

Supersmoothness of the Alfeld split

Michael S. Floater and Kaibo Hu

Abstract We consider spline functions over an Alfeld split in $\mathbb{R}^n$, with smoothness r on the faces. In a recent paper, we derived the maximal order of supersmoothness of such a spline at the interior vertex. The proof was based on a characterization of supersmoothness in terms of the degeneracy of polynomial spline spaces on the split, and on the dimension formula for such spaces due to Foucart, Sorokina, and Schenck. In this note, we present an alternative, inductive proof of supersmoothness using only properties of directional derivatives.

1 Introduction

In this paper we consider splines over a simplicial mesh Δ of n -simplices T_i in $\mathbb{R}^n$, i.e., a triangular mesh in $\mathbb{R}^2$, a tetrahedral mesh in $\mathbb{R}^3$, etc. We let $\Omega = \cup_i T_i$. For an integer $r \geq 0$ we call $s : \Omega \rightarrow \mathbb{R}$ a *spline* of smoothness C^r if $s \in C^r(\Omega)$ and for each simplex $T_i \in \Delta$, there is a function $s_i \in C^\infty(T_i)$ such that $s|_{T_i} = s_i$. Here, by $s_i \in C^\infty(T_i)$ we mean that s_i is the restriction of a function in $C^\infty(U_i)$ to T_i for some open set U_i containing the closed set T_i . We denote the set of all such splines s by $S^r(\Delta)$. A special case of a spline $s \in S^r(\Delta)$ is when all its pieces s_i are polynomials of degree at most d for some fixed $d \geq 0$. These polynomial splines form a linear space, which we will denote by $S_d^r(\Delta)$. However, the pieces of a spline in $S^r(\Delta)$ could in general be polynomials, rational functions, trigonometric functions, and so on.

Splines in $S^r(\Delta)$ often have extra orders of smoothness at a vertex v of Δ , a property known as *supersmoothness*. This was first observed for polynomial splines. For

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example, the Clough-Tocher macroelement, which is a C^1 piecewise cubic polynomial over the split shown in Figure 1, is twice differentiable at the refinement point v , as observed by Farin [9], and is therefore said to have supersmoothness of order 2 at v . For the construction of splines or finite elements with higher orders of continu-

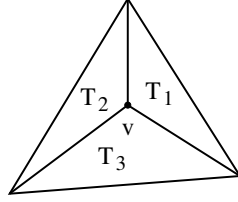


Fig. 1 A Clough-Tocher split.

ity, it is important to recognize and make use of supersmoothness. It plays a role in many of the macroelement constructions surveyed by Lai and Schumaker[15], where applications of splines to approximation theory and computer-aided geometric design are discussed. Recognizing supersmoothness is also important for constructing finite elements. For example, to obtain a C^1 space on an arbitrary triangular mesh in a planar domain, one uses second-order derivatives as degrees of freedom at vertices and requires that they are single-valued among the triangles (Argyris finite elements). An underlying reason for this additional (higher than C^1) derivative degree of freedom at vertices is the supersmoothness. For solving vector- and tensor-valued problems, there is a surge of interest in constructing spline and finite element spaces that fit into sequences in the framework of finite element exterior calculus [4, 3]. See, e.g. [2, 4, 5, 6, 7, 8, 12, 13, 14], and the references therein for some results in this direction. These sequences connect scalar splines to tensor fields. The tensor fields can be obtained by differentiating the scalar splines, which may inherit the supersmoothness of the scalar fields. Thus, supersmoothness is also of importance in the study of these problems.

Since Farin's observation about the Clough-Tocher element, Sorokina, in [18] and [19] has derived further supersmoothness properties of polynomial splines, and in particular higher order supersmoothness in a cell in $\mathbb{R}^2$. More recently, Shekhtman and Sorokina [17] showed that supersmoothness is also inherent in more general splines, such as those in $S^r(\Delta)$.

