zero-intertwiner for all σ -projective representations. In particular, if λ_σ is the left regular σ -projective representation of G , then $T_f: \ell^2(G) \rightarrow \ell^2(G)$ satisfies

$$T_f\delta_1 = \sum_{x \in G} f(x)\sigma(x, 1)\delta_x.$$

This is the zero vector if and only if $f(x) = 0$ for all x , a contradiction. $\square$

Corollary 4.8. *Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. The number of irreducible σ -projective representations of G up to isomorphism equals the number of σ -regular conjugacy classes in G .*

Proof. By Theorem 4.7 we have that the number of irreducible σ -projective representations of G up to isomorphism equals the dimension of $\text{Class}^\sigma(G)$. By Corollary 4.5 we have that the dimension of $\text{Class}^\sigma(G)$ equals the number of σ -regular conjugacy classes in G . $\square$

Definition 4.9. Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. Denote the irreducible σ -characters of G by $\{\chi_1, \dots, \chi_n\}$. The σ -projective representation group of G is the lattice $R^\sigma(G) := \mathbb{Z}\{\chi_1, \dots, \chi_n\}$. In the special case where $\sigma = 1$, the trivial 2-cocycle, we simply use the notation $R(G)$, and $R(G)$ is in this case called the *group ring* of G .

Proposition 4.10. *Let G be a finite group, $\sigma \in Z^2(G; \mathbb{T})$, and $\{\chi_1, \dots, \chi_n\}$ be the irreducible σ -projective characters of G . Then;*

- (1) *for any σ -projective representation π of G we have that $\chi_\pi \in R^\sigma(G)$,*
- (2) *any element $\chi \in R^\sigma(G)$ of the form*

$$\chi = \sum_{j=1}^n m_j \chi_j$$

with $m_j \geq 0$ is the character of a σ -projective representation of G ,

- (3) $\mathbb{C} \otimes R^\sigma(G) \simeq \text{Class}^\sigma(G)$,
- (4) *the representation ring $R(G)$ is a commutative ring with multiplication given by point-wise multiplication, and*
- (5) *$R^\sigma(G)$ is a module over $R(G)$ where $R(G)$ acts on $R^\sigma(G)$ by point-wise multiplication.*

Proof. Let $\{\pi_1, \dots, \pi_n\}$ be irreducible σ -projective representations of G such that $\chi_{\pi_i} = \chi_i$. By Maschke's Theorem, Lemma 3.5, and Proposition 3.8 we have that

$$\chi_\pi = \sum_{j=1}^n m_j \chi_j,$$

where each m_j is the number of subrepresentations of π isomorphic to π_j . Since m_j is necessarily an integer we have that $\chi_\pi \in R^\sigma(G)$, proving the first point. Conversely, if

$$\chi = \sum_{j=1}^n m_j \chi_j$$

then the σ -projective representation

$$\pi = \bigoplus_{j=1}^n m_j \pi_j$$

will by Proposition 3.8 satisfy $\chi_\pi = \chi$, which proves the second point.

The third point is just a restatement of Theorem 4.7.

By linearity of the point-wise product, it is enough to see that $\chi\chi' \in R(G)$ for two irreducible characters χ and χ' of G . If ρ and ρ' are two irreducible representations of G with characters χ and χ' , then we have by Proposition 3.8 that

$$\chi\chi' = \chi_{\rho \otimes \rho'}.$$

By the first point we therefore have that $\chi\chi' \in R(G)$.

To prove the final point it is enough to check that $\chi\chi' \in R^\sigma(G)$ where $\chi \in R(G)$ is an irreducible character, and $\chi' \in R^\sigma(G)$ is an irreducible σ -character. This follows by Proposition 3.8 in the same way as it did for the previous point. $\square$

5. THE TWISTED GROUP ALGEBRA AND (CO)INDUCTION

So far we have explicitly worked with the category $\mathbf{Rep}^\sigma(G)$ as σ -projective representations of G . We can however recognize this category as a module category of a certain ring.

Definition 5.1. Let G be a finite group and $\sigma \in Z^2(G; \mathbb{T})$. We define the σ -twisted group algebra of G to be the complex vector space

$$\mathbb{C}[G]^\sigma := \{f: G \rightarrow \mathbb{C} : f \text{ is a function}\}.$$

This becomes an algebra with the σ -twisted convolution product defined by

$$(f_1 *_\sigma f_2)(x) = \sum_{y \in G} \sigma(y, y^{-1}x) f_1(y) f_2(y^{-1}x).$$

The σ -twisted group algebra of G has a distinguished basis given by $\{\delta_g\}_{g \in G}$, where

$$\delta_g(x) = \begin{cases} 1, & x = g, \\ 0, & x \neq g. \end{cases}$$

It is clear that we may identify $\mathbf{Rep}^\sigma(G)$ with $\mathbf{Mod}(\mathbb{C}[G]^\sigma)$, where the latter is the category of (left) $\mathbb{C}[G]^\sigma$ -modules.