The results of [17] inspired us to look for more supersmoothness properties of splines, leading to a characterization of supersmoothness in terms of the dimension of polynomial spline spaces in [10], which we now describe. For a smooth enough function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote by

$$D^\alpha f(x) = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} f(x)$$

its partial derivative at a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with respect to the multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. If two simplices $T_i, T_j \in \Delta$

share a common face $F = T_i \cap T_j \neq \emptyset$ of any dimension up to $n - 1$ then the corresponding pieces s_i, s_j of a spline $s \in S^r(\Delta)$ have common derivatives on F of all orders up to r , i.e.,

$$D^\alpha s_i|_F = D^\alpha s_j|_F, \quad |\alpha| \leq r,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

We will say that a spline $s \in S^r(\Delta)$ has *supersmoothness* of order $\rho \geq r$ at a vertex v of Δ , and write $s \in C^\rho(v)$, if

$$D^\alpha s_i(v) = D^\alpha s_j(v), \quad |\alpha| \leq \rho,$$

whenever $v \in T_i \cap T_j$, or equivalently, for all pairs of simplices $T_i, T_j \in \Delta_v$, where Δ_v is the *cell* of v , the set of all simplices of Δ that contain v . Using Taylor series we derived the following theorem in [10].

Theorem 1 *All splines $s \in S^r(\Delta)$ have supersmoothness of order $\rho \geq r$ at a vertex v of Δ if and only if $S_\rho^r(\Delta_v)$ is degenerate.*

By degenerate we mean that $S_\rho^r(\Delta_v) = \Pi_\rho$, where Π_ρ is the linear space of polynomials in $\mathbb{R}^n$ of degree at most ρ . Since

$$\dim \Pi_\rho = \binom{\rho + n}{n},$$

this means that $S_\rho^r(\Delta_v)$ is degenerate if

$$\dim S_\rho^r(\Delta_v) = \binom{\rho + n}{n}.$$

2 Supersmoothness for the Alfeld split

An important example of a cell is the Alfeld split Δ_A in $\mathbb{R}^n$, which is a cell Δ_v consisting of just $n + 1$ simplices adjacent to v . An Alfeld split is a Clough-Tocher split in $\mathbb{R}^2$, while in $\mathbb{R}^3$ it consists of four tetrahedra meeting at v , as shown in Figure 2. Using Theorem 1 we were able to deduce, in [10], the precise order of supersmoothness of splines in $S^r(\Delta_A)$ at v using the dimension formula for $S_d^r(\Delta_A)$. This dimension formula, conjectured by Foucart and Sorokina [11] and proved by Schenck [16], is

$$\dim S_d^r(\Delta_A) = \dim \Pi_d + A(n, d, r), \quad (1)$$

where

$$A(n, d, r) = \begin{cases} n \binom{d+n-(r+1)(n+1)/2}{n}, & \text{if } r \text{ is odd;} \\ \binom{d+n-1-r(n+1)/2}{n} + \dots + \binom{d-r(n+1)/2}{n}, & \text{if } r \text{ is even.} \end{cases}$$

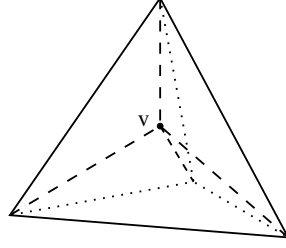


Fig. 2 Alfeld split in $\mathbb{R}^3$.

Using the fact that the binomial coefficient $\binom{n}{k}$ is zero when $k < 0$ or $k > n$, a straightforward calculation shows that $A(n, d, r) = 0$ if and only if $d \leq \rho_{n,r}$, where

$$\rho_{n,r} := r + (n-1) \left\lfloor \frac{r+1}{2} \right\rfloor. \quad (2)$$

By Theorem 1 it follows that the precise supersmoothness at v of splines in $S^r(\Delta_A)$ is $\rho_{n,r}$. As an example, the exact supersmoothness of splines in $S^1(\Delta_A)$ is n .

The purpose of the remainder of this paper is to present an alternative, more elementary proof of the fact that splines in $S^r(\Delta_A)$ have supersmoothness of order at least $\rho_{n,r}$.

3 Alternative proof of supersmoothness

Without using the dimension formula (1) we will show

Theorem 2 *A spline $s \in S^r(\Delta_A)$ has supersmoothness at v of order at least $\rho_{n,r}$.*

We will work with the directional derivatives of s in the $n+1$ directions defined by the interior edges of Δ_A , i.e., the edges that emanate from v . This was the approach used by Alfeld [1] to show that the Clough-Tocher element has supersmoothness of order 2, and by Shekhtman and Sorokina [17] to derive one order of supersmoothness for splines in an arbitrary cell in $\mathbb{R}^2$.

Let $\Delta_A = \{T_1, T_2, \dots, T_{n+1}\}$ and $T = \cup_{i \in I} T_i$, where $I = \{1, 2, \dots, n+1\}$. For each $i \in I$, let v_i be the vertex common to T_j , $j \in I$, $j \neq i$, ‘opposite’ to T_i . For each distinct $i, j \in I$, we denote by $F_{i,j}$ the $(n-1)$ -dimensional face $F_{i,j} = T_i \cap T_j$. The notation for the $\mathbb{R}^2$ case is shown in Figure 3.

Let $\mathcal{D}_i$ denote the directional derivative in the direction $v_i - v$, $i \in I$. Since any n of the $n+1$ vectors $v_i - v$, $i \in I$, are linearly independent, it is sufficient to show that $\mathcal{D}^\alpha s \in C(v)$ for all $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}_0^{n+1}$ with $|\alpha| = \alpha_1 + \dots + \alpha_{n+1} \leq \rho_{n,r}$, where

$$\mathcal{D}^\alpha := \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_{n+1}^{\alpha_{n+1}}.$$

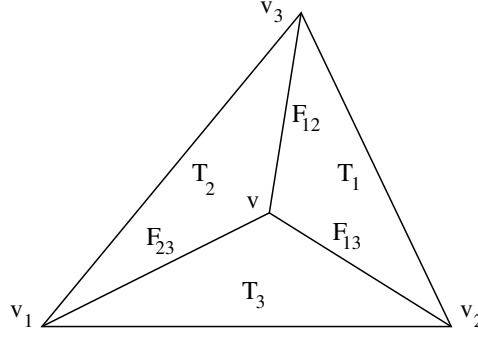


Fig. 3 Notation for the split in $\mathbb{R}^2$.

We start with the observation that for each $i, j \in I$, the $(n-1)$ -dimensional face $F_{i,j} = T_i \cap T_j$ contains all edges $[v, v_k]$, $k \in I$, except $[v, v_i]$ and $[v, v_j]$. Two functions f and g that are C^∞ in a neighborhood of $F_{i,j}$ and are equal on $F_{i,j}$ have the same directional derivative on $F_{i,j}$ in any direction within $F_{i,j}$ because $f - g = 0$ on $F_{i,j}$. This leads to a condition on $\alpha \in \mathbb{N}_0^{n+1}$ for when the pieces s_i and s_j of a spline in $S^r(\Delta_A)$ have the same $\mathcal{D}^\alpha$ derivative at v .

Lemma 1 *Let $s \in S^r(\Delta_A)$ and let $\alpha \in \mathbb{N}_0^{n+1}$ be such that $\alpha_i + \alpha_j \leq r$ for some distinct $i, j \in I$. Then $\mathcal{D}^\alpha s_i(v) = \mathcal{D}^\alpha s_j(v)$.*

Proof Since $s \in C^r(T)$ and $\alpha_i + \alpha_j \leq r$, $\mathcal{D}_i^{\alpha_i} \mathcal{D}_j^{\alpha_j} s \in C(T)$, and in particular,

$$\mathcal{D}_i^{\alpha_i} \mathcal{D}_j^{\alpha_j} s_i|_{F_{i,j}} = \mathcal{D}_i^{\alpha_i} \mathcal{D}_j^{\alpha_j} s_j|_{F_{i,j}}.$$

We can differentiate both sides of this equation in directions contained in the common face $F_{i,j}$. Thus we can differentiate it with respect to

$$\prod_{k \in I \setminus \{i,j\}} \mathcal{D}_k^{\alpha_k},$$

and we deduce that

$$\mathcal{D}^\alpha s_i|_{F_{i,j}} = \mathcal{D}^\alpha s_j|_{F_{i,j}},$$

which gives the result. $\square$

From this we deduce

Lemma 2 *Let $\alpha \in \mathbb{N}_0^{n+1}$ with $\min(\alpha) + \max(\alpha) \leq r$. Then $\mathcal{D}^\alpha s \in C(v)$.*