Remark 5.2. Left multiplication with the basis elements $\{\delta_g\}_{g \in G}$ in $\mathbb{C}[G]^\sigma$ induces a σ -projective representation of G , namely the left regular σ -projective representation of G . This follows because we may identify $\mathbb{C}[G]^\sigma$ with $\ell^2(G)$ since G is finite.

Furthermore, right multiplication also defines a *right* projective representation of G on $\mathbb{C}[G]^\sigma$. We may restrict both this left projective representation and this right projective representation to any subgroup H of G . This observation will be necessary to make sense of induction and coinduction defined below.

The identification of $\mathbf{Rep}^\sigma(G)$ with $\mathbf{Mod}(\mathbb{C}[G]^\sigma)$ allows us to use important ring-theoretic tools, in particular those of *induction* and *coinduction* which we introduce now.

Definition 5.3. Let G be a finite group and H be a subgroup of G . If $\sigma \in Z^2(G; \mathbb{T})$ and π is a σ -projective representation of G , then we get a 2-cocycle $\text{Res}_H^G \sigma$ of H by restricting to H , and we furthermore get a projective representation $\text{Res}_H^G \pi$ of H by restricting π from G to H . This defines a functor $\text{Res}_H^G: \mathbf{Rep}^\sigma(G) \rightarrow \mathbf{Rep}^{\text{Res}_H^G \sigma}(H)$, called the *restriction functor*. We define two functors $\text{Ind}_H^G, \text{Coind}_H^G: \mathbf{Rep}^{\text{Res}_H^G \sigma}(H) \rightarrow \mathbf{Rep}^\sigma(G)$ as follows: If θ is a $\text{Res}_H^G \sigma$ -projective representation of G on V we define

$$\begin{aligned} \text{Ind}_H^G(V) &:= \mathbb{C}[G]^\sigma \otimes_H V, & (\text{Ind}_H^G(\theta)(x))(f \otimes v) &:= (\delta_x *_\sigma f) \otimes v, \\ \text{Coind}_H^G(V) &:= \text{Hom}_H(\mathbb{C}[G]^\sigma, V), & (\text{Coind}_H^G(\theta)(x))(F)(f) &:= F(f *_\sigma \delta_x). \end{aligned}$$

Then the pairs $(\text{Ind}_H^G(\theta), \text{Ind}_H^G(V))$ and $(\text{Coind}_H^G(\theta), \text{Coind}_H^G(V))$ define σ -projective representations of G . Finally, if T is an intertwiner between two $\text{Res}_H^G \sigma$ -projective representations of H , then

$$\begin{aligned} \text{Ind}_H^G(T) &:= \text{id}_{\mathbb{C}[G]^\sigma} \otimes T, \\ \text{Coind}_H^G(T) &:= T_*, \end{aligned}$$

where $T_*(F) = T \circ F$, define intertwiners between the induced σ -projective representations of G , and the coinduced σ -projective representations of G respectively. This data defines the functors Ind_H^G and Coind_H^G .

If χ is the character of the $\text{Res}_H^G \sigma$ -projective representation θ of H we will use the notation $\text{Ind}_H^G(\chi)$ and $\text{Coind}_H^G(\chi)$ to denote the characters of the induced and coinduced representations of θ respectively.

These constructions are natural in the following sense.

Proposition 5.4 (Frobenius reciprocity). *Let G be a finite group and H be a subgroup of G . Then $(\text{Ind}_H^G, \text{Res}_H^G)$ and $(\text{Res}_H^G, \text{Coind}_H^G)$ form adjoint pairs. In other words, for any σ -projective representation π of G on V_π , and any $\text{Res}_H^G \sigma$ -projective representation θ of H on V_θ , we have natural linear isomorphisms*

$$\begin{aligned} \text{Hom}_G(\text{Ind}_H^G(\theta), \pi) &\simeq \text{Hom}_H(\theta, \text{Res}_H^G(\pi)), \\ \text{Hom}_H(\text{Res}_H^G(\pi), \theta) &\simeq \text{Hom}_G(\pi, \text{Coind}_H^G(\theta)). \end{aligned}$$

Proof. We will prove that $(\text{Ind}_H^G, \text{Res}_H^G)$ is an adjoint pair, the proof that $(\text{Res}_H^G, \text{Coind}_H^G)$ is an adjoint pair is similar, so we omit the details.

We want to define a linear map

$$\text{Hom}_G(\text{Ind}_H^G(\theta), \pi) \rightarrow \text{Hom}_H(\theta, \text{Res}_H^G(\pi)).$$

Given a G -intertwiner $T: \text{Ind}_H^G(V_\theta) \rightarrow V_\pi$ define $T': V_\theta \rightarrow V_\pi$ by

$$T'(v) = T(\delta_1 \otimes v).$$

Then T' is an H -intertwiner by the following computation: Suppose that $x \in H$, then

$$\begin{aligned}
\pi(x)T'(v) &= \pi(x)T(\delta_1 \otimes v) \\
&= T(\delta_x *_{\sigma} \delta_1 \otimes v) \\
&= \sigma(x, 1)T(\delta_x \otimes v) \\
&= \frac{\sigma(x, 1)}{\sigma(1, x)}T(\delta_1 *_{\sigma} \delta_x \otimes v) \\
&= T(\delta_1 *_{\sigma} \delta_x \otimes v) && \text{since } \sigma(x, 1) = \sigma(1, x), \\
&= T(\delta_1 \otimes \theta(x)v) && \text{by definition of } \otimes_H, \\
&= T'(\theta(x)v).
\end{aligned}$$

It is clear that the assignment $T \mapsto T'$ is linear.

It remains to define a linear map

$$\text{Hom}_H(\theta, \text{Res}_H^G(\pi)) \rightarrow \text{Hom}_G(\text{Ind}_H^G(\theta), \pi)$$

which is inverse to the assignment $T \mapsto T'$. With that in mind, given an H -intertwiner $F: V_{\theta} \rightarrow V_{\pi}$, define $F': \text{Ind}_H^G(V_{\theta}) \rightarrow V_{\pi}$ by

$$F'(\delta_g \otimes v) = \pi(g)F(v).$$

If F' is a G -intertwiner, we have that (up to a nonzero scalar) the assignment $F \mapsto F'$ is inverse to $T \mapsto T'$, completing the proof. We see that F' is indeed a G -intertwiner by the following computation: Let $y \in G$, then

$$\begin{aligned}
\pi(y)F'(\delta_g \otimes v) &= \pi(y)\pi(g)F(v) \\
&= \sigma(y, g)\pi(yg)F(v) \\
&= F'(\sigma(y, g)\delta_{yg} \otimes v) \\
&= F'(\delta_y *_{\sigma} \delta_g \otimes v) \\
&= F'((\text{Ind}_H^G(\theta)(y))(\delta_g \otimes v)). \quad \square
\end{aligned}$$

To distinguish between the inner products of $\text{Class}^{\sigma}(G)$ and $\text{Class}^{\text{Res}_H^G \sigma}(H)$ we will use the notation $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle_H$ respectively. Then Frobenius reciprocity may be stated in terms of characters as follows.

Corollary 5.5. *Let G be a finite group and H be a subgroup of G . If π is a σ -projective representation of G and θ is a $\text{Res}_H^G \sigma$ -projective representation of H , then*

$$\begin{aligned}
\langle \text{Ind}_H^G(\chi_{\theta}), \chi_{\pi} \rangle_G &= \langle \chi_{\theta}, \chi_{\pi} \rangle_H, \\
\langle \chi_{\pi}, \chi_{\theta} \rangle_H &= \langle \chi_{\pi}, \text{Coind}_H^G(\chi_{\theta}) \rangle_G.
\end{aligned}$$

Proof. This follows immediately by Corollary 3.9. $\square$

Note that Frobenius reciprocity is not unique to finite groups G , however, what is unique for finite groups is the following result.

Proposition 5.6. *Let G be a finite group and H be a subgroup of G . Suppose that $\sigma \in Z^2(G; \mathbb{T})$. Then for any $\text{Res}_H^G \sigma$ -projective representation θ of H on V we have that $\text{Ind}_H^G(\theta) \simeq \text{Coind}_H^G(\theta)$ as σ -projective representations.*

Proof. Let $S \subset G$ be a set of coset representatives of G/H . Then any element in $\text{Ind}_H^G(V)$ can uniquely be written as a sum

$$\sum_{s \in S} \delta_s \otimes v_s,$$

where each $v_s \in V$. Define a linear map $T: \text{Ind}_H^G(V) \rightarrow \text{Coind}_H^G(V)$ by

$$T\left(\sum_{s \in S} \delta_s \otimes v_s\right) = \sum_{s \in S} F_{v_s},$$

where $F_{v_s} \in \text{Coind}_H^G(V)$ is the H -intertwiner defined by

$$F_{v_s}(\delta_x) = \begin{cases} \sigma(x, s)\theta(xs)v_s, & xs \in H, \\ 0, & \text{otherwise.} \end{cases}$$

This is indeed an H -intertwiner: If $h \in H$, then $hxs \in H$ if and only if $xs \in H$, it follows that

$$F_{v_s}(\delta_h * \sigma \delta_x) = \theta(h)F_{v_s}(\delta_x)$$

for every $x \in G$.