Proof Choose some element α_i of α with $\alpha_i = \min(\alpha)$, and let $j \in I$, $j \neq i$. Then, $\alpha_j \leq \max(\alpha)$ and so $\alpha_i + \alpha_j \leq r$ and by Lemma 1, $\mathcal{D}^\alpha s_i(v) = \mathcal{D}^\alpha s_j(v)$ for all $j \neq i$. $\square$

3.1 Partial ordering

To prove Theorem 2, we need to show that $D^\alpha s \in C(v)$ for all $\alpha \in \mathbb{N}_0^{n+1}$ with $|\alpha| \leq \rho_{n,r}$. To do this we will use induction among all sequences $\alpha \in \mathbb{N}_0^{n+1}$ with the same norm $|\alpha|$, with respect to a certain partial ordering. Let $m(\alpha) \geq 1$ denote the multiplicity of $\min(\alpha)$ in the sequence α . For $\alpha, \beta \in \mathbb{N}_0^{n+1}$ with $|\alpha| = |\beta|$ we will write $\alpha < \beta$ if α has less variation than β in the sense that either

1. $\min(\alpha) > \min(\beta)$, or
2. $\min(\alpha) = \min(\beta)$ and $m(\alpha) < m(\beta)$.

If both $\min(\alpha) = \min(\beta)$ and $m(\alpha) = m(\beta)$ then we write $\alpha = \beta$.

3.2 Sequences of least variation

With respect to the partial ordering defined above, consider an element $\alpha \in \mathbb{N}_0^{n+1}$ with least variation, i.e., such that $\alpha \leq \beta$ for all $\beta \in \mathbb{N}_0^{n+1}$ with $|\beta| = |\alpha|$. Then α has the property that $\max(\alpha) - \min(\alpha) \leq 1$. To see this, suppose, on the contrary, that $\max(\alpha) - \min(\alpha) \geq 2$, and let i, j be such that $\alpha_i = \max(\alpha)$ and $\alpha_j = \min(\alpha)$. Then define $\beta \in \mathbb{N}_0^{n+1}$ by $\beta_i = \alpha_i - 1$, $\beta_j = \alpha_j + 1$, and $\beta_k = \alpha_k$ for $k \neq i, j$. Then $|\beta| = |\alpha|$ and $\beta < \alpha$ and α does not have least variation.

Thus the elements of a sequence $\alpha \in \mathbb{N}_0^{n+1}$ of least variation differ by at most 1. Suppose then that $\alpha \in \mathbb{N}_0^{n+1}$ has least variation and that $|\alpha| = \rho_{n,r}$, and recalling (2), observe that we can express $\rho_{n,r}$ as

$$\rho_{n,r} = \begin{cases} n \left(\frac{r+1}{2} \right) + \frac{r-1}{2}, & \text{if } r \text{ is odd;} \\ (n+1) \frac{r}{2}, & \text{if } r \text{ is even.} \end{cases} \quad (3)$$

We then deduce that if r is odd, α is a permutation of the sequence

$$\underbrace{\left(\frac{r+1}{2}, \dots, \frac{r+1}{2}, \frac{r-1}{2} \right)}_n \quad (4)$$

and if r is even,

$$\alpha = \underbrace{\left(\frac{r}{2}, \dots, \frac{r}{2} \right)}_{n+1} \quad (5)$$

Clearly, for both odd and even r we have $\min(\alpha) + \max(\alpha) = r$. Moreover, if $\alpha \in \mathbb{N}_0^{n+1}$ has least variation and $|\alpha| \leq \rho_{n,r}$, then its elements, in decreasing order will be equal to or less than those in (4) or (5), and therefore $\min(\alpha) + \max(\alpha) \leq r$. By Lemma 2, it thus follows that $\mathcal{D}^\alpha s \in C(v)$ for all $\alpha \in \mathbb{N}_0^{n+1}$ of least variation with $|\alpha| \leq \rho_{n,r}$.

3.3 Completing the proof

To complete the proof of Theorem 2 we will use the following lemma.

Lemma 3 Suppose $\alpha \in \mathbb{N}_0^{n+1}$ with $|\alpha| \leq \rho_{n,r}$ and that $\min(\alpha) + \max(\alpha) > r$. Then $2 \min(\alpha) < r$.