To see that T is a G -intertwiner, we fix $x, y \in G$ and $s \in S$. We want to show that

$$T((\text{Ind}_H^G(\theta)(x))(\delta_s \otimes v_s))(\delta_y) = (\text{Coind}_H^G(\theta)(x))T(\delta_s \otimes v_s)(\delta_y).$$

Note that there exists a unique $s_x \in S$ and a unique $h_x \in H$ such that $xs = s_x h_x$. Suppose that $yx s \in H$ and note that this assumption is equivalent to $ys_x \in H$. We then compute that

$$\begin{aligned} T((\text{Ind}_H^G(\theta)(x))(\delta_s \otimes v_s))(\delta_y) &= T((\delta_x * \sigma \delta_s) \otimes v_s)(\delta_y) \\ &= \sigma(x, s)T(\delta_{xs} \otimes v_s)(\delta_y) \\ &= \sigma(x, s)T(\delta_{s_x h_x} \otimes v_s)(\delta_y) \\ &= \frac{\sigma(x, s)}{\sigma(s_x, h_x)}T(\delta_{s_x} \otimes (\theta(h_x)v_s))(\delta_y) \\ &= \frac{\sigma(x, s)\sigma(y, s_x)}{\sigma(s_x, h_x)}\theta(ys_x)(\theta(h_x)v_s) \\ &= \frac{\sigma(x, s)\sigma(y, s_x)\sigma(ys_x, h_x)}{\sigma(s_x, h_x)}\theta(yxs)v_s \\ &= \sigma(x, s)\sigma(y, xs)\theta(yxs)v_s \\ &= \sigma(y, x)\sigma(yx, s)\theta(yxs)v_s \\ &= \sigma(y, x)T(\delta_s \otimes v_s)(\delta_{yx}) \\ &= T(\delta_s \otimes v_s)(\delta_y * \sigma \delta_x) \\ &= (\text{Coind}_H^G(\theta)(x))T(\delta_s \otimes v_s)(\delta_y). \end{aligned}$$

If $yx s \notin H$, then we also have that $ys_x \notin H$. In this case

$$T((\text{Ind}_H^G(\theta)(x))(\delta_s \otimes v_s))(\delta_y) = 0 = (\text{Coind}_H^G(\theta)(x))T(\delta_s \otimes v_s)(\delta_y).$$

Since $x, y \in G$, $s \in S$, and $v_s \in V$ where arbitrary, this shows that T is indeed a G -intertwiner. To see that T is an isomorphism, note that the map defined by

$$F \mapsto \sum_{s \in S} \frac{1}{\sigma(s^{-1}, s)\sigma(1, 1)} \delta_s \otimes F(\delta_{s^{-1}})$$

is the inverse map to T . □

With this we will specialize to only talking about induction. Before we start talking about Clifford theory we will prove a final result on the transitivity of induction.

Proposition 5.7. *Let G be a finite group and $H \subset L \subset G$ be nested subgroups of G . Suppose that $\sigma \in Z^2(G; \mathbb{T})$ and that θ is a $\text{Res}_H^G \sigma$ -projective representation of H on V . Then*

$$\text{Ind}_L^G(\text{Ind}_H^L(\theta)) \simeq \text{Ind}_H^G(\theta).$$

Proof. Define $T: \text{Ind}_L^G(\text{Ind}_H^L(V)) \rightarrow \text{Ind}_H^G(V)$ by

$$T(\delta_g \otimes \delta_l \otimes v) = \sigma(g, l) \delta_{gl} \otimes v,$$

where $g \in G$, $l \in L$, and $v \in V$. By the properties of $\otimes_L$ we have that

$$\begin{aligned} \delta_g \otimes \delta_l \otimes v &= \frac{1}{\sigma(l, 1)} \delta_g \otimes \delta_l *_{\sigma} \delta_1 \otimes v \\ &= \frac{1}{\sigma(l, 1)} \delta_g *_{\sigma} \delta_l \otimes \delta_1 \otimes v \\ &= \frac{\sigma(g, l)}{\sigma(l, 1)} \delta_{gl} \otimes \delta_1 \otimes v, \end{aligned}$$

thus we conclude that T is an isomorphism because an inverse map is determined by

$$\delta_g \otimes v \mapsto \frac{1}{\sigma(g, 1)} \delta_g \otimes \delta_1 \otimes v.$$

Finally, T is a G -intertwiner: Let $x \in G$, then

$$\begin{aligned} T(\delta_x *_{\sigma} \delta_g \otimes \delta_l \otimes v) &= \sigma(x, g) T(\delta_{xg} \otimes \delta_l \otimes v) \\ &= \sigma(x, g) \sigma(xg, l) \delta_{xgl} \otimes v \\ &= \sigma(x, gl) \sigma(g, l) \delta_{xgl} \otimes v \\ &= \sigma(g, l) \delta_x *_{\sigma} \delta_{gl} \otimes v. \end{aligned}$$

□

6. CLIFFORD THEORY

Having introduced induction and coinduction, we want to understand the irreducible projective representations of a group G , by understanding the irreducible projective representations of subgroups of G . We attain the most powerful results if we only consider normal subgroups of G . Before we continue we need to introduce some notation.

Definition 6.1. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is a $\text{Res}_N^G \sigma$ -projective representation of N on V , then for $g \in G$ we define the projective representation $\theta^g: N \rightarrow U(V)$ by

$$\theta^g(x) = \sigma(g^{-1}, x) \overline{\sigma(g^{-1}xg, g^{-1})} \theta(g^{-1}xg).$$

We furthermore define the set

$$I_G(\theta) = \{g \in G : \theta^g \simeq \theta\}.$$

This is called the *inertia subgroup of G associated to θ* .

Proposition 6.2. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is a $\text{Res}_N^G \sigma$ -projective representation of N on V , then θ^g is a $\text{Res}_N^G \sigma$ -projective representation of N for every $g \in G$. Furthermore, the inertia subgroup of G , $I_G(\theta)$, is a subgroup of G .

Proof. That θ^g is $\text{Res}_N^G \sigma$ -projective follows by standard 2-cocycle manipulations. To see that $I_G(\theta)$ is a subgroup of G we need to check that $1 \in I_G(\theta)$, and that $I_G(\theta)$ is closed under the group operation and inverse operation of G . It is clear that $1 \in I_G(\theta)$, as $\theta^1 = \theta$. For the other statements, the crucial observation is to notice that if $\theta_1 \simeq \theta_2$, are two isomorphic $\text{Res}_N^G \sigma$ -projective representations of N , then

$$(\theta_1)^g \simeq (\theta_2)^g$$

for any $g \in G$. This is indeed the case, if $T \in \text{Hom}_N(\theta_1, \theta_2)$ is an isomorphism, then $T \in \text{Hom}_N((\theta_1)^g, (\theta_2)^g)$, and since T was a linear isomorphism to begin with, this shows that $(\theta_1)^g \simeq (\theta_2)^g$.

Then we get that $I_G(\theta)$ is closed under the group operation of G as follows: Let $g_1, g_2 \in I_G(\theta)$. Then,

$$\theta^{g_1 g_2} = (\theta^{g_1})^{g_2} \simeq \theta^{g_2} \simeq \theta.$$

To see that $I_G(\theta)$ is closed under the inverse operation of G , suppose that $g \in I_G(\theta)$. Then

$$\theta = \theta^1 = \theta^{g g^{-1}} = (\theta^g)^{g^{-1}} \simeq \theta^{g^{-1}}. \quad \square$$

Lemma 6.3. *Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is a $\text{Res}_N^G \sigma$ -projective representation of N on V , then N is a subgroup of $I_G(\theta)$.*

Proof. Let $n \in N$. We claim that $\theta(n^{-1}): V \rightarrow V$ is an intertwiner from θ^n to θ . Indeed, for any $x \in N$ we have that

$$\begin{aligned} \theta^n(x)\theta(n^{-1}) &= \sigma(n^{-1}, x)\overline{\sigma(n^{-1}xn, n^{-1})}\theta(n^{-1}xn)\theta(n^{-1}) \\ &= \sigma(n^{-1}, x)\theta(n^{-1}x) \\ &= \theta(n^{-1})\theta(x). \end{aligned} \quad \square$$

In particular, Lemma 6.3 tells us that there is a well-defined action of the quotient group G/N on the category $\mathbf{Rep}^{\text{Res}_N^G \sigma}(N)$.

Note that the following result holds in a larger generality; it is not necessary to assume that N is a normal subgroup. For our purposes we only need the result in the stated case, this setup furthermore avoids some cluttered notation.

Theorem 6.4 (Mackey's formula). *Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If $S \subset G$ is a choice of representatives of the (N, N) -double cosets of G , then we have that*

$$\text{Res}_N^G(\text{Ind}_N^G(\theta)) \simeq \bigoplus_{s \in S} \theta^s$$

for any $\text{Res}_N^G \sigma$ -projective representation θ of N .

Proof. Let θ be a $\text{Res}_N^G \sigma$ -projective representation of N on V . Define the following subspace of $\text{Ind}_N^G(V)$:

$$V(s) := \text{span}\{(\text{Ind}_N^G(\theta)(x))(\delta_1 \otimes v) : v \in V, x \in NsN\}.$$