Proof Suppose for the sake of contradiction that $2 \min(\alpha) \geq r$.

If r is even then $\min(\alpha) \geq r/2$ and so $\alpha_i \geq r/2$ for $i \in I$, and moreover since $\min(\alpha) + \max(\alpha) > r$, there is at least one index i such that $\alpha_i \geq r/2 + 1$. Therefore, recalling (3),

$$|\alpha| > (n+1)\frac{r}{2} = \rho_{n,r},$$

which is a contradiction.

Otherwise, r is odd. Then $\min(\alpha) \geq (r+1)/2$ and $\alpha_i \geq (r+1)/2$ for all $i \in I$, and so

$$|\alpha| \geq (n+1)\frac{r+1}{2} > \rho_{n,r},$$

which is again a contradiction. $\square$

Now let $I_\rho = \{\alpha \in \mathbb{N}_0^{n+1} : |\alpha| = \rho\}$.

Proof of Theorem 2. Choose $\rho \leq \rho_{n,r}$ and let $\alpha \in I_\rho$. We will show that $\mathcal{D}^\alpha s \in C(v)$ by induction over sequences in I_ρ . If α has least variation then $\mathcal{D}^\alpha s \in C(v)$ as previously shown. Thus we may now suppose that α does not have least variation and that $\mathcal{D}^\beta s \in C(v)$ for any $\beta \in I_\rho$ such that $\beta < \alpha$.

Suppose next that $\min(\alpha) + \max(\alpha) \leq r$. Then $\mathcal{D}^\alpha s \in C(v)$ by Lemma 2. Otherwise, $\min(\alpha) + \max(\alpha) > r$. Then let $i \in I$ such that $\alpha_i = \min(\alpha)$, and let $j \in I, j \neq i$. If $\alpha_i + \alpha_j \leq r$, then $\mathcal{D}^\alpha s_i(v) = \mathcal{D}^\alpha s_j(v)$ by Lemma 1. Otherwise, $\alpha_i + \alpha_j > r$. Then we use the fact that since the n vectors $v_k - v, k \in I \setminus \{j\}$, are linearly independent, we can express the directional derivative $\mathcal{D}_j$ as

$$\mathcal{D}_j = \sum_{\substack{k=1 \\ k \neq j}}^{n+1} c_k \mathcal{D}_k, \quad (6)$$

for constants c_k . Applying this substitution $(\alpha_i + \alpha_j - r)$ times to $\mathcal{D}^\alpha$ shows that

$$\mathcal{D}^\alpha \in \text{span}\{\mathcal{D}^\beta : \beta \in J\}, \quad (7)$$

where

$$J = \{\beta \in I_\rho : \beta_j = r - \alpha_i, \beta_k \geq \alpha_k, k \neq j\}.$$

We now consider two types of β in J , (i) $\beta_i = \alpha_i$, and (ii) $\beta_i > \alpha_i$.

In case (i), $\beta_i + \beta_j = r$, and by Lemma 1, $\mathcal{D}^\beta s_i(v) = \mathcal{D}^\beta s_j(v)$. In case (ii), observe that by Lemma 3, $2\alpha_i < r$. Therefore, $\beta_j = r - \alpha_i > \alpha_i$. Since also $\beta_i > \alpha_i$ and $\beta_k \geq \alpha_k$ for all $k \in I \setminus \{i, j\}$ it follows that $\min(\beta) \geq \alpha_i$ and the number of occurrences of α_i in β is at most $m(\alpha) - 1$. Thus $\beta < \alpha$, and we may assume by the

induction hypothesis that $\mathcal{D}^\beta s \in C(v)$. In particular, $\mathcal{D}^\beta s_i(v) = \mathcal{D}^\beta s_j(v)$. Thus we deduce that $\mathcal{D}^\beta s_i(v) = \mathcal{D}^\beta s_j(v)$ for all $\beta \in J$, and so $\mathcal{D}^\alpha s_i(v) = \mathcal{D}^\alpha s_j(v)$. Since this argument applies to all $j \neq i$, it follows that $\mathcal{D}^\alpha s \in C(v)$. $\square$

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