Since

$$G = \bigcup_{s \in S} NsN$$

is a partition of G , we have that each $V(s)$ only intersect at the origin, and that

$$\text{Ind}_N^G(V) = \bigoplus_{s \in S} V(s).$$

Since N is a normal subgroup of G , we furthermore have that

$$V(s) = \text{span}\{\delta_s \otimes v : v \in V\},$$

and it is clear from the definition of $V(s)$ that this is an N -invariant subspace of $\text{Ind}_N^G(V)$. The linear map $T: V \rightarrow V(s)$ defined by $T(v) = \delta_s \otimes v$ defines an intertwiner from $\theta^s \rightarrow \text{Res}_N^G(\text{Ind}_N^G(\theta)|_{V(s)})$: Notice that every element in N can uniquely be written in the form

sxs^{-1} , then

$$\begin{aligned}
T(\theta^s(sxs^{-1})v) &= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}T(\theta(x)v) \\
&= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}(\delta_s \otimes \theta(x)v) \\
&= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}(\delta_s *_\sigma \delta_x \otimes v) \\
&= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}\sigma(s, x)(\delta_{sx} \otimes v) \\
&= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}\sigma(s, x)(\delta_{sxs^{-1}s} \otimes v) \\
&= \sigma(s^{-1}, sxs^{-1})\overline{\sigma(x, s^{-1})}\sigma(s, x)\overline{\sigma(sxs^{-1}, s)}(\delta_{sxs^{-1}} *_\sigma \delta_s \otimes v) \\
&= (\delta_{sxs^{-1}} *_\sigma \delta_s \otimes v) \\
&= (\text{Ind}_N^G(\theta)(sxs^{-1}))(\delta_s \otimes v) \\
&= (\text{Ind}_N^G(\theta)(sxs^{-1}))(T(v)).
\end{aligned}$$

It is furthermore clear that T is a linear isomorphism. Taking the direct sum over all $s \in S$ yields the desired result. $\square$

Remark 6.5. By Lemma 6.3 we have that Mackey's formula is independent of the choice of representatives of the (N, N) -double cosets.

Lemma 6.6. *Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is an irreducible $\text{Res}_N^G \sigma$ -projective representation of N , and $g \in G$, then θ^g is irreducible.*

Proof. By Corollary 3.10, we have that θ^g is irreducible if and only if

$$\langle \chi_{\theta^g}, \chi_{\theta^g} \rangle_N = 1.$$

This is indeed the case:

$$\begin{aligned}
\langle \chi_{\theta^g}, \chi_{\theta^g} \rangle_N &= \frac{1}{|N|} \sum_{x \in N} \chi_{\theta^g}(x) \overline{\chi_{\theta^g}(x)} \\
&= \frac{1}{|N|} \sum_{x \in N} \chi_\theta(g^{-1}xg) \overline{\chi_\theta(g^{-1}xg)} \\
&= \frac{1}{|N|} \sum_{x \in N} \chi_\theta(x) \overline{\chi_\theta(x)} \\
&= \langle \chi_\theta, \chi_\theta \rangle_N \\
&= 1.
\end{aligned}$$

$\square$

Theorem 6.7. *Let G be a finite group, N be a normal subgroup of G , π be an irreducible σ -projective representation of G , and θ be an irreducible $\text{Res}_N^G \sigma$ -projective representation of N such that*

$$l := \dim \text{Hom}_N(\theta, \text{Res}_N^G \pi)$$

is nonzero. Let $c_\theta := [I_G(\theta) : N]$. Then,

- (1) $\text{Res}_N^G(\text{Ind}_N^G(\theta))$ has the following decomposition into irreducible $\text{Res}_N^G \sigma$ -projective representations:

$$\text{Res}_N^G(\text{Ind}_N^G(\theta)) \simeq c_\theta \bigoplus_{r \in G/I_G(\theta)} \theta^r,$$

- (2) the induced character of θ satisfies:

$$\langle \text{Ind}_N^G(\chi_\theta), \text{Ind}_N^G(\chi_\theta) \rangle_G = c_\theta,$$

- (3) $\text{Res}_N^G(\pi)$ has the following decomposition into irreducible $\text{Res}_N^G \sigma$ -projective representations:

$$\text{Res}_N^G(\pi) \simeq l \bigoplus_{r \in G/I_G(\theta)} \theta^r,$$

(4) the degrees of π and θ satisfy the following equality:

$$d_\pi = l[G : I_G(\theta)]d_\theta.$$

Proof. Since N is a normal subgroup of G , any choice of coset representatives of G/N will also be representatives of the (N, N) -double cosets. By Lemma 6.3 we have that N is also a normal subgroup of $I_G(\theta)$, and we may think of the quotient group $I_G(\theta)/N$ as a subgroup of G/N . Hence, by Mackey's formula and Remark 6.5 we have that

$$\begin{aligned} \text{Res}_N^G(\text{Ind}_N^G(\theta)) &= \bigoplus_{s \in G/N} \theta^s \\ &= \bigoplus_{t \in I_G(\theta)/N} \left(\bigoplus_{r \in (G/N)/(I_G(\theta)/N)} \theta^{tr} \right) \\ &\simeq \bigoplus_{t \in I_G(\theta)/N} \left(\bigoplus_{r \in (G/N)/(I_G(\theta)/N)} \theta^r \right) \\ &= c_\theta \bigoplus_{r \in (G/N)/(I_G(\theta)/N)} \theta^r \\ &= c_\theta \bigoplus_{r \in G/I_G(\theta)} \theta^r, \end{aligned}$$

where the last equality follows by the third isomorphism theorem. By Lemma 6.6 we have that each θ^r is irreducible, proving the first statement.

For the second statement observe that $\theta^{r_1} \simeq \theta^{r_2}$, for $r_1, r_2 \in G/I_G(\theta)$ if and only if $r_1 = r_2$. Hence, each representation appearing in

$$\bigoplus_{r \in G/I_G(\theta)} \theta^r$$

are inequivalent. Using the first statement along with Frobenius reciprocity and the orthogonality relations, we get that

$$\begin{aligned} \langle \text{Ind}_N^G(\chi_\theta), \text{Ind}_N^G(\chi_\theta) \rangle_G &= \langle \chi_\theta, \text{Res}_N^G(\text{Ind}_N^G(\chi_\theta)) \rangle_N \\ &= c_\theta \sum_{r \in G/I_G(\theta)} \langle \chi_\theta, \chi_{\theta^r} \rangle_N \\ &= c_\theta. \end{aligned}$$

For the third statement we note that for any $g \in G$, the map $\text{Hom}_N(\theta, \text{Res}_N^G(\pi)) \rightarrow \text{Hom}_N(\theta^g, \text{Res}_N^G(\pi))$, $T \mapsto \pi(g)T$ is an isomorphism (checking that this is true is routine work at this point so we omit the details). This in particular implies that each projective representation θ^g is an irreducible projective sub-representation of $\text{Res}_N^G(\pi)$ with multiplicity l . It remains to say that these are the *only* irreducible projective sub-representations of $\text{Res}_N^G(\pi)$. By Frobenius reciprocity we have that π is a projective sub-representation of $\text{Ind}_N^G(\theta)$, hence $\text{Res}_N^G(\pi)$ is a projective sub-representation of $\text{Res}_N^G(\text{Ind}_N^G(\theta))$, the latter of which we know by the first statement only contains projective sub-representations of the form θ^g . Hence,

$$\text{Res}_N^G(\pi) \simeq l \bigoplus_{r \in G/I_G(\theta)} \theta^r.$$

The final statement follows immediately by the third statement. $\square$

For the final few results we want to discuss it will be useful to establish some notation.

Definition 6.8. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. Let

$$\begin{aligned}\widehat{G}^\sigma &:= \{[\pi] : \pi \text{ is an irreducible } \sigma\text{-projective representation of } G\}, \\ \widehat{N}^\sigma &:= \{[\theta] : \theta \text{ is an irreducible } \text{Res}_N^G \sigma\text{-projective representation of } N\},\end{aligned}$$

where the bracket $[\pi]$ means that we only consider π up to isomorphism.

If θ is an irreducible $\text{Res}_N^G \sigma$ -projective representation of N , we set

$$\begin{aligned}\widehat{I_G(\theta)}^\sigma &:= \{[\gamma] : \gamma \text{ is an irreducible } \text{Res}_{I_G(\theta)}^G \sigma\text{-projective representation of } I_G(\theta)\}, \\ \widehat{G}^\sigma(\theta) &:= \{[\pi] \in \widehat{G}^\sigma : \text{Hom}_N(\theta, \text{Res}_N^G(\pi)) \neq 0\}, \\ \widehat{I_G}^\sigma(\theta) &:= \{[\gamma] \in \widehat{I_G(\theta)}^\sigma : \text{Hom}_N(\theta, \text{Res}_N^{I_G(\theta)}(\gamma)) \neq 0\}.\end{aligned}$$

For an element $[\pi] \in \widehat{G}^\sigma(\theta)$ we define $l_\pi := \dim \text{Hom}_N(\theta, \text{Res}_N^G(\pi))$ and call it the *inertia index* of π .

The map $G \times \widehat{N}^\sigma \rightarrow \widehat{N}^\sigma$, $(g, \theta) \mapsto \theta^g$ defines a G -action on $\widehat{N}^\sigma$. From this perspective we have that $I_G(\theta)$ is the stabilizer subgroup of G with respect to θ . We will use the notation Σ_N^G for a choice of representatives of the G -orbits of $\widehat{N}^\sigma$.

Corollary 6.9. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. Then

$$\widehat{G}^\sigma = \bigcup_{\theta \in \Sigma_N^G} \widehat{G}^\sigma(\theta)$$

is a partition of $\widehat{G}^\sigma$.

Proof. This is an immediate consequence of Theorem 6.7 point three. $\square$

Lemma 6.10. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is a $\text{Res}_N^G \sigma$ -projective representation of N , then $\widehat{G}^\sigma(\theta) = \widehat{G}^\sigma(\theta^g)$ for every $g \in G$. Furthermore,

$$(2) \quad \text{Ind}_N^G(\theta) \simeq \bigoplus_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi \pi \simeq \text{Ind}_N^G(\theta^g).$$

Proof. By Theorem 6.7 point three, we have that θ^g is a projective sub-representation of $\text{Res}_N^G(\pi)$ if and only if θ is a projective sub-representation of $\text{Res}_N^G(\pi)$. Hence, $\widehat{G}^\sigma(\theta) = \widehat{G}^\sigma(\theta^g)$, furthermore, θ and θ^g have the same multiplicity within $\text{Res}_N^G(\pi)$. From this (2) follows. $\square$

Proposition 6.11. Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is an irreducible $\text{Res}_N^G \sigma$ -projective representation of N , then

$$\sum_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi^2 = [I_G(\theta) : N].$$

Proof. By (2) and Theorem 6.7 point one, we have that

$$\bigoplus_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi \text{Res}_N^G(\pi) \simeq \text{Res}_N^G(\text{Ind}_N^G(\theta)) \simeq [I_G(\theta) : N] \bigoplus_{r \in G/I_G(\theta)} \theta^r.$$

It follows that

$$\begin{aligned}
[I_G(\theta) : N] &= \dim \operatorname{Hom}_N \left(\theta, \bigoplus_{r \in G/I_G(\theta)} \theta^r \right) \\
&= \dim \operatorname{Hom}_N \left(\theta, \bigoplus_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi \operatorname{Res}_N^G(\pi) \right) \\
&= \sum_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi \dim \operatorname{Hom}_N (\theta, \operatorname{Res}_N^G(\pi)) \\
&= \sum_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi^2.
\end{aligned}$$

□

Theorem 6.12 (Clifford correspondence). *Let G be a finite group, N be a normal subgroup of G , and $\sigma \in Z^2(G; \mathbb{T})$. If θ is an irreducible $\operatorname{Res}_N^G \sigma$ -projective representation of N , then $\operatorname{Ind}_{I_G(\theta)}^G : \widehat{I_G(\theta)}^\sigma(\theta) \rightarrow \widehat{G}^\sigma(\theta)$ is well-defined and a bijection.*

Proof. If we apply Lemma 6.10 with $I_G(\theta)$ in place of G we get that

$$\operatorname{Ind}_N^{I_G(\theta)}(\theta) \simeq \bigoplus_{\gamma \in \widehat{I_G(\theta)}^\sigma(\theta)} l_\gamma \gamma.$$

By Proposition 5.7 we get that

$$(3) \quad \operatorname{Ind}_N^G(\theta) \simeq \bigoplus_{\gamma \in \widehat{I_G(\theta)}^\sigma(\theta)} l_\gamma \operatorname{Ind}_{I_G(\theta)}^G(\gamma).$$

Note that the inertia group $I_{I_G(\theta)}(\theta) = I_G(\theta)$. Hence, by Proposition 6.11 and Theorem 6.7 point two, we have that

$$\begin{aligned}
\sum_{\gamma \in \widehat{I_G(\theta)}^\sigma(\theta)} l_\gamma^2 &= [I_G(\theta) : N] \\
&= \langle \operatorname{Ind}_N^G(\theta), \operatorname{Ind}_N^G(\theta) \rangle_G \\
&= \sum_{\gamma \in \widehat{I_G(\theta)}^\sigma(\theta)} \sum_{\omega \in \widehat{I_G(\theta)}^\sigma(\theta)} l_\gamma l_\omega \langle \operatorname{Ind}_{I_G(\theta)}^G(\gamma), \operatorname{Ind}_{I_G(\theta)}^G(\omega) \rangle_G.
\end{aligned}$$

This equation can only be true if

$$\langle \operatorname{Ind}_{I_G(\theta)}^G(\gamma), \operatorname{Ind}_{I_G(\theta)}^G(\omega) \rangle_G = \begin{cases} 1, & \gamma \simeq \omega, \\ 0, & \gamma \not\simeq \omega. \end{cases}$$

Which, by Corollary 3.10 tells us that $\operatorname{Ind}_{I_G(\theta)}^G(\gamma)$ is an irreducible σ -projective representation of G . Furthermore, this equation also tells us that

$$\operatorname{Ind}_{I_G(\theta)}^G : \widehat{I_G(\theta)}^\sigma(\theta) \rightarrow \widehat{G}^\sigma(\theta)$$

is injective. To see that it is surjective notice that by (2) and (3) that

$$\bigoplus_{\pi \in \widehat{G}^\sigma(\theta)} l_\pi \pi \simeq \operatorname{Ind}_N^G(\theta) \simeq \bigoplus_{\gamma \in \widehat{I_G(\theta)}^\sigma(\theta)} l_\gamma \operatorname{Ind}_{I_G(\theta)}^G(\gamma).$$

In particular, every summand on the left hand side also appears on the right hand side, which proves surjectivity. □

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